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Algebraic independence of the values of certain functions at a transcendental number

by

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1. Introduction. Throughout the present paper, we denote by K an algebraic number field of finite degree, and denote by I_K its integer ring. Let $\boldsymbol{f}(z) = (f_1(z), \ldots, f_m(z))^t$ be a column vector of m holomorphic functions in the unit disk whose coefficients in their Taylor series expansions at the origin all lie in the field K. Suppose that $\boldsymbol{f}(z)$ satisfies the functional equation

$$\text{(M)} \qquad \boldsymbol{f}(z) = \boldsymbol{A}(z)\boldsymbol{f}(z^r) + \boldsymbol{B}(z) \qquad (r \in \boldsymbol{N},\ r \geqslant 2),$$

where $\boldsymbol{A}(z)$ is an $m \times m$ non-singular matrix with entries in $K[z]$ and $\boldsymbol{B}(z)$ is a column vector of degree m with entries in $K[z]$. In [9], Mahler first studied the algebraic independence of the values of the above type functions at an algebraic number in the unit disk, and later, several mathematicians improved his results. For such studies, we refer the reader to the papers by Mahler [9], Loxton and van der Poorten ([7], [8]), Kubota [6], Nesterenko [12] and Nishioka [14]. At the present stage, we have the following result as a special case of the recent result by Nishioka [14] (see also [6] and [12]).

THEOREM. *In the notation as above, put* $a(z) = \det \boldsymbol{A}(z)$. *Suppose that* $f_1(z), \ldots, f_m(z)$ *are algebraically independent over the field* $K(z)$. *Let* α *be a nonzero algebraic number in the unit disk satisfying* $a(\alpha^{r^l}) \neq 0$ *for any* l $(l = 0, 1, 2, \ldots)$. *Then the numbers* $f_1(\alpha), \ldots, f_m(\alpha)$ *are algebraically independent.*

In connection with this theorem, we study in the present paper the transcendence degree of the field $\boldsymbol{Q}(\omega, f_1(\omega), \ldots, f_m(\omega))$ over the field $\boldsymbol{Q}$, where ω is a transcendental number in the unit disk. Our main result is the following

THEOREM 1. *Let* $f_1(z), \ldots, f_m(z)$ *be m holomorphic functions in the unit disk whose coefficients in their Taylor series expansions at the origin all lie in the field* K. *Suppose that* $f_1(z), \ldots, f_m(z)$ *are algebraically independent over the field* $K(z)$ *and* $\boldsymbol{f}(z) = (f_1(z), \ldots, f_m(z))^t$ *satisfies the functional equation* (M). *Let* ω

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be a transcendental number in the unit disk. Then we have

$$\operatorname{tr}\deg_{\boldsymbol{Q}} \boldsymbol{Q}(\omega, f_1(\omega), \ldots, f_m(\omega)) \geqslant [(m+1)/2].$$

Let $F_r(z)$ be the function defined by

$$F_r(z) = \sum_{\nu=0}^{\infty} z^{r^\nu} \quad (r \in \boldsymbol{N},\ r \geqslant 2). \tag{1.1}$$

By Loxton and van der Poorten [8], the $r-1$ functions $F_r(z)$, $F_r(z^2)$, …, $F_r(z^{r-1})$ are algebraically independent over the field $\boldsymbol{C}(z)$. Since $F_r(z^i)$ satisfies the functional equation $F_r(z^i) = F_r(z^{ri}) + z^i$ we have the following

Corollary. *Let $F_r(z)$ be the function defined by* (1.1), *and let ω be a transcendental number in the unit disk. Then we have*

$$\operatorname{tr}\deg_{\boldsymbol{Q}} \boldsymbol{Q}(\omega, F_r(\omega), F_r(\omega^2), \ldots, F_r(\omega^{r-1})) \geqslant [r/2].$$

Remark 1.1. Theorem 1 and its corollary are analogous to the results on the special values of exponential and elliptic functions (see Chudnovsky [3], Nesterenko [11], Philippon [16] and Diaz [4]).

In Theorem 1, if $m = 1$ or 2, we have only the trivial lower bound because of the transcendence of ω. In these cases, however, we can prove the following results (Theorems 2 and 3 below).

Theorem 2. *Let $f(z)$ be a transcendental holomorphic function in the unit disk whose coefficients in its Taylor series expansion at the origin all lie in the field K. Suppose that $f(z)$ satisfies the functional equation*

$$f(z) = a(z) f(z^r) + b(z) \quad (r \in \boldsymbol{N},\ r \geqslant 2),$$

where $a(z), b(z) \in K[z]$. Let ω be a nonzero complex number in the unit disk. If $f(\omega) \in \overline{\boldsymbol{Q}(\omega)}$ and $a(\omega^{r^l}) \neq 0$ for any l ($l = 0, 1, 2, \ldots$), then ω is a transcendental number and we have the following measure of transcendence of ω: Let $P(x) \in \boldsymbol{Z}[x]$ be a nonzero polynomial whose degree is at most d and whose height is at most H. Then, assuming $dH \geqslant 3$, we have

$$|P(\omega)| > \exp\left\{-C_1 \log(dH) \log\log(dH) \left(d \log(d+1) + (\log(dH))^2\right)\right\},$$

where C_1 is a positive constant depending only on $K, r, \omega, a(z), b(z)$ and $f(z)$. In particular, the type of transcendence of ω is at most $3+\varepsilon$ for any $\varepsilon > 0$.

