ACTA ARITHMETICA
LIX.1 (1991)

The k-functions in multiplicative number theory, V
Changes of sign of some arithmetical error terms

by

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1. Introduction and statement of results. The principal goal of this paper is to prove an estimate from below for the number $V(T, q, a)$ of changes of sign of the remainder term in Dirichlet's prime number formula:

$$\Delta(x, q, a) = \psi(x, q, a) - \frac{1}{\varphi(q)}x = \sum_{\substack{n \leqslant x \\ n \equiv a \pmod q}} \Lambda(n) - \frac{1}{\varphi(q)}x, \tag{1.1}$$

$$(x \geqslant 1,\ (a, q) = 1,\ q \neq 2)$$

in the interval $[1, T]$. Before stating the first theorem let us introduce some subsidiary notation.

For a natural number $q \geqslant 1$ and an arbitrary prime p let us write

$$q_p = qp^{-k}$$

where $p^k \| q$, $k \geqslant 0$. Thus $q_p = q$ whenever $p \nmid q$. Let $g_{p,q}$ denote the order of $p \pmod{q_p}$. Moreover, for every real λ, we define the function $F(z, q, a, \lambda)$ for $z = x+iy$, $x \geqslant 1$, $y > 0$, by the formula

$$F(z, q, a, \lambda) = i\lambda - 2F_1(z, q, a) + h(z, q, a), \tag{1.2}$$

where

$$F_1(z, q, a) = \frac{e^{-z/2}}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi(a)} K(z, \chi'), \tag{1.3}$$

(χ' denotes the primitive character induced by χ; K is defined in [5], part I, § 1),

$$h(z, q, a) = \alpha_1 z e^{-z/2} + \alpha_2 e^{-z/2} + e^{-z/2} h_1(z, q, a) + e^{-z/2} h_2(z, q, a), \tag{1.4}$$

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* This paper has been written during the author's Research Fellowship from the Alexander von Humboldt Foundation.

$$(1.5)\qquad \alpha_1=\alpha_1(q,a)=\begin{cases}0 & \text{if } q=1,\\ \dfrac{1}{\varphi(q)}-\sum'_{p^k\|q}\dfrac{1}{g_{p,q}\varphi(p^k)} & \text{if } q\geqslant 3,\ a\not\equiv\pm 1 \pmod q,\\ \dfrac{1}{\varphi(q)}-\dfrac{1}{2}-\sum'_{p^k\|q}\dfrac{1}{g_{p,q}\varphi(p^k)} & \text{if } q\geqslant 3,\ a\equiv\pm 1 \pmod q,\end{cases}$$

$$(1.6)\qquad \alpha_2=\alpha_2(q,a)=\frac{1}{\varphi(q)}\sum_{\chi(\mathrm{mod}\, q)}\overline{\chi(a)}B(\chi)-\sum'_{p^k\|q}\frac{\log p}{\varphi(p^k)}\left(\frac{1}{2}-\frac{l_p}{g_{p,q}}\right)$$

($B(\chi)$ is a constant defined by (4.6) of [5], part I),

$$(1.7)\qquad h_1(z,q,a)=\begin{cases}-R(z,0) & \text{if } q=1,\\ 0 & \text{if } q\geqslant 3,\ a\not\equiv\pm 1 \pmod q,\\ -\frac{1}{2}(R(z,0)+R(z,1)) & \text{if } q\geqslant 3,\ a\equiv 1 \pmod q,\\ -\frac{1}{2}(R(z,0)-R(z,1)) & \text{if } q\geqslant 3,\ a\equiv -1 \pmod q,\end{cases}$$

where as in [5], part I,

$$(1.8)\qquad R(z,0)=\tfrac{1}{2}\log(1-e^{-2z}),\qquad R(z,1)=\tfrac{1}{2}\log\frac{e^z-1}{e^z+1}$$

$$(z=x+iy,\ y>0,\ x\geqslant 1),$$

and finally

$$(1.9)\qquad h_2(z,q,a)=-\frac{1}{\pi i}\sum'_{p^k\|q}\frac{\log p}{\varphi(p^k)}\sum_{n=1}^{\infty}\frac{1}{n}e(nZ),$$

$$(1.10)\qquad Z=\frac{z}{g_{p,q}\log p}-\frac{l_p}{g_{p,q}},\qquad e(\theta)=e^{2\pi i\theta}.$$

The dash in formulae (1.5), (1.6) and (1.9) indicates that the summation is restricted to prime divisors p of q for which the residue class $a \pmod{q_p}$ belongs to the cyclic subgroup generated by $p \pmod{q_p}$; the empty sum equals zero. The numbers l_p appearing in (1.6) and (1.10) are determined uniquely by the conditions

$$(1.11)\qquad 0<l_p\leqslant g_{p,q},\qquad p^{l_p}\equiv a \pmod{q_p}.$$

Let us denote by $N(T,Y,\lambda)$ the number of zeros of $F(z,q,a,\lambda)$ in the region

$$z=x+iy,\quad 1<x\leqslant T,\quad 0<y\leqslant Y.$$

Moreover, we write

$$\delta_\lambda=\begin{cases}1 & \text{for } \lambda=0,\\ 0 & \text{for } \lambda\neq 0.\end{cases}$$

THEOREM 1.1. *Let $q\geqslant 1$, $q\neq 2$ and let us assume the Generalized Riemann Hypothesis for L-functions* (mod q). *Moreover, let*

$$\prod_{\chi(\mathrm{mod}\, q)}L(1/2,\chi)\neq 0$$

and let $Y_\lambda>1$ denote a real number such that

$$\lambda i+F_1(z,q,a)\neq 0\quad \textit{for } y>Y_\lambda.$$

Then for $Y>Y_\lambda$ and $T\geqslant T_0(Y,\lambda)$ we have

$$(1.12)\qquad V(T,q,a)\geqslant\frac{\delta_\lambda\gamma_0}{\pi}\log T+2N(\log T-c_1\log\log T,\,Y,\lambda)+O_{Y,\lambda}(\log\log T)$$

with a suitable positive constant $c_1=c_1(Y)$; γ_0 denotes here the minimal positive number γ for which

$$\sum_{\chi(\mathrm{mod}\, q)}\overline{\chi(a)}\,m(1/2+i\gamma,\chi)\neq 0$$

($m(\varrho,\chi)$ being the multiplicity of the zero of $L(s,\chi)$ at $s=\varrho$).

Estimate (1.12) is in a sense the best possible. To see this let us consider $F(z,q,a,0)$. We have (cf. Lemma 4.1 below)

$$\lim_{y\to 0^+}\operatorname{Re}F(x+iy,q,a,0)=e^{-x/2}\Delta_0(e^x,q,a),\qquad x\geqslant 1.$$

