ACTA ARITHMETICA
LIX.1 (1991)

Iwaniec's bilinear form of the error term in the Selberg sieve

by

SAVERIO SALERNO (Salerno)

1. Introduction. Let $\mathcal{N}$ be a finite sequence of integers and let $\mathcal{P}$ be a set of primes. We assume that

$$\mathcal{N}_d \overset{\text{def}}{=} \sum_{\substack{n\in\mathcal{N} \\ n\equiv 0 \,(\mathrm{mod}\, d)}} 1 = \frac{\omega(d)}{d} X + R(\mathcal{N}, d) \tag{1.1}$$

where $\omega(d)$ is a positive function with $\omega(d) < d$ for any d, multiplicative (that is, $\omega(nm) = \omega(n)\omega(m)$ if $(n, m) = 1$), X is an approximation of $|\mathcal{N}|$ and $R(\mathcal{N}, d)$ is a remainder term, which has to be small in average. This fact is usually expressed by saying that $\mathcal{N}$ is well distributed in the arithmetic progressions.

The final goal of sieve methods would be to deduce from this information the presence of the expected quantity of primes in the sequence $\mathcal{N}$. Unfortunately, the so-called "parity phenomenon" (see the Selberg example in Bombieri [2]) shows that sieve methods alone cannot produce primes, but only almost-primes P_r, that is, numbers with at most r prime factors, counting the multiplicity.

We are concerned in this paper with the application of the Selberg sieve to these problems. It can be applied directly to find upper bound for the sifting function

$$S(\mathcal{N}, \mathcal{P}, z) = \left|\{n \in \mathcal{N} \mid (n, P(z)) = 1\}\right| \tag{1.2}$$

where

$$P(z) = \prod_{\substack{p\in\mathcal{P} \\ p<z}} p. \tag{1.3}$$

For this purpose, one observes indeed that

$$S(\mathcal{N}, \mathcal{P}, z) \leqslant \sum_{n\in\mathcal{N}} \Big(\sum_{\substack{\nu | n, \nu < D \\ \nu | P(z)}} \lambda_\nu \Big)^2 \tag{1.4}$$

for any choice of the coefficients λ_ν with $\lambda_1 = 1$.

To treat the problem of the representation of P_r, it is convenient to use the weighted sieve, for instance Kuhn's weights or Richert's logarithmic weights.

The general idea is that

$$\sum_{\substack{n\in\mathcal{N}\\ n=P_r}} 1 \geqslant \sum_{\substack{n\in\mathcal{N}\\ (n,P(z))=1}} h(n) \tag{1.5}$$

where $h(n)$ is positive only if $n = P_r$ and $h(n) \leqslant 1$.

It turns out that the final results of the type (1.4), (1.5) are better if we can choose larger α in the error term coming from (1.1)

$$\sum_{\substack{d<X^\alpha\\ d\mid P(z)}} |R(\mathcal{N}, d)|, \tag{1.6}$$

of course, still keeping it under control.

For example if $\mathcal{N} = \{p < x\}$, we have $\alpha = 1/2-\varepsilon$ by the Bombieri–Vinogradov theorem (see [1]), whilst it has been conjectured by Elliott and Halberstam [6] that $\alpha = 1-\varepsilon$.

If $\mathcal{N} = \{n^2+1 < x\}$, then $\alpha = 1-\varepsilon$, and it is easily seen that it is not possible to improve it.

In order to overcome these limitations, Iwaniec [10] introduced new ideas, in particular the bilinear form of the error term in the linear sieve (Rosser's sieve)

$$\sum_{m<M}\sum_{\substack{n<N\\ mn\mid P(z)}} a_m b_n R(\mathcal{N}, mn). \tag{1.7}$$

For (1.7) it is often possible to prove the desired bound for $MN = X^{\alpha'}$, with $\alpha' > \alpha$. In this manner Iwaniec [11] succeeded in proving that $n^2+1 = P_2$ infinitely often.

We also assume that the sequence $\mathcal{N}$ satisfies

$$\sum_{p<x} \frac{\omega(p)\log p}{p} = k\log x + O(1) \tag{1.8}$$

and we say that our sieve has dimension k.