Remark 1.2. The transcendence of ω in the theorem is an easy consequence of a special case of the Theorem quoted above. But we shall prove it without using the Theorem. Of course our main purpose is to obtain a good lower bound for $|P(\omega)|$.

The above measure is very sharp with respect to the degree of the polynomial $P(x)$. To emphasize this fact, we state the following

Corollary. *In the notation and the assumptions of Theorem 2, let $P(x) \in \boldsymbol{Z}[x]$ be a nonzero polynomial whose degree is at most d and whose height is at most H. Then, assuming $d \geqslant H \geqslant 3$, we have*

$$|P(\omega)| > \exp\{-C_2 d (\log d)^2 \log\log d\},$$

where C_2 is a positive constant depending only on K, r, ω, $a(z)$, $b(z)$ and $f(z)$.

Remark 1.3. The above corollary implies that any complex number ω satisfying the assumptions of Theorem 2 is an $\tilde{S}$-number of order 1 in the sense of Sprindžuk [17]. In this connection, we also note that the following result of Chudnovsky is known: The number e^π is an $\tilde{S}$-number of order 1 (see Chudnovsky [3], Chap. 1, Theorem 2.8).

Using Theorem 2, we can prove the following

Theorem 3. *Let $f_1(z)$ and $f_2(z)$ be two holomorphic functions in the unit disk whose coefficients in their Taylor series expansions at the origin all lie in the field K. Suppose that $f_1(z)$ and $f_2(z)$ are algebraically independent over the field $K(z)$, and satisfy the functional equations*

$$f_i(z) = a_i(z) f_i(z^r) + b_i(z) \quad (r \in \boldsymbol{N},\ r \geqslant 2)$$

respectively, where $a_i(z), b_i(z) \in K[z]$. Let ω be a transcendental number in the unit disk. Then we have

$$\operatorname{tr}\deg_{\boldsymbol{Q}} \boldsymbol{Q}(\omega, f_1(\omega), f_2(\omega)) \geqslant 2.$$

If ω and $f_2(\omega)$ are algebraically dependent, then we have further the following measure of algebraic independence of ω and $f_1(\omega)$: Let $P(x, y) \in \boldsymbol{Z}[x, y]$ be a nonzero polynomial whose total degree is at most d and whose height is at most H. Put $t = d + \log H$, and assume $t > 1$. Then we have

$$|P(\omega, f_1(\omega))| > \exp\{-C_2 d^3 t^6 (\log t)^{13}\},$$

where C_2 is a positive constant depending only on K, r, ω, $a_i(z)$, $b_i(z)$ and $f_i(z)$. In particular, the type of algebraic independence of ω and $f_1(\omega)$ is at most $9+\varepsilon$ for any $\varepsilon > 0$.

2. Notation and lemmas. Hereafter we use the usual notation as follows. For any algebraic number α with minimal defining polynomial

$$Q(x) = a_0(x-\alpha)(x-\alpha^{(1)}) \ldots (x-\alpha^{(\chi-1)}) \in \boldsymbol{Z}[x] \quad (a_0 > 0),$$

we denote by $\overline{|\alpha|}$ the *house* of α, i.e. the maximum of the absolute values of the roots of $Q(x)$, and by $M(\alpha)$ the *Mahler measure* of α, i.e. the number which is defined by

$$M(\alpha) = a_0 \prod_{i=0}^{\chi-1} \max(1, |\alpha^{(i)}|), \quad \alpha^{(0)} = \alpha.$$

For any polynomial P (in any number of variables) whose coefficients are algebraic numbers, we denote by $\deg_x P$ the *degree of P in the variable x*, by $\deg P$ the *total degree* of P, by $H(P)$ the *height* of P, i.e. the maximum of the houses of the coefficients of P, and by $L(P)$ the *length* of P, i.e. the sum of the houses of the coefficients of P. For a formal power series $f(z)\in C[[z]]$, we denote by $\operatorname{ord} f(z)$ the *order* of zeros of $f(z)$ at $z=0$.

Now we give several lemmas. The first two lemmas are basic tools for the proof of all our theorems. The following result gives the construction of the (so-called) auxiliary function.

LEMMA 2.1. *Let $f_1(z), \ldots, f_m(z)$ be m holomorphic functions in the unit disk whose coefficients of their Taylor series expansions at the origin all lie in the field K. Suppose that $\boldsymbol{f}(z) = (f_1(z), \ldots, f_m(z))^t$ satisfies the functional equation* (M) *in Section* 1. *Let N be a positive integer. Then there are algebraic integers $a(i_0, i_1, \ldots, i_m)\in I_K$ $(0\leqslant i_0, i_1, \ldots, i_m \leqslant N)$, not all zero, satisfying the following properties: For all $i_0, i_1, \ldots, i_m$ with $0\leqslant i_0, i_1, \ldots, i_m\leqslant N$, we have*

$$\overline{|a(i_0, i_1, \ldots, i_m)|} \leqslant \exp(C_4 N\log N),$$

where C_4 is a positive constant depending only on $K, m, r, \boldsymbol{A}(z), \boldsymbol{B}(z)$ and $\boldsymbol{f}(z)$, and the function