Changes of sign of $\Delta(x,q,a)$ from $(-)$ to $(+)$ are discontinuous and appear at points of the form p^k, $p^k\equiv a \pmod q$; changes of sign in the opposite direction are continuous and can appear at points which are not natural numbers. Let p^k be a change of sign of $\Delta(x,q,a)$ and let $x_0=k\log p$. Then x_0 is a logarithmic branch point for $F_1(z,q,a)$ (cf. [5], part I) and for

$$(1.13)\qquad z=x_0+re^{i\varphi},\qquad 0<r<r_0,\qquad 0<\varphi<\pi,$$

we have

$$(1.14)\qquad \operatorname{Re}F(z,q,a,0)=\frac{k\log p}{\pi}\operatorname{Re}[ie^{-z/2}\log(z-x_0)+h(z)]$$

$$=-\frac{k\log p}{\pi p^{k/2}}\varphi+A+O(r\log(1/r)),$$

where h is holomorphic in a sufficiently small neighbourhood of x_0 and A is a constant such that

$$0<A<(k\log p)/p^{k/2}.$$

Let us consider solutions of the equation

$$\operatorname{Re}F(z,q,a,0)=0$$

in the region (1.13) with r_0 small enough. They form a curve L ending at $z = x_0$ and at this point tangent to the half-line

$$\varphi = \pi \frac{A p^{k/2}}{k \log p}, \qquad r > 0.$$

Since

$$\operatorname{Im} F(z, q, a, 0) = -\frac{k \log p}{\pi p^{k/2}} \log\frac{1}{r} + O(1), \qquad r \to 0^+,$$

we have for z lying on L

$$F(z, q, a, 0) = -i\lambda$$

where $\lambda \to \infty$ as $z \to x_0$. We see that for λ large enough $F(z, q, a, \lambda)$ has exactly one zero in a (small) neighbourhood of each change of sign from $(-)$ to $(+)$. Since every other change of sign is of this form, there exists $\lambda_0 = \lambda_0(T)$ such that for every $\lambda > \lambda_0$ and arbitrary $Y \geqslant 1$ we have

(1.15) $$2N(T, Y, \lambda) - 1 \leqslant V(T, q, a) \leqslant 2N(T, Y, \lambda).$$

Hence the quantity $2N(T, Y, \lambda)$ approximates $V(T, q, a)$ very precisely. The right-hand side of (1.12) contains a summand of this form and in this sense (1.12) is the best possible. Moreover, the advantage of (1.12) over (1.15) is that it holds for all large T whereas the range of T in (1.15) is restricted by the condition $\lambda_0(T) < \lambda$.

COROLLARY 1.1. *Let $q \geqslant 1$, $q \neq 2$ and let us assume the Generalized Riemann Hypothesis for L-functions* (mod q). *Moreover, let*

$$\prod_{\chi (\mathrm{mod}\, q)} L(1/2, \chi) \neq 0$$

and let us define $\varkappa = \varkappa(q, a)$ by the formula

(1.16) $$\varkappa = \lim_{Y \to 0^+} \lim_{T \to \infty} \frac{1}{T} \# \{z = x+iy \mid F_1(z, q, a) = 0,\ 0 < x < T,\ y \geqslant Y\}.$$

Then

(1.17) $$\liminf_{T \to \infty} \frac{V(T, q, a)}{\log T} \geqslant \frac{\gamma_0}{\pi} + 2\varkappa.$$

Although Theorem 1.1 and Corollary 1.1 are conditional they lead to some unconditional results for special q and a (mod q). For simplicity let us consider the case of $q = 1$ and the corresponding remainder

(1.18) $$\Delta_3(x) = \psi(x) - x = \sum_{n \leqslant x} \Lambda(n) - x, \qquad x \geqslant 1.$$

Changes of sign of this function have been considered by many authors and there exists a remarkable amount of work on this topic (cf. [3], part I for

a detailed history of researches). To date, the best estimate for the number $V_3(T)$ of changes of sign of $\Delta_3(x)$ in the interval $[1, T]$ has been proved in [3], part II:

(1.19) $$\liminf_{T \to \infty} \frac{V_3(T)}{\log T} \geqslant \frac{\gamma_0}{\pi},$$

where $\gamma_0 = 14.13\ldots$ denotes the imaginary part of "the lowest zero" of the Riemann zeta function. Corollary 1.1 combined with numerical computations involving non-trivial zeta zeros and other results from [3] leads to the following improvement of (1.19).

THEOREM 1.2. *We have*

(1.20) $$\liminf_{T \to \infty} \frac{V_3(T)}{\log T} \geqslant \frac{\gamma_0}{\pi} + 10^{-250}.$$

Let us remark that although the improvement is not large it was not possible to achieve it using previously known methods (cf. [4], [3], part III, [8]). The exponent 250 is by no means the best possible and it can be improved at the cost of more elaborate calculations; no serious attempt has been made in this paper to do this. The problem of the best constant in (1.20) seems to be both important and interesting and the author hopes to consider it in a forthcoming paper.

The basic tool used in the proof of Theorem 1.1 is a result concerning generalized Dirichlet series which seems to be of independent interest. We consider the function

(1.21) $$F(z) = \sum_{n=1}^{\infty} a_n e^{i w_n z}, \qquad z = x+iy,\ y > 0,$$

and let us assume that its coefficients and exponents are subject to the following restrictions:

(1.22) $$w_n \in \boldsymbol{R}, \quad a_n \in \boldsymbol{C}, \quad n = 1, 2, \ldots, \quad 0 < w_1 < w_2 < \ldots,$$

(1.23) $$\sum_{n=1}^{\infty} |a_n| e^{-w_n y} < \infty \quad \text{for every } y > 0.$$

For $x \in \boldsymbol{R}$ we write

(1.24) $$P(x) = \lim_{y \to 0^+} P(x, y),$$

where

(1.25) $$P(x, y) = \operatorname{Re} F(x+iy).$$

We assume that $P(x)$ exists for each x or at least for $x \geqslant x_0$ (x_0 is fixed). We are interested in solutions of the equation

(1.26) $$P(x, y) = 0 \quad (y > 0).$$

The set of $z = x+iy$ satisfying (1.26) is the union of a number of curves lying on the upper half-plane. We call them the nodal lines for F. To avoid misunderstandings we formulate the corresponding definition, preceded by a few general remarks.

Let

$$\operatorname{grad} P_{|(x,y)=(x_0,y_0)} \neq 0.$$

Then (x_0, y_0) is a regular solution of (1.26); solutions near (x_0, y_0) form a smooth curve. If (x_0, y_0) is singular, i.e.

$$\operatorname{grad} P_{|(x,y)=(x_0,y_0)} = 0,$$

the situation is more complicated. Since

$$\|\operatorname{grad} P\|^2 = |F'|^2,$$

we have $F'(z_0) = 0$, $z_0 = x_0+iy_0$. Let k denote the order of the zero of F' at z_0. Then, as can easily be seen, (x_0, y_0) is a point of intersection of $k+1$ smooth curves.

Let $D \subset H = \{z \in \boldsymbol{C} \mid \operatorname{Im} z > 0\}$ be an open domain and let L_1 and L_2 denote two smooth curves in D satisfying (1.26), i.e.

$$L_i\colon I \ni t \mapsto L_i(t) \in D, \quad I = (0, 1),$$
$$P(\operatorname{Re} L_i(t), \operatorname{Im} L_i(t)) = 0, \quad i = 1, 2.$$

The set of such curves is partially ordered by $\dashv$ defined as follows:

$$L_1 \dashv L_2 \Leftrightarrow \{L_1(t) \mid 0 < t < 1\} \subset \{L_2(t) \mid 0 < t < 1\}.$$

DEFINITION 1.1. The curve $L\colon I \to D$ is a *nodal line for* F *in* D if it is smooth, satisfies (1.26) and is maximal in the sense of $\dashv$.

Most often we consider nodal lines in H and we call them simply "nodal lines". In some cases we use nodal lines in certain proper subdomains $D \subsetneq H$; in such occurrences D will always be precisely defined and no confusion can appear.