The purpose of this paper is to show that the bilinear form of the error term is also available for the Selberg sieve. This is of particular importance when k is large (indeed, when $k > 1$), when the Selberg sieve is known to give the best results (see Selberg [15], Salerno [14]). Also, after some recent works (see Bombieri, Friedlander and Iwaniec [3], [4], [5]), a general approach to bilinear forms, giving a higher exponent than the usual one, seems to be possible.

I am deeply indebted to Professor Iwaniec for his generous help and important suggestions, and I thank him here. Also, I take the pleasure to thank Professor Bombieri for helpful discussions on these topics. Finally, I thank the referee for helpful remarks.

2. Statement of the results. We define the function $\sigma_k(s)$ as the continuous solution of the difference-differential equation

$$\begin{aligned} \frac{s^k}{\sigma_k(s)} &= 2^k e^{k\gamma}\Gamma(k+1) \quad \text{if } 0 < s \leqslant 2,\\ \{s^{-k}\sigma_k(s)\}' &= -ks^{-k-1}\sigma_k(s-2) \quad \text{if } s > 2. \end{aligned} \tag{2.1}$$

We introduce also, for $s > 1$, the function $\eta_k(s)$ as

$$\eta_k(s) = ks^{-k}\int_s^\infty \left\{\frac{1}{\sigma_k(t-1)} - 1\right\} dt. \tag{2.2}$$

One can show that the functions $\sigma_k(s)$ and $1-\eta_k(s)$ are nonnegative, increasing,

$$1-\eta_k(s) < 1 < 1/\sigma_k(s) \quad \text{for } s > 1, \tag{2.3}$$

and for large s

$$\sigma_k(s) = 1+O(e^{-s}), \quad \eta_k(s) = O(e^{-s}). \tag{2.4}$$

For these and other properties of $\sigma_k(s)$ and $\eta_k(s)$, we refer to the book of Halberstam and Richert [8].

Our results are the following:

THEOREM 1. *Assume that $\mathcal{N}$ satisfies (1.1), (1.8). Let*

$$V(z) = \prod_{p<z}\left(1-\frac{\omega(p)}{p}\right).$$

Let $D \geqslant z$ and $s = \log D^2/\log z$. We have, for any integer q,

$$S(\mathcal{N}_q, \mathcal{P}, z) \leqslant \frac{\omega(q)}{q} V(z)X\left\{\frac{1}{\sigma_k(s)} + o(1)\right\} + R \tag{2.5}$$

where, with $L = (\log D)$

$$|R| \ll L^c \sum_{m<\zeta D}\sum_{\substack{n<D\\ mn\mid P(z)}} \alpha(m)\beta(n)R(\mathcal{N}, mnq), \tag{2.6}$$

with $\zeta = D^{\varepsilon^2}$ $(\varepsilon \to 0)$, and $\alpha(m)$, $\beta(m)$ are suitable coefficients such that

$$\alpha(m) \ll \tau^c(m), \quad |\beta(n)| \ll \tau^c(n). \;\blacksquare \tag{2.7}$$

Here and in the sequel, we denote by c a positive constant which can assume different values at various appearences.

THEOREM 2. *Assume that $\mathcal{N}$ satisfies (1.1), (1.8). Let $D \geqslant z$ and $s = \log D^2/\log z$. Then we have*

$$S(\mathcal{N}, \mathcal{P}, z) \geqslant V(z)X\{1-\eta_k(s) + o(1)\} - R \tag{2.8}$$

where

$$|R| \ll L^c \sum_{m<\zeta D\sqrt{z}}\sum_{\substack{n<D\\ mn\mid P(z),\, mn<D^2}} \alpha(m)\beta(n)R(\mathcal{N}, mnq) \tag{2.9}$$

with $\zeta = D^{\varepsilon^2}$ $(\varepsilon \to 0)$, and $\alpha(m)$, $\beta(m)$ satisfying (2.7). $\blacksquare$

One can remark that the upper bound (Th. 1) is given for $S(\mathcal{N}_q, \mathcal{P}, z)$, whilst the lower bound (Th. 2) concerns only $S(\mathcal{N}, \mathcal{P}, z)$. There is no intrinsic reason for this difference, but this is what we need for the weighted sieve (Th. 3 below).