$$E(z) = \sum_{i_0=0}^{N}\sum_{i_1=0}^{N}\cdots\sum_{i_m=0}^{N} a(i_0, i_1, \ldots, i_m) z^{i_0} f_1(z)^{i_1}\ldots f_m(z)^{i_m} := \sum_{h=0}^{\infty} b_h z^h$$

satisfies the inequality $\operatorname{ord} E(z)\geqslant (1/2)N^{m+1}$, i.e. $b_h=0$ for all h with $h<(1/2)N^{m+1}$. Further, for all h with $h\geqslant (1/2)N^{m+1}$, we have

$$\overline{|b_h|}\leqslant \exp(C_5 N\log h) \quad \text{and} \quad D_0^{N[\log h]+1} b_h \in I_K,$$

where $C_5>0$ and $D_0\in N$ are constants depending only on $K, m, r, \boldsymbol{A}(z), \boldsymbol{B}(z)$ and $\boldsymbol{f}(z)$.

Proof. For each i, we write the Taylor series expansion for $f_i(z)$ at the origin as

$$f_i(z) = \sum_{h=0}^{\infty} f_{i,h} z^h.$$

Then, similarly to the proof of Lemma 1 of Becker-Landeck [1] we can show

$$\overline{|f_{i,h}|}\leqslant C_6 \exp\{C_7 \log(h+1)\} \quad \text{and} \quad D_1^{[\log(h+1)]+1} f_{j,h}\in I_K$$

for all i and h, where $C_6>0$, $C_7>0$ and $D_1\in N$ are constants depending only on $K, m, r, \boldsymbol{A}(z), \boldsymbol{B}(z)$ and $\boldsymbol{f}(z)$. Using Siegel's lemma (see for example, Waldschmidt [18], Lemma 1.3.1) together with these facts, we can construct, similarly to the proof of Proposition 1 of Nishioka [14], the auxiliary function $E(z)$ which satisfies all the properties stated in the lemma. We omit the details.

The following result is a special case of the result of Nishioka [13] (see also Becker and Nishioka [2]), which gives an estimate for the orders of zeros of certain functions.

LEMMA 2.2 (Nishioka's estimate for the zero order). *Let $f_1(z), \ldots, f_m(z)\in C[[z]]$ be m formal power series, and put $\boldsymbol{f}(z) = (f_1(z), \ldots, f_m(z))^t$. Suppose that $\boldsymbol{f}(z)$ satisfies the functional equation*

$$\boldsymbol{f}(z) = \boldsymbol{A}(z)\boldsymbol{f}(z^r)+\boldsymbol{B}(z) \quad (r\in N,\ r\geqslant 2),$$

where $\boldsymbol{A}(z)$ is an $m\times m$ non-singular matrix with entries in $C(z)$ and $\boldsymbol{B}(z)$ is a column vector of degree m with entries in $C(z)$. Let $Q(z, x_1, \ldots, x_m)\in C[z, x_1, \ldots, x_m]$ be a polynomial with $\deg_z Q\leqslant M$ and $\deg_{x_i} Q\leqslant N$ $(1\leqslant i\leqslant m)$ where $M\geqslant N\geqslant 1$. If $Q(z, f_1(z), \ldots, f_m(z))\neq 0$, then we have

$$\operatorname{ord} Q(z, f_1(z), \ldots, f_m(z))\leqslant C_8 M N^m,$$

where C_8 is a positive constant depending only on m, r, $\boldsymbol{A}(z)$, $\boldsymbol{B}(z)$ and $\boldsymbol{f}(z)$.

Remark 2.1. In the case $m=1$, the result of the above type was first proved independently by Galochkin [5] and by Miller [10].

Our proof of Theorem 1 depends deeply on a result of Philippon [16] which gives a criterion for algebraic independence of several numbers. We owe the following formulation of the result to Diaz [4]. For any $\theta\in C^n$ and $c>0$, we denote by $B(\theta, c)$ the open ball in C^n whose centre is θ and whose radius is c.

LEMMA 2.3 (Philippon's criterion for algebraic independence). *Let $\theta = (\theta_1, \ldots, \theta_n)$ be an element of C^n, and let σ, δ, R and S be four increasing functions on $\boldsymbol{N}$ whose values are at least 1. Suppose that the value $\sigma(l)+\delta(l)$ tends to infinity together with l. Suppose that there is a sequence of families of polynomials in $K[x_1, \ldots, x_n]$, say $\{(Q_{l,1}, \ldots, Q_{l,\varphi(l)})\}_{l\in \boldsymbol{N}}$ (where $\varphi(l)$ is a function of $l\in \boldsymbol{N}$), satisfying the following conditions: For any sufficiently large l,*

(a) *$Q_{l,1}, \ldots, Q_{l,\varphi(l)}$ have only finitely many common zeros in $B(\theta, \exp(-R(l)))$;*

(b) *$\deg Q_{l,i}\leqslant \delta(l)$ and $\overline{h}(Q_{l,i})\leqslant \sigma(l)$ for $1\leqslant i\leqslant \varphi(l)$;*