The extremum principle for harmonic functions implies that there are no closed nodal lines. Moreover, by Taylor's expansion we see that a nodal line does not end at a point belonging to H. Hence every such curve ends on the real axis or at infinity. From (1.21)–(1.23) we see that

$$(1.27) \qquad P(x, y) = e^{-w_1 y}|a_1|\{\cos(w_1 x+\varphi_1)+r(x, y)\}, \qquad \varphi_1 = \arg a_1,$$

and for sufficiently large y we have

$$|r(x, y)| \leqslant 1/2, \qquad -\infty < x < \infty.$$

Hence each half-line

$$z = x_k+iy, \quad y > 0, \quad x_k = ((2k+1)\pi-2\varphi_1)/2w_1, \quad k \in \boldsymbol{Z},$$

is a vertical asymptote for a certain nodal line. We say that these nodal lines are *infinite* and they *begin* at the points $x_k+i\infty$, $k \in \boldsymbol{Z}$. The basic result about nodal lines reads as follows.

THEOREM 1.3. *Let F be a non-zero function such as in* (1.21)–(1.23). *We suppose moreover that*

$$(1.28) \qquad w_n \geqslant b_1 n^{\delta}, \quad n = 1, 2, \ldots,$$

$$(1.29) \qquad \sum_{n \leqslant x} |a_n| \ll (\log x)^{c_2}, \quad x \geqslant 2,$$

$$(1.30) \qquad P(x, y) = P(x)+O\big(y^{b_2}e^{c_3|x|}+h(x, y)\big), \quad x \in \boldsymbol{R},$$

where b_1, b_2, c_2, c_3 are positive constants (depending on F) and h is a positive function tending to zero as $|x| \to \infty$ uniformly for $y > 0$.

Then there exists a constant $c_4 = c_4(F) > 0$ such that every infinite nodal line beginning at $x_{k_0}+i\infty$, $k_0 \in \boldsymbol{Z}$, is contained in the region

$$(1.31) \qquad z = x+iy, \quad |x-x_{k_0}| \leqslant c_4 \log(|x_{k_0}|+2), \quad y > 0.$$

In particular, it ends at a point on the real axis. If we suppose that the condition (1.30) *is satisfied for $x \geqslant \bar{x}_0$ only, then the assertion is still valid for nodal lines beginning at the points $x_{k_0}+i\infty$, $x_{k_0} > \bar{x}_1$ with a certain $\bar{x}_1 \geqslant \bar{x}_0$.*

2. Proof of Theorem 1.3. Let L be the infinite nodal line beginning at $x_{k_0}+i\infty$. We can assume that $|x_{k_0}|$ is sufficiently large. Let L' denote the infinite nodal line beginning at $x_{k_0}+(2\pi)/w_1+i\infty$. Let us fix a real number $y_0 > 1$ such that all the nodal lines lying on the half-plane $y = \operatorname{Im} z > y_0$ are infinite and lie in vertical strips of the form

$$z = x+iy, \quad |x-x_k| \leqslant \pi/(10w_1), \quad k \in \boldsymbol{Z}, \quad y > y_0.$$

Such y_0 does exist. Indeed, $r(x, y)$ in (1.27) tends to zero as $y \to \infty$ uniformly with respect to x. Hence nodal lines in the half-plane $y > y_0$ lie in the strips $|x-x_k| < \pi/(10w_1)$, $k \in \boldsymbol{Z}$, if only y_0 is sufficiently large. Moreover,

$$\frac{\partial}{\partial x}P(x, y) = -e^{-w_1 y}|w_1|\{\sin(w_1 x+\varphi_1)+r_1(x, y)\}$$

and again $|r_1(x, y)| \to 0$ as $y \to \infty$ uniformly with respect to x. Hence, near x_k, $(\partial/\partial x)P(x, y)$ has constant sign. Thus in each strip $|x-x_k| < \pi/(10w_1)$, $y > y_0$, there is at most one nodal line. Of course it has to be the infinite nodal line beginning at $x_k+i\infty$.

Next, we fix a constant λ_0 satisfying

$$0 < \lambda_0 < \sup_{x \in \boldsymbol{R}} |P(x)|.$$

We can find a real number t such that

(2.1) $$(3\pi)/w_1 \leqslant |t-x_{k_0}| \leqslant |x_{k_0}|/2,$$

(2.2) $$|P(t)| \geqslant \lambda_0,$$

P being defined by (1.24). According to the Theorem of [5], part IV, t exists for all sufficiently large x_{k_0}.

Let us denote by l the half-line

$$z = t+iy, \quad y > 0.$$

We shall prove that for $|x_{k_0}| \geqslant c_5$

(2.3) $$L \cap l \neq \emptyset$$

implies

(2.4) $$|t-x_{k_0}| \ll \log|x_{k_0}|.$$

It is obvious that then the Theorem follows. We prove (2.3) and (2.4) for $t > x_{k_0}$. The proof in the case $t < x_{k_0}$ is analogous.

We need two simple but useful lemmata, which easily follow from the observation that there are no closed nodal lines or from the extremum principle for harmonic functions.

LEMMA 2.1. *Let L be a nodal line and U an open set satisfying*

$$\bar{U} \subset \{z = x+iy \mid y > 0, |x| < c_6\}, \quad L \cap U \neq \emptyset.$$

Then denoting by ∂U the boundary of U we have

$$L \cap \partial U \neq \emptyset.$$

LEMMA 2.2. *Different, infinite nodal lines are disjoint.*

Let $A_0 \in L \cap l$ and let L_0 denote the part of L beginning at $x_{k_0}+i\infty$ and ending at A_0. We can choose A_0 in such a way that

$$L_0 \cap l = \{A_0\}.$$

Let A_0' and L_0' have the analogous meaning for the nodal line L'. A_0' exists. Indeed, L' has common points with the open domain bounded by $L_0 \cup l$. Hence, by Lemma 2.1 it has common points with $L_0 \cup l$. Since, by Lemma 2.2, $L_0 \cap L' = \emptyset$ we get $L' \cap l \neq \emptyset$.

Let

$D =$ the open domain bounded by L_0, L_0' and the line segment $[A_0, A_0']$,

$$D_1 = \{z \in D \mid 0 < \operatorname{Im} z < y_0\},$$

$$D_2 = \{z \in D \mid x_{k_0} - \pi/(10w_1) < \operatorname{Re} z < x_{k_0} + (2\pi)/w_1\}.$$

We consider the function

$$g(z) = \exp(F(z)), \quad z \in H.$$

Let

$$b_3 = \sup_{z \in D_2} |g(z)|.$$

We have $b_3 > 1$ if y_0 is sufficiently large. Let us fix a point $z_0 \in D_2$ such that

$$|g(z_0)| \geqslant b_4, \quad \text{where} \quad b_4 = (1+b_3)/2.$$

Moreover, we write

$$b_5 = (3+b_3)/4.$$

Then

(2.5) $$1 < b_5 < b_4 < b_3.$$

Let us fix $y_1 > 2y_0$ such that

$$\sup_{x \in \mathbf{R}} |g(x+iy_1)| \leqslant b_5$$

and write

$$D_3 = \{z \in D \mid 0 < \operatorname{Im} z < y_1\},$$

$$\alpha = \pi/(2y_1), \quad \varepsilon = 4c_7 |t|^{2c_2} \exp(-\alpha|t-x_{k_0}|)$$

where c_7 is such that

(2.6) $$|P(x, y)| \leqslant c_7 \log^{c_2}(1/y) \quad \text{for } 0 < y < 1/2, \ x \in \mathbf{R}.$$

The constant c_7 exists. Indeed, using (1.28) and (1.29) we obtain

$$|P(x, y)| \leqslant \sum_{n \leqslant X} |a_n| + \sum_{n \geqslant X} |a_n| e^{-w_n y}$$
$$\ll_\delta (\log X)^{c_2} + \frac{1}{y^{2/\delta}} \sum_{n \geqslant X} \frac{|a_n|}{w_n^{2/\delta}} \ll (\log X)^{c_2} + \frac{1}{y^{2/\delta}} \frac{(\log X)^{c_2}}{X}.$$

Choosing $X = y^{-2/\delta}$ we get

$$|P(x, y)| \ll (\log(1/y))^{c_2}.$$

Let us introduce the other subsidiary function by the formula

(2.7) $$G(z) = g(z) \exp\{-\varepsilon \cos(i\alpha(z-x_{k_0}))\}$$

and let us write

$$\lambda_1 = \sup_{z \in \partial D_3} |G(z)|.$$

We estimate λ_1 from above.