THEOREM 3. *Assume that* $\mathcal{N}$ *satisfies* (1.1), (1.8) *and let* $D \geqslant z$ *and* $s = \log D^2/\log z$, *so that Theorems* 1 *and* 2 *hold.*

Let u, v *and* α *be real numbers such that*

(2.10) $$z = X^{1/v} < X^{1/u} < D^2 = X^{\alpha}.$$

Assume also that $|n| < X^{\alpha\mu}$. *Let* r *be a positive integer such that*

(2.11) $$r > u\alpha\mu - 1 + \frac{1}{1-\eta_k(\alpha v)}\left\{k\int_u^v \frac{1}{\sigma_k}\left(v\left(\alpha-\frac{1}{t}\right)\right)\left(1-\frac{u}{t}\right)\frac{dt}{t}\right\}.$$

Then we have

(2.12) $$\sum_{\substack{n\in\mathcal{N}\\ n=P_r}} 1 \geqslant c|\mathcal{N}|V(z)+O(R)$$

where

(2.13) $$|R| \ll L^c \sum_{\substack{m<\zeta D\sqrt{z}\\ mn\mid P(z),mn<D^2}} \sum_{n<D} \alpha(m)\beta(n)R(\mathcal{N}, mn)$$

with $\zeta = D^{\varepsilon^2}$ $(\varepsilon \to 0)$, *and* $\alpha(m)$, $\beta(m)$ *satisfying* (2.7). ■

We point out that, by using Buchstab's identity

(2.14) $$S(\mathcal{N}, \mathcal{P}, z) = S(\mathcal{N}, \mathcal{P}, z_1) - \sum_{\substack{z_1<p<z\\ p\in\mathcal{P}}} S(\mathcal{N}_p, \mathcal{P}, p),$$

if one has an upper bound and a lower bound with functions $F_0(s)$ and $f_0(s)$ respectively, one obtains the upper and lower bounds with new functions $F_1(s)$, $f_1(s)$, given by

(2.15)
$$F_1(s) = 1 - ks^{-k}\int_s^{\infty}\{f_0(t-1)-1\}t^{k-1}\,dt,$$
$$f_1(s) = 1 - ks^{-k}\int_s^{\infty}\{F_0(t-1)-1\}t^{k-1}\,dt.$$

Formulas (2.15) are called Buchstab's transform. Iterating the procedure, one obtains limit functions $F(s)$ and $f(s)$ which are invariant under Buchstab's transform. Of course, one obtains different functions $F(s)$, $f(s)$ starting with different $F_0(s)$, $f_0(s)$. For example, starting from Brun's sieve, one gets Rosser's sieve (see Iwaniec [9], [12]). The iteration starting from a Selberg sieve has been studied by Iwaniec, van de Lune and te Riele [13].

It is clear that the bilinear form of the error (2.6), (2.13) is essentially preserved when we perform a single iteration by means of (2.14). We shall study elsewhere the general problem of iteration of the Selberg sieve, as well as arithmetical applications. Here we only remark that by the results of Bombieri, Friedlander and Iwaniec [3] we are allowed to take in (2.13) $D^2 = X^{29/56-\varepsilon}$ with the error term still under control, and this improves the constant in (2.12) for $\mathcal{N} = \{p+2 \mid p < x\}$. We point out that $p+2 = P_2$ infinitely often has been proved by Fouvry and Grupp [7] without Chen's device, but using different weights and sharp estimates on bilinear forms.

3. Proof of Theorem 1. We assume in the sequel that k, the dimension of the sieve, is $\geqslant 1$. We introduce the Selberg weights λ_ν of level D and dimension k, that is, $\lambda_1 = 1$ and for square-free ν,

(3.1) $$\lambda_\nu = \mu(\nu)\prod_{p\mid\nu}\left(1-\frac{\omega(p)}{p}\right)^{-1} \sum_{\substack{r<D/\nu\\ (r,\nu)=1,r\mid P(z)}} g(r)\Big(\sum_{\substack{r<D\\ r\mid P(z)}} g(r)\Big)^{-1}$$

with g multiplicative and

$$g(p) = \frac{\omega(p)}{p-\omega(p)}.$$

By the usual analytical techniques, one obtains

(3.2) $$\sum_{\substack{r<D/\nu\\ (r,\nu)=1\\ r\mid P(z)}} \mu^2(r)g(r) \sim \prod_{p\mid\nu}(1-g(p)) \sum_{\substack{r<D/\nu\\ r\mid P(z)}} \mu^2(r)g(r).$$