(c) *$0<\max\{|Q_{l,i}(\theta)|;\ 1\leqslant i\leqslant \varphi(l)\}\leqslant \exp(-S(l))$.*

Let E be the ideal in $K[x_1, \ldots, x_n]$ defined by $E = \{Q\in K[x_1, \ldots, x_n];\ Q(\theta)=0\}$. Then there exists a positive constant C_9 depending only on n, E and $[K:\boldsymbol{Q}]$, satisfying the following property: If k is an integer such that the function $S/(\sigma+\delta)\delta^k$ is increasing for sufficiently large l, and such that the inequality

$$S(l)^{k+2}\geqslant C_9\{\sigma(l+1)+\delta(l+1)\}\delta(l+1)^k\{S(l)^{k+1}+R(l+1)^{k+1}\}$$

holds for any sufficiently large l, then we have

$$\operatorname{tr}\deg_{\boldsymbol{Q}}\boldsymbol{Q}(\theta)\geqslant k+1.$$

Remark 2.2. In the above lemma, $\bar{h}$ is the height function defined by Philippon [16]. We do not recall the definition, because we use only the following relation between $\bar{h}$ and the ordinary height defined at the beginning of the present section: For any nonzero polynomial $P \in I_K[x_1, \ldots, x_n]$, we have

$$\bar{h}(P) \leqslant \log H(P) + n \log(\deg P + 1). \tag{2.1}$$

The following result is a variant of Lemma 10 of Nesterenko [11], which is necessary for the proof of Theorems 2 and 3.

LEMMA 2.4. *Let $\omega_1, \ldots, \omega_m$ be complex numbers, and let γ_1, γ_2, X, $Y_{1,i}$ $(0 \leqslant i \leqslant m)$ and Y_2 be positive numbers. Put $Y_1 = \max\{Y_{1,i};\ 0 \leqslant i \leqslant m\}$. Let ζ be a complex number which is integral over the ring $\mathbf{Z}[\omega_1, \ldots, \omega_m]$, and whose degree over the field $\mathbf{Q}(\omega_1, \ldots, \omega_m)$ is $\varkappa$. Let P be a polynomial in $\mathbf{Z}[x_0, x_1, \ldots, x_m, y]$ which is homogeneous in the variables $x_0, x_1, \ldots, x_m$ and whose degree in the variable y is at most $\varkappa - 1$. Suppose that the polynomial P satisfies the conditions*

$$\deg_{x_i} P \leqslant Y_{1,i} \quad (0 \leqslant i \leqslant m), \qquad \log H(P) \leqslant Y_2,$$
$$-\gamma_1 X \leqslant \log|P(\omega_0, \omega_1, \ldots, \omega_m, \zeta)| \leqslant -\gamma_2 X,$$

where $\omega_0 = 1$. Then there are positive numbers $\gamma_3, \ldots, \gamma_8$ depending only on $\omega_1, \ldots, \omega_m, \zeta, \varkappa, \gamma_1$ and γ_2 satisfying the following property: If $Y_1, Y_2 > \gamma_3$ and $X > \gamma_4 Y$, where $Y = Y_1 + Y_2$, then there exists a homogeneous polynomial Q in $\mathbf{Z}[x_0, x_1, \ldots, x_m]$ which satisfies the conditions

$$\deg_{x_i} Q \leqslant \gamma_5 Y_{1,i} \quad (0 \leqslant i \leqslant m), \qquad \log H(Q) \leqslant \gamma_6 Y_2,$$
$$-\gamma_7 X \leqslant \log|Q(\omega_0, \omega_1, \ldots, \omega_m)| \leqslant -\gamma_8 X.$$

Remark 2.3. In Lemma 10 of [11], Nesterenko only assumes the estimate for the value $\deg P + \log H(P)$. On the other hand, in the above formulation of the lemma, we assume separately the estimate for $\deg_{x_i} P$ $(0 \leqslant i \leqslant m)$ and for $\log H(P)$ because of our purpose. This change does not affect essentially the structure of the proof. We also note that, in [11], Nesterenko noticed that the argument upon which the proof of his lemma was based had first been used by Chudnovsky.

Remark 2.4. In Lemma 10 of [11], Nesterenko assumes that $\omega_1, \ldots, \omega_m$ are algebraically independent over the field $\mathbf{Q}$. We owe the removal of this assumption to Nishioka [15].

3. Proof of Theorem 1. In what follows, we denote by $c_1, c_2, \ldots$ positive constants depending only on K, m, r, ω, $\mathbf{A}(z)$, $\mathbf{B}(z)$ and $\mathbf{f}(z)$. Let ε and q be positive numbers satisfying

$$\frac{m+2-\varepsilon}{2+4\varepsilon} > \left[\frac{m+1}{2}\right] \quad \text{and} \quad 1 < \lambda := \frac{\log r}{\log q} < 1 + \varepsilon. \tag{3.1}$$

For any $l \in \mathbf{N}$, we put $N_l = [q^l]$. For $\mathbf{f}(z)$ and $N = N_l$, we take the auxiliary function which satisfies the properties stated in Lemma 2.1, and denote by $E_l(z)$ this function. Since the functions $f_1(z), \ldots, f_m(z)$ are algebraically independent over the field $K(z)$, by Lemmas 2.1 and 2.2, we have

$$\tfrac{1}{2} N_l^{m+1} \leqslant \operatorname{ord} E_l(z) \leqslant c_1 N_l^{m+1}. \tag{3.2}$$