For $z\in(L\cup L')\cap\partial D_3$ we have

(2.8) $$|G(z)| = \exp\{-\varepsilon\operatorname{Re}\tfrac{1}{2}(e^{\alpha(z-x_{k_0})}+e^{-\alpha(z-x_{k_0})})\} = \exp\{-(\varepsilon/2)(e^{\alpha(x-x_{k_0})}+e^{-\alpha(x-x_{k_0})})\cos(\alpha y)\} \leqslant 1.$$

For $z\in\partial D_3$, $\operatorname{Im} z = y_1$ we have

(2.9) $$|G(z)| = |g(z)| \leqslant b_5.$$

For $z\in\partial D_3\cap l = [A_0, A_0']$ we have

(2.10) $$y = \operatorname{Im} z \geqslant e^{-t^2} \quad \text{if } |t| \text{ is sufficiently large.}$$

Indeed, writing $A_0 = (t, y_3)$ we have using (1.30)

$$0 = |P(t, y_3)| \geqslant |P(t)|-|P(t, y_3)-P(t)| \geqslant \lambda_0 - O(y_3^{b_2}e^{c_3|t|}+h(t, y_3)) \geqslant \lambda_0/2 - c_8 y_3^{b_2}e^{c_3|t|} \geqslant \tfrac{1}{2}\lambda_0(1-y_3^{b_2}e^{b_2t^2})$$

if $|x_{k_0}|$ is sufficiently large; (2.10) therefore follows.

Hence using (2.6) and (2.10) we obtain for $z\in[A_0, A_0']$, $y<1/2$

(2.11) $$|G(z)| \leqslant \exp\{c_7|t|^{2c_2}-\tfrac{1}{2}\varepsilon e^{\alpha|t-x_{k_0}|}\cos(\alpha y)\} \leqslant \exp\{c_7|t|^{2c_2}-\tfrac{1}{4}\varepsilon e^{\alpha|t-x_{k_0}|}\} = 1$$

because

$$0<\alpha y<(\pi y_0)/(2y_1)<\pi/4 \quad \text{and} \quad 1>\cos(\alpha y)>1/2.$$

For $z\in[A_0, A_0']$, $\operatorname{Im} z > 1/2$ we have

(2.12) $$|G(z)| \leqslant \exp\{\sum_{n=1}^{\infty}|a_n|e^{-w_n/2}-\tfrac{1}{4}\varepsilon e^{\alpha|t-x_{k_0}|}\} \leqslant 1$$

for sufficiently large $|x_{k_0}|$.

Collecting (2.8), (2.9), (2.11) and (2.12) we obtain $\lambda_1 \leqslant b_5$ and from the extremum principle

$$\sup_{z\in D_3}|G(z)| \leqslant b_5.$$

Since $z_0\in D_3$, we get

$$b_4 \leqslant |g(z_0)| = |G(z_0)\exp\{\varepsilon\cos(i\alpha(z-x_{k_0}))\}| \leqslant b_5\exp\{O(|t|^{2c_2}\exp(-\alpha|t-x_{k_0}|))\}.$$

Consequently, using (2.5),

$$|t|^{2c_2} \gg \exp(\alpha|t-x_{k_0}|),$$

and finally,

$$|t-x_{k_0}| \ll \log|t| \ll \log|x_{k_0}|;$$

(2.4) therefore follows. The proof is complete.

3. A subsidiary result.

THEOREM 3.1. *Let F be such as in Theorem 1.3 and let g be holomorphic for $z = x+iy$, $y>0$, $x>\bar{x}_0$, and such that $\operatorname{Re} g(z)\to 0$ as $x\to\infty$ uniformly in every horizontal strip $z = x+iy$, $0<y<y_0$, $y_0>0$. Then, for every $Y>0$ there exists $c_8 = c_8(Y)$ such that in every rectangle*

$$z = x+iy, \quad 0\leqslant y\leqslant Y,$$

$$|x-x_0| \leqslant c_9\log x_0, \quad x_0 \geqslant c_8,$$

there exists a curve $\mathscr{C}$ joining the lines $y = Y$ and $y = 0$ and such that

(3.1) $$|\Delta_{\mathscr{C}}\arg(i\lambda+F(z)+g(z))| \leqslant \pi$$

for every real constant λ.

Proof. It is enough to prove the theorem for Y large enough. Let us consider the function

$$P_Y(x) = P(x, Y) = \operatorname{Re} F(x+iY), \quad x\in\mathbf{R}.$$

It is almost periodic and not identically zero. Hence there exists a relatively dense set of points ξ_ν, $\nu = 1, 2, \ldots$, and a positive constant λ_0 such that

$$|P_Y(\xi_\nu)| \geqslant \lambda_0$$

for every $\nu\geqslant 1$. Let $Y_1>Y$ be so large that

$$|P(x, Y_1)| < \tfrac{1}{3}\lambda_0$$

for all $x\in\mathbf{R}$. Moreover, let c_{10} be such that

$$|\operatorname{Re} g(z)| < \tfrac{1}{3}\lambda_0$$

for $x\geqslant c_{10}$ uniformly for $0<y\leqslant Y_1$.

Let L be a nodal line for F beginning at $x_0+O(1)+i\infty$, $x_0\geqslant c_{11}>c_{10}$ and let $\xi_{\nu_0}+iY$ be a point lying to the right of L and such that

$$\xi_{\nu_0} = x_0+O(\log x_0), \quad |P_Y(\xi_{\nu_0})| \geqslant \lambda_0.$$

Let L' be another nodal line for F lying to the right of $\xi_{\nu_0}+iY$. By Theorem 1.3 we can choose L' such that $L\cup L'$ is contained in the region

$$z = x+iy, \quad |x-x_0| \ll \log x_0, \quad y>0.$$

Let

$$\lambda_1 = \operatorname{Re}(F+g)(\xi_{\nu_0}+iY).$$

Then

$$|\lambda_1| \geqslant \lambda_0-\tfrac{1}{3}\lambda_0 = \tfrac{2}{3}\lambda_0.$$

We consider the function

$$G(z) = F(z)+g(z)-\lambda_1.$$

Let L'' denote the nodal line for G passing through $\xi_{\nu_0}+iY$. For $z\in L\cup L'$ we have

$$|\mathrm{Re}(F+g)(z)| = |\mathrm{Re}\, g(z)| \leqslant \tfrac{1}{3}\lambda_0 < \lambda_1$$

and thus

$$L''\cap(L\cup L') = \emptyset.$$

Moreover, for $\mathrm{Im}\, z = Y_1$

$$|\mathrm{Re}(F+g)(z)| \leqslant \tfrac{1}{3}\lambda_0+\tfrac{1}{3}\lambda_0 < |\lambda_1|.$$

Thus L'' lies in the region bounded by L, L', the line $y = Y_1$ and the real axis. By Lemma 2.1 it has a common point with the real axis. Let $\mathscr{C}$ be a part of L'' joining $\xi_{\nu_0}+iY$ with $\boldsymbol{R}$. Then

$$\left|\mathrm{Re}\big(i\lambda+F(z)+g(z)\big)\right| = |\lambda_1|$$

for $z\in\mathscr{C}$ and (3.1) follows.