Let $\mathbf{f}(z^r) = \tilde{\mathbf{A}}(z)\mathbf{f}(z) + \tilde{\mathbf{B}}(z)$, and let $D \in \mathbf{N}$ be the least positive integer such that $a(z) := D \det \mathbf{A}(z) \in I_K[z]$ and such that the entries of $a(z)\tilde{\mathbf{A}}(z)$, $a(z)\tilde{\mathbf{B}}(z)$ lie in $I_K[z]$. Put

$$\Gamma_l = \Big\{\prod_{v=0}^{l-1} a(\omega^{r^v})\Big\}^{mN_l} E_l(\omega^{r^l}). \tag{3.3}$$

LEMMA. *For any sufficiently large l, we have*

$$-c_2 r^l N_l^{m+1} \leqslant \log|\Gamma_l| \leqslant -c_3 r^l N_l^{m+1}. \tag{3.4}$$

Proof. For the number $E_l(\omega^{r^l})$ with sufficiently large l, using (3.2) and the properties of $E_l(z)$ stated in Lemma 2.1, we can prove, similarly to the proof of Proposition 2 of Nishioka [14], the inequalities

$$-c_4 r^l N_l^{m+1} \leqslant \log|E_l(\omega^{r^l})| \leqslant -c_5 r^l N_l^{m+1}. \tag{3.5}$$

Put $s = \operatorname{ord} a(z)$. Then we have

$$c_6 |\omega|^{sr^v} \leqslant |a(\omega^{r^v})| \leqslant c_7$$

for all $v \geqslant 0$. Hence we have

$$-c_8(sr^l + l) \leqslant \sum_{v=0}^{l-1} \log|a(\omega^{r^v})| \leqslant c_9 l.$$

We can deduce (3.4) from (3.5) and the above inequalities. This completes the proof of the lemma.

Put $\omega_i = f_i(\omega)$ $(i = 1, \ldots, m)$, and put $\theta = (\omega, \omega_1, \ldots, \omega_m)$. Then, by (3.3) and the functional equation (M), we can write $\Gamma_l = Q_l(\omega, \omega_1, \ldots, \omega_m)$ for some polynomial $Q_l(x_0, x_1, \ldots, x_m) \in I_K[x_0, x_1, \ldots, x_m]$. Using the estimates in Lemma 2.1, we have

$$\deg Q_l \leqslant c_{10} r^l N_l \quad \text{and} \quad \log H(Q_l) \leqslant c_{11} N_l \log N_l. \tag{3.6}$$

Let $R(l)$ be the function of $l \in \mathbf{N}$ defined by

$$R(l) = q^{(m+1+\lambda+\varepsilon)l}. \tag{3.7}$$

Then, by (3.4) and (3.6), we obtain

$$Q_l(\gamma_0, \gamma_1, \ldots, \gamma_m) \neq 0 \quad \text{if} \quad (\gamma_0, \gamma_1, \ldots, \gamma_m) \in B\big(\theta, \exp(-R(l))\big) \tag{3.8}$$

for sufficiently large l.

Let $\delta(l)$, $\sigma(l)$ and $S(l)$ be the functions of $l\in\mathbf{N}$ defined by

(3.9) $$\delta(l)=q^{(1+\lambda+\varepsilon)l},\quad \sigma(l)=q^{(1+\varepsilon)l}\quad\text{and}\quad S(l)=q^{(m+1+\lambda-\varepsilon)l}.$$

Then, by (3.4), (3.6) and (2.1), we have

(3.10) $$\deg Q_l\leqslant\delta(l),\quad \bar{h}(Q_l)\leqslant\sigma(l)\quad\text{and}\quad 0<|Q_l(\theta)|\leqslant\exp(-S(l))$$

for sufficiently large l. Now we apply Lemma 2.3 to the sequence of polynomials $\{Q_l\}_{l\in\mathbf{N}}$ (taking $\varphi(l)=1$ in Lemma 2.3) and the functions δ, σ, R, S. By (3.4), (3.8) and (3.10), the conditions (a)–(c) in Lemma 2.3 are satisfied. Put $k=[(m+1)/2]-1$. Then, by (3.1) and (3.9), the function $S/(\sigma+\delta)\delta^k$ is increasing for sufficiently large l. Further, by (3.1), (3.7) and (3.9), for any given sufficiently large positive constant C,

$$S(l)^{k+2}>C\{\sigma(l+1)+\delta(l+1)\}\delta(l+1)^k\{S(l)^{k+1}+R(l+1)^{k+1}\}$$

for sufficiently large l. Hence, by applying Lemma 2.3, we obtain

$$\operatorname{tr}\deg_{\mathbf{Q}}\mathbf{Q}(\theta)\geqslant[(m+1)/2].$$

This completes the proof of the theorem.

4. Proof of Theorem 2. Theorem 2 can be proved similarly to the proof of the result of Becker-Landeck [1] which improves earlier results of Galochkin [5] and Miller [10] on transcendence measures of $f(\alpha)$ for an algebraic number α in the unit disk. In what follows, we denote by $c_1, c_2, \ldots$ positive constants depending only on K, r, ω, $a(z)$, $b(z)$ and $f(z)$.