4. Boundary values of $\mathrm{Re}\, F(z, q, a, \lambda)$.

LEMMA 4.1. *Let* $q\geqslant 1$, $q\neq 2$, *and let us assume that*

$$\prod_{\chi\,(\mathrm{mod}\, q)} L(\sigma, \chi)\neq 0 \quad \text{for } 0<\sigma<1.$$

Then for $x>1$ *we have*

$$\lim_{y\to 0^+} \mathrm{Re}\, F(x+iy, q, a, \lambda) = e^{-x/2}\Delta_0(e^x, q, a), \tag{4.1}$$

where for real $u\geqslant 1$ *we write*

$$\Delta_0(u, q, a) = \tfrac{1}{2}\big(\Delta(u-0, q, a)+\Delta(u+0, q, a)\big). \tag{4.2}$$

Proof. Suppose first that $q\geqslant 3$ and let $x\neq k\log p$, p prime, $k\in\boldsymbol{Z}$. Then

$$\lim_{y\to 0^+}\big(-2\mathrm{Re}\, F_1(x+iy, q, a)\big) = -\frac{e^{-x/2}}{\varphi(q)}\sum_{\chi\,(\mathrm{mod}\, q)}\overline{\chi(a)}\, F(x, \chi') \tag{4.3}$$

where as in [5], part I,

$$F(x, \chi') = \lim_{y\to 0^+}\big(K(x+iy, \chi')+\overline{K(x+iy, \bar\chi')}\big).$$

We have (Theorem 4.1 in [5], part I)

$$F(x, \chi') = -\psi(e^x, \chi')+e(\chi')e^x-e_1(\chi')x-R(x, d)+B(\chi), \tag{4.4}$$

where

$$e(\chi) = \begin{cases}1 & \text{if } \chi = \chi_0,\\ 0 & \text{if } \chi\neq\chi_0,\end{cases}\qquad e_1(\chi) = \begin{cases}1 & \text{if } \chi\neq\chi_0,\ \chi(-1) = 1,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\chi(-1) = (-1)^d, \quad d\in\{0, 1\}.$$

Inserting (4.4) into (4.3) and writing

$$S_+(q, a) = \frac{1}{\varphi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\, q)\\ d=0}}\chi(a), \qquad S_-(q, a) = \frac{1}{\varphi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\, q)\\ d=1}}\chi(a)$$

we deduce that the limit in (4.3) equals

$$\begin{aligned} & e^{-x/2}\Delta(e^x, q, a)+\frac{e^{-x/2}}{\varphi(q)}\sum_{\chi\,(\mathrm{mod}\, q)}\overline{\chi(a)}\sum_{\substack{p^\alpha\leqslant e^x\\ p\mid q}}\chi'(p^\alpha)\log p \\ &\quad +xe^{-x/2}\Big(S_+(q, a)-\frac{1}{\varphi(q)}\Big)+e^{-x/2}S_+(q, a)R(x, 0) \\ &\quad +e^{-x/2}S_-(q, a)R(x, 1)-\frac{e^{-x/2}}{\varphi(q)}\sum_{\chi\,(\mathrm{mod}\, q)}\overline{\chi(a)}B(\chi) \\ &\quad = e^{-x/2}\Delta(e^x, q, a)+E+F+G+H-I, \end{aligned}\tag{4.5}$$

say. We have to prove that

$$E+F+G+H-I = -h(x, q, a), \tag{4.6}$$

$h(x, q, a)$ being defined by (1.4)–(1.11). Using the orthogonality law for Dirichlet characters it is easy to see that (for $q\geqslant 3$, $q\neq 1$) we have

$$S_+(q, a) = \begin{cases}0 & \text{for } a\not\equiv\pm 1 \pmod q,\\ 1/2 & \text{otherwise;}\end{cases} \tag{4.7}$$

$$S_-(q, a) = \begin{cases}0 & \text{for } a\not\equiv\pm 1 \pmod q,\\ 1/2 & \text{for } a\equiv 1 \pmod q,\\ -1/2 & \text{for } a\equiv -1 \pmod q.\end{cases} \tag{4.8}$$

Moreover, for $(a, q) = 1$ and arbitrary prime number p, $p^k\| q$, $k\geqslant 0$, we have

$$\frac{1}{\varphi(q)}\sum_{\chi\,(\mathrm{mod}\, q)}\overline{\chi(a)}\chi'(p^\alpha) = \begin{cases}1/\varphi(p^k) & \text{if } p^\alpha\equiv a \pmod{q_p},\\ 0 & \text{otherwise.}\end{cases} \tag{4.9}$$

Indeed, the sum in (4.9) equals

$$\frac{1}{\varphi(q)}\sum_{d\mid q}\ \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\, d)}\overline{\chi(a)}\chi(p^\alpha), \tag{4.10}$$

the star indicates that the summation is restricted to the primitive characters (mod d) only. For $d|q$, $p|d$ we have $\chi(p)=0$ for each Dirichlet character (mod d). Hence (4.10) equals

$$\frac{1}{\varphi(q)}\sum_{d|q_p}\sum_{\chi(\mathrm{mod}\, d)}^{*}\overline{\chi(a)}\chi(p^{\alpha})=\frac{1}{\varphi(q)}\sum_{\chi(\mathrm{mod}\, q_p)}\overline{\chi(a)}\chi(p^{\alpha})$$

and (4.9) follows by an application of the orthogonality law for characters.

Using (4.9) we obtain

$$(4.11)\qquad E=e^{-x/2}\sum_{\substack{p^{\alpha}\leqslant e^{x}\\ p|q,\, p^{\alpha}\equiv a (\mathrm{mod}\, q_p)}}\frac{\log p}{\varphi(p^k)}=e^{-x/2}\sum_{p^k\|q}{}'\frac{\log p}{\varphi(p^k)}\sum_{\substack{m\geqslant 0\\ p^{mg_{p,q}+l_p}\leqslant e^{x}}}1$$

$$=e^{-x/2}\sum_{p^k\|q}{}'\frac{\log p}{\varphi(p^k)}\Psi(X)+xe^{-x/2}\sum_{p^k\|q}{}'\frac{1}{g_{p,q}\varphi(p^k)}$$

$$-e^{-x/2}\sum_{p^k\|q}{}'\frac{l_p\log p}{g_{p,q}\varphi(p^k)}+\tfrac{1}{2}e^{-x/2}\sum_{p^k\|q}{}'\frac{\log p}{\varphi(p^k)},$$

where

$$\Psi(X)=[X]-X+\frac{1}{2},\qquad X\in\boldsymbol{R},\qquad X=X_p=\frac{x}{g_{p,q}\log p}-\frac{l_p}{g_{p,q}},$$

and the dash has the same meaning as in (1.5) and (1.9). Using (4.7) and (4.8) we obtain

$$(4.12)\qquad G+H=-h_1(x,q,a)e^{-x/2},$$

h_1 being defined by (1.7).

Combining (4.7)–(4.12) and taking into account that

$$\Psi(X)=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nX)}{n}=\lim_{y\to 0^+}\mathrm{Re}\frac{1}{\pi i}\sum_{n\geqslant 1}\frac{1}{n}e(nZ)$$

we obtain (4.6).

The case $q=1$, $x\neq k\log p$ is very easy. We proceed as before, but instead of (4.7) and (4.8) we use the obvious identities

$$S_+(1,1)=1,\qquad S_-(1,1)=0$$

(S_- being the empty sum).

The proof in case $x=k\log p$ follows from what we have already proved. We have only to observe that both sides of (4.1) have the following property:

$$g(x)=\tfrac{1}{2}\big(g(x+0)+g(x-0)\big),\qquad x\geqslant 1$$

(cf. (4.4) in [5], part I). The proof is complete.

LEMMA 4.2. *For $z=x+iy$, $x\geqslant 1$, $y>0$ we have*

$$(4.13)\qquad \mathrm{Re}\,h(z,q,a)=O(|z|e^{-x/2}).$$

Moreover, for every fixed $Y>0$ we have

$$(4.14)\qquad |h(z,q,a)|=O_Y(|z|e^{-x/2})$$

for $x\geqslant 1$ and $y>Y$.

Proof. From (1.4) we obtain

$$(4.15)\qquad \mathrm{Re}\,h(z,q,a)=O\Big(\big(|z|+|h_1(z,q,a)|+|\mathrm{Re}\,h_2(z,q,a)|+|(\sin y)\mathrm{Im}\,h_2(z,q,a)|\big)e^{-x/2}\Big).$$

It can easily be seen that (1.7) and (1.8) imply that

$$(4.16)\qquad |h_1(z,q,a)|\ll 1.$$

Moreover, writing $u=u_p=(2\pi y)/(g_{p,q}\log p)$ we have, by partial summation,

$$(4.17)\qquad |\mathrm{Re}\,h_2(z,q,a)|\ll\sum_{p|q}\Big|\sum_{n\geqslant 1}(1/n)e^{-un}\sin(2\pi nX)\Big|\ll\sup_{\substack{t\in\boldsymbol{R}\\ N\in\boldsymbol{N}}}|S_N(t)|\ll 1$$

where

$$S_N(t)=\sum_{n=1}^{N}(1/n)\sin(2\pi nt),$$

because the series $\sum_{n=1}^{\infty}(1/n)\sin(nt)$ is boundedly convergent.

We have

$$(4.18)\qquad |\mathrm{Im}\,h_2(z,q,a)|\leqslant|h_2(z,q,a)|\ll\sum_{p|q}\sum_{n\geqslant 1}\frac{1}{n}e^{-nu}\ll\sum_{p|q}\frac{1}{u}\sum_{n\geqslant 1}\frac{1}{n^2}\ll\frac{1}{y}$$

and thus

$$(4.19)\qquad |(\sin y)\mathrm{Im}\,h_2(z,q,a)|\ll 1.$$

Inserting (4.16), (4.17) and (4.19) into (4.15) we obtain (4.13).

The proof of (4.14) is simple: it is enough to use (1.4), (4.16) and (4.18).

5. Proof of Theorem 1.1. The function $i\lambda+F_1(z,q,a)$ is almost periodic and not identically zero on the upper half-plane. For $y>Y_\lambda$ we have

$$\inf_{x\in\boldsymbol{R}}|i\lambda+F_1(x+iy,q,a)|>0$$

(cf. [1], Chapitre IV, § 19). Hence, using the celebrated Bohr theorem (cf. [2]) we can write

$$(5.1)\qquad \log\big(i\lambda+F_1(z,q,a)\big)=i\delta_\lambda\gamma_0 z+g(z,q,a,\lambda),$$

g being almost periodic in the sense of Bohr. Putting

$$M=M_\lambda=\sup_{x\in\boldsymbol{R}}|g(x+iY_\lambda,q,a,\lambda)|$$

we have

$$(5.2)\qquad |i\lambda+F_1(z,q,a)|\geqslant e^{-M-\gamma_0 y}\qquad \text{for } y\geqslant Y_\lambda.$$

By Lemma 4.2 there exists a constant $c_{12} > 1$ such that

(5.3) $$|\mathrm{Re}\, h(z, q, a)| < \tfrac{1}{2} e^{-M-\gamma_0 Y}$$

uniformly for $z = x+iy$, $x \geqslant c_{12}$, $0 < y < Y$.

Let C_1 be a curve contained in the rectangle

$$z = x+iy, \quad c_{12} \leqslant x \leqslant c_{13}, \quad 0 < y \leqslant Y$$

such that

$$|\Delta_{C_1} \arg F(z, q, a, \lambda)| \leqslant \pi.$$

Moreover, let C_2 be an analogous curve contained in

$$z = x+iy, \quad T - c_{14} \log T \leqslant x \leqslant T \ (T > c_{15}), \quad 0 < y < Y.$$

These curves exist according to Theorem 3.1. The hypotheses of Theorem 3.1 are easy to verify except the existence of the limits $P(x)$ in (1.24), which, however, follows from [5], part I, and except (1.30), which is proved in the Appendix (Lemma A3).

Let C be the closed, positively oriented curve consisting of C_1, C_2 and two line segments lying on the real axis and the line $y = Y$ respectively. We modify C in the following way. If C passes through a singularity or a zero of $F(z, q, a, \lambda)$ (it is possible on the real axis only) then we substitute a small part of C near this point by the half-circle lying on the upper half-plane. We can do it in such a way that $F(z, q, a, \lambda)$ does not vanish in the region contained between these half-circles and the real axis. We denote this modified curve by C_0. The variation of the argument of $F(z, q, a, \lambda)$ along this (positively oriented) curve does not depend on the particular choice of radii of the half-circles (if they are sufficiently small). Let us denote by A and B the end points of C_1 and C_2 on $\boldsymbol{R}$ respectively. Since, according to Lemma 4.1,

$$\lim_{y \to 0^+} \mathrm{Re}\, F(x+iy, q, a, \lambda) = e^{-x/2} \Delta_0(e^x, q, a)$$

and for every increase of argument by 2π there are at least two changes of sign of the real part, we have

(5.4) $$\begin{aligned} V(e^T, q, a) &\geqslant V(e^B, q, a) - V(e^A, q, a) \\ &\geqslant \frac{1}{\pi} \Delta_{\substack{z \in C_0 \\ z \text{ between } A \text{ and } B}} \arg F(z, q, a, \lambda) \\ &= \frac{1}{\pi} \Delta_{C_0} \arg F(z, q, a, \lambda) - \frac{1}{\pi} \Delta_{C_3} \arg F(z, q, a, \lambda) + O(1), \end{aligned}$$

where C_3 denotes the line segment joining C_2 to C_1 on the line $y = Y$.

But

(5.5) $$\begin{aligned} \frac{1}{2\pi} \Delta_{C_0} \arg F(z, q, a, \lambda) &= \text{the number of zeros of } F \text{ inside } C_0 \\ &\geqslant N(T - c_{14} \log T, Y, \lambda) - N(c_{13}, Y, \lambda). \end{aligned}$$

Using (5.2) and (5.3) we have

$$F(z, q, a, \lambda) = \big(i\lambda + F_1(z, q, a)\big)\big(1 + \theta(z)\big),$$

where $|\theta(z)| \leqslant 1/2$. Hence, using (5.1),

(5.6) $$\Delta_{C_3} \arg F(z, q, a, \lambda) = \Delta_{C_3} \arg\big(i\lambda + F_1(z, q, a)\big) + O(1) = -\delta_\lambda \gamma_0 T + O(\log T).$$

Collecting (5.4)–(5.6) we obtain (1.12), which ends the proof.