Our main purpose is to prove the following proposition.

PROPOSITION. *In the notation and the assumptions of Theorem 2, for any algebraic number α of degree d, we have*

$$|\omega-\alpha|>\exp\left\{-c_1\log\tilde{M}(\alpha)\log\log\tilde{M}(\alpha)-\left(d\log(d+1)+(\log\tilde{M}(\alpha))^2\right)\right\},$$

where $\tilde{M}(\alpha)=\max(3, M(\alpha))$. In particular, ω is a transcendental number.

Proof. Our proof of the Proposition includes several lemmas. We first prove the following:

LEMMA 4.1. *For any $N\in\mathbf{N}$ $(N\geqslant 2)$ and $l\in\mathbf{N}$ with $r^l\geqslant c_2N\log N$, there exists a nonzero polynomial $Q(x)\in\mathbf{Z}[x]$ satisfying the conditions*

(4.1) $$\deg Q\leqslant c_3Nr^l,\quad \log H(Q)\leqslant c_4N(\log N+l),$$

(4.2) $$-c_5N^2r^l\leqslant\log|Q(\omega)|\leqslant -c_6N^2r^l.$$

Proof. Put $\omega_1=f(\omega)$. For $f(z)$ and N, we take the auxiliary function $E(z)$ which satisfies the properties stated in Lemma 2.1. Let D be the least positive integer such that $\tilde{a}(z):=Da(z)$, $Db(z)\in I_K[z]$. For $l\in\mathbf{N}$ with $r^l\geqslant c_2N\log N$, put

$$\Gamma=\{\prod_{\nu=0}^{l-1}\tilde{a}(\omega^{r^\nu})\}^NE(\omega^{r^l}).$$

Then, as in the proof of Theorem 1, we can write $\Gamma=Q_0(\omega,\omega_1)$ for some polynomial $Q_0(x,y)\in I_K[x,y]$ which satisfies the conditions

$$\deg_xQ_0\leqslant c_7Nr^l,\quad \deg_yQ_0\leqslant N,\quad \log H(Q_0)\leqslant c_8N(\log N+l),$$
$$-c_9N^2r^l\leqslant\log|Q_0(\omega,\omega_1)|\leqslant -c_{10}N^2r^l.$$

We now modify Q_0 to a polynomial $Q(x)\in\mathbf{Z}[x]$ with the required properties. We follow an argument used by Nesterenko [11] and by Nishioka [14]. Let $A=\mathbf{Z}[\omega]$ and $\varkappa=[K(\omega,\omega_1):\mathbf{Q}(\omega)]$. Then there exists an element ζ which generates the field $K(\omega,\omega_1)$ over the field $\mathbf{Q}(\omega)$ and which is integral of degree $\varkappa$ over A. Let $\chi=[K:\mathbf{Q}]$ and let $\beta_1,\ldots,\beta_\chi$ constitute a $\mathbf{Z}$-basis of I_K. There exists a nonzero element γ of A such that the numbers $\gamma\beta_1,\ldots,\gamma\beta_\chi$, $\gamma\omega_1$ all lie in $A[\zeta]$. Then the number $\gamma^{1+N}Q_0(\omega,\omega_1)$ is an element of $A[\zeta]$ and there exists a polynomial $\tilde{Q}_0\in\mathbf{Z}[x_0,x_1,y]$ which is homogeneous in the variables x_0, x_1 and satisfies $\gamma^{1+N}Q_0(\omega,\omega_1)=\tilde{Q}_0(1,\omega,\zeta)$, and further satisfies the conditions

$$\deg_y\tilde{Q}_0\leqslant\varkappa-1,\quad \deg_x\tilde{Q}_0\leqslant c_{11}Nr^l,\quad \log H(\tilde{Q}_0)\leqslant c_{12}N(\log N+l),$$
$$-c_{13}N^2r^l\leqslant\log|\tilde{Q}_0(1,\omega,\zeta)|\leqslant -c_{14}N^2r^l,$$

where $\deg_x\tilde{Q}_0$ is the total degree of $\tilde{Q}_0$ in the variables x_0, x_1. We can apply Lemma 2.4 to the above situation, and obtain a polynomial $Q(x)\in\mathbf{Z}[x]$ with the required properties. This completes the proof of the lemma.

Let $Q(x)$ be the polynomial which satisfies the properties stated in the above lemma (the parameters N and l will be defined at the end of the proof). Put $\Delta=|\omega-\alpha|$. We may assume $\Delta\leqslant 1$, and hence we have $|\alpha|<2$.

LEMMA 4.2. *There exists a positive constant c_{15} such that, if*

(4.3) $$\Delta\leqslant\exp(-c_{15}N^2r^l),$$

then $Q(\alpha)\neq 0$ and we also have the lower estimate

(4.4) $$|Q(\alpha)|>\exp\{-c_{16}N(d(\log N+l)+r^l\log M(\alpha))\}.$$

Proof. Using (4.1), the lower estimate of (4.2) and the inequality $|\alpha|<2$, we have

$$\begin{aligned}|Q(\alpha)|&\geqslant|Q(\omega)|-|Q(\omega)-Q(\alpha)|\\&\geqslant\exp(-c_5N^2r^l)-L(Q)\max\{|\omega^j-\alpha^j|;\ 1\leqslant j\leqslant\deg Q\}\\&\geqslant\exp(-c_5N^2r^l)-\Delta\exp(c_{17}Nr^l).\end{aligned}$$

Hence, if $c_{15}>c_5+c_{17}$, then we have $Q(\alpha)\neq 0$. This proves the first part of the lemma. The lower estimate (4.4) follows from (4.1) and from the following (so-called) Liouville estimate (see Chudnovsky [3], Chap. 1, Lemma 1.7):

$$|Q(\alpha)|\geqslant L(Q)^{1-d}M(\alpha)^{-\deg Q}.$$

This completes the proof of the lemma.

As in the proof of the above lemma, the triangle inequality

$$|Q(\omega)| \geqslant |Q(\alpha)| - |Q(\omega) - Q(\alpha)|,$$

together with (4.1), the upper estimate of (4.2) and (4.4) leads to the following

LEMMA 4.3. *If the inequality* (4.3) *in Lemma* 4.2 *holds, then we have*

$$(4.5) \qquad c_{18} N r^l \leqslant c_{19} \{d(\log N + l) + r^l \log M(\alpha)\}.$$

We now finish the proof of the Proposition. By Lemmas 4.2 and 4.3, if we take N and l (with $r^l \geqslant c_2 N \log N$) such that the inequality (4.5) does not hold, then we have

$$\Delta > \exp(-c_{15} N^2 r^l).$$

This can be done first by taking $N = [c_{20} \log \tilde{M}(\alpha)]$ and next by taking

$$l = \min\{l \in \boldsymbol{N};\ r^l \geqslant c_{21}(N \log N + d \log(d+1) N^{-1} \log N)\}.$$

This establishes the Proposition.

We need the following result by Chudnovsky (see Chudnovsky [3], Chap. 1, Lemma 1.12).

LEMMA 4.4. *Let $P(x) \in \boldsymbol{Z}[x]$ be a nonzero polynomial of degree d, and let θ be a complex number. Then there exists a root α of P of multiplicity s such that*

$$|P(\theta)| \geqslant |\theta - \alpha|^s \{2d^2 H(P) L(P)\}^{-d+1}.$$

Proof of the theorem. Let $P(x) \in \boldsymbol{Z}[x]$ be a nonzero polynomial whose degree is at most d and whose height is at most H. We assume $dH \geqslant 3$. Then, by Lemma 4.4, we can take a root α of P of multiplicity s such that

$$|P(\omega)| \geqslant |\omega - \alpha|^s \{2d^2 H(d+1) H\}^{-d+1} \geqslant |\omega - \alpha|^s \exp(-4d \log(dH)).$$

Put $d_0 = \deg \alpha$; then $sd_0 \leqslant d$. By an inequality due to Mahler, we also have $s \log M(\alpha) \leqslant \log L(P) \leqslant \log\{(d+1)H\}$. The Proposition together with these estimates yields the theorem.

5. Proof of Theorem 3.

Put $\omega_i = f_i(\omega)$ $(i = 1, 2)$. Assume that ω and ω_2 are algebraically dependent. Under this assumption, we prove the algebraic independence of ω and ω_1 with their measure of the algebraic independence stated in the theorem. This clearly includes the proof of the lower bound for the transcendence degree of the field $\boldsymbol{Q}(\omega, \omega_1, \omega_2)$ over the field $\boldsymbol{Q}$ stated in the theorem. In what follows, we denote by $c_1, c_2, \ldots$ positive constants depending only on K, r, ω, $a_i(z)$, $b_i(z)$ and $f_i(z)$.

Let $P(x, y) \in \boldsymbol{Z}[x, y]$ be a nonzero irreducible polynomial with $\deg P \leqslant d$ and $H(P) \leqslant H$. Put $t = d + \log H$, and assume $t > 1$. Our assertion is proved by the usual method, originated by Gel'fond, which uses elimination theory (see, for example, Chudnovsky [3], Chap. 7, § 3). If we apply Lemma 2.1 for $f_1(z)$ and $f_2(z)$, and replace $\boldsymbol{Z}[\omega]$, $\boldsymbol{Q}(\omega)$ and $K(\omega, \omega_1)$ appearing in the proof of Lemma 4.1 by $\boldsymbol{Z}[\omega, \omega_1]$, $\boldsymbol{Q}(\omega, \omega_1)$ and $K(\omega, \omega_1, \omega_2)$ respectively, then we can prove the following lemma similarly to the proof of Lemma 4.1.