6. Proof of Corollary 1.1. Let $Y_0 > 0$ be such that

$$F_1(w, q, a) \neq 0 \quad \text{for } \mathrm{Im}\, w > Y_0.$$

For every Y, $0 < Y < Y_0$ we have

(6.1) $$F(z, q, a, 0) = -2F_1(z, q, a) + O(x e^{-x/2})$$

uniformly for $x \geqslant 1$ (cf. Lemma 4.2). We encircle each zero of $F_1(z, q, a)$ in the region

(6.2) $$z = x+iy, \quad Y < y < Y_0, \quad x \geqslant 1,$$

by a small circle of fixed radius $r > 0$. Then there exists a positive constant m such that

(6.3) $$|F_1(z, q, a)| \geqslant m$$

for each z outside these circles and lying in the region (6.2). From Rouché theorem it follows using (6.1)–(6.3) that for sufficiently large c_{15} functions F and F_1 have the same number of zeros in the region

$$z = x+iy, \quad c_{15} \leqslant x \leqslant T, \quad Y < y < Y_0.$$

Let ε be positive. We fix Y, $0 < Y < Y_0$, and U_0 such that

$$\#\{z = x+iy \mid F_1(z, q, a) = 0,\ 0 < x < U,\ y > Y\} \geqslant (\varkappa - \varepsilon) U$$

for $U \geqslant U_0$. Then, taking $U = \log T - c_1 \log\log T$, we have

(6.4) $$\begin{aligned} \frac{N(U, Y, 0)}{\log T} &\geqslant \frac{N(U, T, 0)}{U} \left(1 + O\left(\frac{\log\log T}{\log T}\right)\right) \\ &\geqslant \left(\varkappa - \varepsilon + O\left(\frac{1}{\log T}\right)\right)\left(1 + O\left(\frac{\log\log T}{\log T}\right)\right) = \varkappa - \varepsilon + O\left(\frac{\log\log T}{\log T}\right). \end{aligned}$$

Hence, applying Theorem 1.1 with $\lambda = 0$ we get

$$\liminf_{T \to \infty} \frac{V(T, q, a)}{\log T} \geqslant \frac{\gamma_0}{\pi} + 2 \liminf_{T \to \infty} \frac{N(U, Y, 0)}{\log T} \geqslant \frac{\gamma_0}{\pi} + 2(\varkappa - \varepsilon),$$

and the corollary follows.

7. The density of zeros of a general almost periodic function on the upper half-plane. Let us consider a function of the form

$$F(z) = \sum_{n=1}^{\infty} a_n e^{iw_n z}, \quad a_n \in \mathbf{C},\ w_n \in \mathbf{R},$$

which is absolutely convergent for $y = \operatorname{Im} z > 0$.

Let $\xi = u + iw$, $w > 0$, $u > 0$ be a zero of F. Let us write

$$F_N(z) = \sum_{n=1}^{N} a_n e^{iw_n z}$$

and suppose C to be a Jordan curve on the upper half-plane encircling ξ. We put

$$\alpha = \inf_{z\in C} F_N(z), \quad y_0 = \inf_{z\in C} \operatorname{Im} z, \quad x_0 = \inf_{z\in C} \operatorname{Re} z, \quad x_1 = \sup_{z\in C} \operatorname{Re} z,$$

$$a = \sum_{n=1}^{N} |a_n| e^{-w_n y_0}, \quad b = \sum_{n=N+1}^{\infty} |a_n| e^{-w_n y_0}.$$

THEOREM 7.1. *If* $\alpha > 3b$ *and* $x_1 - x_0 < 1$ *then*

$$\varkappa \geqslant q_0^{-N},$$

where

$$\varkappa = \lim_{Y\to 0^+} \lim_{T\to\infty} \frac{1}{T} \# \{z = x+iy \mid F(z) = 0,\ 0 < x < T,\ y > Y\},$$

$$q_0 = [4\pi a/(\alpha - 3b)] + 1.$$

Proof. We want to make use of the following version of Dirichlet's theorem about diophantine approximations.

THEOREM (cf. [8], p. 153). *Given N natural numbers $x_1, x_2, \ldots, x_N$ and two positive integers q and M, we can find M integers $1 \leqslant n_1 < n_2 < \ldots < n_M \leqslant Mq^N$, such that*

$$\|n_l x_j\| \leqslant 1/q \quad \text{for } l = 1, 2, \ldots, M \text{ and } j = 1, 2, \ldots, N.$$

From this theorem it follows that there exist

$$M = \left[\frac{T - x_0 - 1}{q_0^N}\right] = q_0^{-N} T + O(1)$$

natural numbers n_l such that

$$1 \leqslant n_l \leqslant T - 1 - x_0 \quad (l = 1, 2, \ldots, M)$$

and

$$\left\| n_l \frac{w_j}{2\pi} \right\| < \frac{1}{q_0} \quad (l = 1, 2, \ldots, M,\ j = 1, 2, \ldots, N).$$

For $1 \leqslant l \leqslant M$ we write

$$C_l = n_l + C = \{z + n_l \mid z \in C\}.$$

Then for $z \in C_l$ we have

$$|F(z)| \geqslant |F_N(z)| - b \geqslant |F_N(z - n_l)| - \frac{2\pi}{q_0} a - b \geqslant \alpha - \frac{2\pi a}{q_0} - b.$$

On the other hand,

$$|F(\xi + n_l)| \leqslant |F(\xi)| + |F_N(\xi) - F_N(\xi + n_l)| + 2b = (2\pi/q_0) a + 2b.$$

Hence, taking into account $q_0 > (4\pi a)/(\alpha - 3b)$ we conclude that

$$|F(\xi + n_l)| < \inf_{z\in C_l} |F(z)|.$$

Using the maximum-modulus principle for $1/F$ we see that F has a zero inside C_l. Since $x_1 - x_0 < 1$ zeros inside distinct curves C_l are distinct. Hence

$$\frac{1}{T} \# \{z = x+iy \mid F(z) = 0,\ 0 < x < T,\ y \geqslant y_0\} \geqslant M/T = q_0^{-N} + O(1/T)$$

and the result follows.

8. Proof of Theorem 1.2. We suppose first that the Riemann Hypothesis is not true. Then, according to Theorem 2 in [3], part II, we have

$$\liminf_{T\to\infty} \frac{V_3(T)}{\log T} \geqslant \frac{\gamma_1}{\pi},$$

where γ_1 is defined as follows. If $\zeta(s)$ has any zeros $\varrho = \theta + i\gamma$ on the line $\sigma = \theta = \sup_{\zeta(\varrho)=0} \operatorname{Re} \varrho$, then γ_1 denotes the least positive γ corresponding to these zeros; otherwise $\gamma_1 = +\infty$. Hence, if the Riemann Hypothesis is not true we have

$$\gamma_1 \geqslant 10^8$$

(cf. [7]). This is of course much more than enough to prove our theorem. The real difficulty appears in the case when the Riemann Hypothesis is correct. In this case we apply Theorem 7.1 to

$$F(z) = e^{-z/2} K(z, \chi_0).$$

We take

$$N = 85, \quad \xi = 9.539\ldots + i\,0.06\ldots$$

and let C be the boundary of the rectangle with vertices

$$9.5 + i\,0.034, \quad 9.5 + i\,0.07, \quad 9.57 + i\,0.07, \quad 9.57 + i\,0.034.$$

The author used a desk computer to verify that ξ is a zero of F. Moreover, it can be seen that

$$y_0 = 0.034, \quad \alpha > 0.008, \quad a < 1/3, \quad b < 1/4000,$$

$$x_1 - x_0 = 0.07 < 1.$$

We have

$$\alpha - 3b \geqslant 0.007.$$

Theorem 7.1 yields

$$\varkappa \geqslant \left(\left[\frac{4\pi \cdot 1000}{3\cdot 7}\right]+1\right)^{-85} > (200\pi)^{-85} = 10^{-85(2+\log_{10}(2\pi))} > 10^{-248}.$$

Using Corollary 1.1 we obtain

$$\liminf_{T\to\infty} \frac{V_3(T)}{\log T} \geqslant \frac{\gamma_0}{\pi} + 2\cdot 10^{-248},$$

which is slightly better than required.

APPENDIX

We prove here a general estimate needed in the proof of Theorem 1.1. Let χ denote a primitive Dirichlet character (mod q), $q \geqslant 1$, and let for $z = x+iy$, $y>0$

$$F(z, \chi) = K(z, \chi) + \overline{K(z, \bar{\chi})},$$

$$F(x, \chi) = \lim_{y\to 0^+} F(x+iy, \chi), \qquad x \in \boldsymbol{R}.$$

The last limit exists for every x (cf. [5], part I).

LEMMA A1 ([6], Theorem 2.2). *For every* $z = x+iy$, $x \geqslant 1$, $|y| \leqslant 1$, *we have*

$$K(z, \chi) = \frac{e^{3z/2}}{2\pi i} \sum_{\substack{n \\ |\log n - x| \leqslant 1}} \frac{\Lambda(n)\chi(n)}{n^{3/2}(z-\log n)} + O(e^{3x/2}).$$

LEMMA A2. *For* $z = x+iy$, $x>1$, $y>0$, *we have*

$$F(z, \chi) = F(x, \chi) + O(x + ye^{2x}).$$

Proof. For natural n and real x, y let us write

$$\lambda(n, x, y) = \begin{cases} \arctan \dfrac{y}{x-\log n}, & x \neq \log n, \\ 0, & x = \log n. \end{cases}$$

Using Lemma A1 we obtain for $y>0$

$$K(x+it, \chi) = \frac{e^{3x/2}}{2\pi i}(1+O(t)) \sum_{\substack{n \\ |\log n - x| \leqslant 1}} \frac{\Lambda(n)\chi(n)}{n^{3/2}(x-\log n+it)} + O(e^{3x/2})$$

$$= \frac{e^{3x/2}}{2\pi i} \sum_{\substack{n \\ |\log n - x| \leqslant 1}} \frac{\Lambda(n)\chi(n)}{n^{3/2}(x-\log n+it)} + O(e^{3x/2}).$$

Hence

$$F(z, \chi) - F(x, \chi) = i\int_0^y \left(k(x+it, \chi) + \overline{k(x+it, \bar{\chi})}\right)dt$$

$$= \frac{e^{3x/2}}{2\pi i} \sum_{\substack{n \\ |\log n - x| \leqslant 1}} \frac{\Lambda(n)\chi(n)}{n^{3/2}} \int_0^y \left(\frac{1}{x-\log n+it} + \frac{1}{x-\log n-it}\right)dt$$

$$+ O(ye^{3x/2})$$

$$= \frac{e^{3x/2}}{2\pi} \sum_{\substack{n \\ |\log n - x| \leqslant 1}} \frac{\Lambda(n)\chi(n)}{n^{3/2}} \lambda(n, x, y) + O(ye^{3x/2}).$$

Contribution of terms with $|\log n - x| \geqslant b_0 e^{-x}$ does not exceed $O(ye^{2x})$. Contribution of the remaining terms (for sufficiently small b_0 and large x there is at most one such term) is $O(x)$. Lemma A2 therefore follows.

LEMMA A3. *Let* $q \geqslant 1$, $q \neq 2$, $(a, q) = 1$ *and let us assume the Generalized Riemann Hypothesis for L-functions* (mod q). *Then for* $z = x+iy$, $x \geqslant 1$, $y>0$, *we have*

$$\operatorname{Re} F_1(z, q, a) = \operatorname{Re} F_1(x, q, a) + O(xe^{-x/2} + y^{1/2}e^{3x/2}).$$

Proof. The assertion is obvious for $y \geqslant 1/2$. Indeed, in this case $\operatorname{Re} F_1(z, q, a) = O(1)$ and $\operatorname{Re} F_1(x, q, a) = O(e^x)$.

Let us assume that $0 < y < 1/2$. Then

$$e^{-x/2}|K(z, \chi)| \ll \sum_{0<\gamma<1/y} \frac{1}{|\varrho|} + \sum_{\gamma \geqslant 1/y} \frac{e^{-\gamma y}}{\gamma} \ll \log^2\frac{1}{y} + \frac{1}{y}\sum_{\gamma \geqslant 1/y} \frac{1}{\gamma^2} \ll \log^2\frac{1}{y}.$$

Hence

$$\operatorname{Re} F_1(z, q, a) = e^{-x/2} \operatorname{Re}\left\{\frac{1}{\varphi(q)} \sum_{\chi (\text{mod } q)} \overline{\chi(a)} K(z, \chi')\right\}$$

$$+ O\left(e^{-x/2}|e^{iy/2}-1| \max_{\chi (\text{mod } q)} |K(z, \chi')|\right)$$

$$= e^{-x/2} \frac{1}{2\varphi(q)} \sum_{\chi (\text{mod } q)} \overline{\chi(a)} F(z, \chi') + O\left(y\log^2\frac{1}{y}\right)$$

$$= \operatorname{Re} F_1(x, q, a) + O\left(e^{-x/2} \max_{\chi (\text{mod } q)} |F(z, \chi') - F(x, \chi')| + y\log^2\frac{1}{y}\right).$$

We now apply Lemma A2; the remainder term is

$$O\left(xe^{-x/2} + ye^{3x/2} + y\log^2\frac{1}{y}\right) = O(xe^{-x/2} + y^{1/2}e^{3x/2})$$

and the lemma follows.

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Received on 19.10.1989
and in revised form on 20.7.1990 (1980)

ACTA ARITHMETICA
LIX.1 (1991)

Necessary conditions for distinct covering systems with square-free moduli

by

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A *distinct covering system* (henceforth DCS) is a set of congruences

$$a_1 \pmod{d_1},\quad a_2 \pmod{d_2},\quad \ldots,\quad a_k \pmod{d_k};\qquad d_1 < d_2 < \ldots < d_k$$

that cover the integers. For example

$$0 \pmod 2,\quad 0 \pmod 3,\quad 1 \pmod 4,\quad 5 \pmod 6,\quad 7 \pmod{12}$$

is such a system. Guy (Section F13 of [5]) gives many fascinating problems on DCS's. For instance, does a DCS exist with all moduli odd? In this paper we shall be mainly concerned with DCS's whose moduli are square free. Such DCS's exist (see [5], p. 140) but none are known to exist with moduli odd and square free. This is in spite of Erdős's conjecture [4] that for every t there is a distinct covering system in which all moduli are square-free integers all of whose prime factors are greater than p_t, the tth prime. We shall prove that if a DCS exists with all moduli odd and square-free, then the least common multiple of the moduli must be the product of at least 18 primes. This improves a result of Berger, Felzenbaum and Fraenkel [2] who showed that at least 13 primes were necessary.

The paper contains three theorems. With the first of these we show that if a DCS exists whose moduli are divisible by the primes $p_1, p_2, \ldots, p_k$, then a DCS exists in which $p_1, p_2, \ldots, p_k$ are the first k primes. If $p_1, p_2, \ldots, p_k$ are required to satisfy some constraint, such as all being odd, then we may assume that these are the k smallest primes satisfying this constraint.

In the second theorem of the paper we give a sieve theoretic lower bound on the number of integers which are left uncovered by a set of congruences with given square-free moduli.

In the third theorem we use notions connected with set partitions and Bell numbers to simplify the bound given in Theorem 2. This gives a result which can be easily applied to questions about DCS's with square-free moduli.

THEOREM 1. *Let q be a prime and suppose that $\{a_i \pmod{q^{\alpha_i}d_i}$: $i = 1, \ldots, k\}$, where $(q, d_i) = 1$ for each i, is a DCS, and let $q^{\alpha}P$ be the lowest common multiple of $q^{\alpha_1}d_1, \ldots, q^{\alpha_k}d_k$. Suppose that p is a prime such that $p < q$,*