LEMMA 5.1. *For any $N \in \boldsymbol{N}$ $(N \geqslant 2)$ and $l \in \boldsymbol{N}$ with $r^l \geqslant c_1 N \log N$, there exists a nonzero polynomial $Q(x, y) \in \boldsymbol{Z}[x, y]$ satisfying the conditions*

$$(5.1) \qquad \deg_x Q \leqslant c_2 N r^l, \quad \deg_y Q \leqslant c_3 N, \quad \log H(Q) \leqslant c_4 N(\log N + l),$$

$$(5.2) \qquad -c_5 N^3 r^l \leqslant \log|Q(\omega, \omega_1)| \leqslant -c_6 N^3 r^l.$$

Let $Q(x, y)$ be the polynomial constructed in the above lemma. As in the proof of Theorem 2, the parameters N and l will be defined at the end of the proof. Hereafter we assume

$$(5.3) \qquad N \geqslant c_7 d \quad \text{and} \quad r^l \geqslant c_8 \log H.$$

LEMMA 5.2. *There exists a positive constant c_9 such that, if*

$$(5.4) \qquad |P(\omega, \omega_1)| \leqslant \exp(-c_9 N^3 r^l),$$

then the polynomials P and Q are relatively prime, and further their resultant $R(x)$ with respect to the variable y satisfies the conditions

$$(5.5) \qquad \deg R \leqslant c_{10} d N r^l, \quad \log H(R) \leqslant c_{11} N\{d(\log N + l) + \log H\},$$

$$(5.6) \qquad \log|R(\omega)| < -c_{12} N^3 r^l.$$

Proof. Take c_9 large enough compared with c_5 in (5.2); then by Gel'fond's Lemma (see, for example, Chudnovsky [3], Chap. 1, Lemma 1.4), P cannot divide Q, and hence P and Q are relatively prime because of the irreducibility of P. We can easily deduce (5.5) from (5.1) and from the definition of $R(x)$. Put $p = \deg_y P$ and $q = \deg_y Q$. Since

$$|R(\omega)| \leqslant (1 + |\omega|)^{p+q} H(P)^q H(Q)^p (p+q)^{p+q} \max(|P(\omega, \omega_1)|, |Q(\omega, \omega_1)|),$$

using (5.1)–(5.4), we obtain (5.6) and the lemma is proved.

Using Theorem 2 together with (5.5), we can prove the following:

LEMMA 5.3. *If the inequality* (5.4) *in Lemma* 5.2 *holds, then we have*

$$(5.7) \qquad |R(\omega)| > \exp\{-c_{13} T \log T (dNr^l \log(dNr^l) + T^2)\},$$

where $T = N(\log N + l)t$.

By Lemmas 5.2 and 5.3, if we take N and l (with $r^l \geqslant c_1 N \log N$ and with (5.3)) such that (5.6) is inconsistent with (5.7), then the inequality (5.4) does not hold. This can be done by taking

$$N = [c_{14} dt (\log t)^3] \quad \text{and} \quad l = \min\{l \in \boldsymbol{N};\ r^l \geqslant c_{15} t^3 (\log t)^4\},$$

and we obtain

$$|P(\omega, \omega_1)| > \exp(-c_9 N^3 r^l) > \exp(-c_{16} d^3 t^6 (\log t)^{13}).$$

For a reducible polynomial $P \in \mathbf{Z}[x, y]$, we first factorize it into irreducible polynomials, and obtain a lower estimate as above for each of their values. Using these estimates and Gel'fond's Lemma, we obtain the desired lower estimate for $|P(\omega, \omega_1)|$. This completes the proof of the theorem.

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On Fermat's equation with prime power exponents

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1. Introduction. There are many important and interesting results for the general equation

$$x^{p^n} + y^{p^n} + z^{p^n} = 0.$$

One is the following: In 1933, using the method of singular integers, Moriya [5] extended the theorem of Furtwängler to the Fermat equation with power exponent p^n to show that:

Suppose that the equation $x^{p^n} + y^{p^n} + z^{p^n} = 0$ with $n \geq 1$ and p an odd prime has a non-trivial solution x, y, z such that some integer r satisfies one of the following conditions: (i) $r \mid x$, $p \nmid x$, (ii) $r \mid x-y$, $p \nmid x^2-y^2$. *Then* $r^{p-1} \equiv 1 \pmod{p^{n+1}}$.

Again with the same method, Inkeri [2] proved the following generalization of a theorem due to Vandiver [8]:

With the same assumption as in the last theorem,

$$x^p \equiv x \pmod{p}, \quad y^p \equiv y \pmod{p}, \quad z^p \equiv z \pmod{p}.$$

However, this is far from perfect. The author will improve this result in the next section by making use of the recent result of Azuhata [1] of Science University of Tokyo who proved in 1984 that:

If p is an odd prime and there exist pairwise relatively prime integers x, y, z satisfying one of the following conditions: (i) $r \mid x$, $p \nmid x$, (ii) $r \mid x-y$, $p \nmid x^2-y^2$, (iii) $r \mid x^2-yz$, $p \nmid xy+yz+zx$, (iv) $r \mid x^2+yz$, $p \nmid x(y-z)(x^2+yz)$, *then*

$$r^{p-1} \equiv 1 \pmod{p^{2n}}.$$

This is a considerable generalization of Moriya's theorem. The result of the author is the following

THEOREM. *If p is an odd prime and there are relatively prime integers x, y, z satisfying $x^{p^n} + y^{p^n} + z^{p^n} = 0$, then*

$$x^p \equiv x \pmod{p^{3n}}, \quad y^p \equiv y \pmod{p^{3n}}, \quad z^p \equiv z \pmod{p^{3n}},$$

and $x+y+z \equiv 0$ (p^{3n}). *Moreover, if $p \mid z$ then $p^{3n} \mid z$.*