Moreover, setting

$$G(x,z) = \sum_{\substack{d<x\\ d\mid P(z)}} g(d), \quad W(z) = \prod_{p<\nu}\left(1-\frac{\omega(p)}{p}\right), \quad \tau = \frac{\log x}{\log z},$$

we have by Lemma 6.1 of [5]

(3.3) $$\frac{1}{G(x,z)} = W(z)\left\{\frac{1}{\sigma_k(2\tau)} + O\left(\frac{\tau^{2k+1}}{\log z}\right)\right\}.$$

We infer from (3.1), (3.2) and (3.3) that

(3.4) $$\lambda_\nu = \mu(\nu)\left(\frac{\sigma_k\left(2\dfrac{\log^+ D/\nu}{\log z}\right)}{\sigma_k\left(2\dfrac{\log D}{\log z}\right)}\right)^k + O\left(\frac{1}{\log D}\right)$$

and using suitable information on the behaviour of the function $\sigma_k(s)$ in the range of our interest, we finally obtain by (3.4)

$$\lambda_\nu = \mu(\nu)\left(\frac{\log^+ D/\nu}{\log D}\right)^k + O\left(\frac{1}{\log D}\right) = \mu(\nu)\lambda(\nu) + O\left(\frac{1}{\log D}\right), \text{ say,} \tag{3.5}$$

where, as usual, $f^+ = \max(f, 0)$.

Now, let $\zeta < D^{\varepsilon^2}$, to be chosen in the sequel. We have

$$S(\mathcal{N}_q, \mathcal{P}, z) \leqslant \sum_{\substack{n\in\mathcal{N}_q \\ (n,P(\zeta))=1}} \Big(\sum_{\substack{\nu|n,\nu<D \\ \nu|P(z,\zeta)}} \lambda_\nu\Big)^2 \tag{3.6}$$

$$= \sum_{\substack{n\in\mathcal{N}_q \\ (n,P(\zeta))=1}} \Big(\sum_{\substack{\nu|n,\nu<D \\ \nu|P(z,\zeta)}} \mu(\nu)\lambda(\nu)\Big)^2 + O\big(2^{\log z/\log\zeta} S(\mathcal{N}, \mathcal{P}, \zeta)(\log D)^{-1}\big).$$

It is well known that, since ζ is small,

$$S(\mathcal{N}, \mathcal{P}, \zeta) \ll V(\zeta)X \ll (\log\zeta)^{-k}X \tag{3.7}$$

with a remainder controlled by (2.6).

In order to evaluate the main term of (3.6), we proceed as follows:

$$S = \sum_{\substack{n\in\mathcal{N} \\ (n,P(\zeta))=1}} \Big(\sum_{\substack{\nu|n,\nu<D \\ \nu|P(z,\zeta)}} \mu(\nu)\lambda(\nu)\Big)^2 \tag{3.8}$$

$$\leqslant \sum_{n\in\mathcal{N}} \Big(\sum_{\substack{l|n,l<L \\ l|P(\zeta)}} p^+(l)\Big)\Big(\sum_{\substack{\nu|n,\nu<D \\ \nu|P(z,\zeta)}} \mu(\nu)\lambda(\nu)\Big)^2$$

where $p^+(l)$ are upper bound sieve weights of level $L = D^\varepsilon$ and dimension k. Hence

$$S \leqslant \sum_{\substack{l<L \\ l|P(\zeta)}} p^+(l) \sum_{\nu_1,\nu_2|P(z,\zeta)} \mu(\nu_1)\mu(\nu_2)\lambda(\nu_1)\lambda(\nu_2)|\mathcal{N}_{l[\nu_1,\nu_2]}|. \tag{3.9}$$

Using (1.1), we obtain for the main term of (3.9), say S_1,

$$S_1 = X\Big\{\sum_{\substack{l<L \\ l|P(\zeta)}} \frac{p^+(l)\omega(l)}{l}\Big\}\Big\{\sum_{\nu_1\nu_2|P(z,\zeta)} \frac{\mu(\nu_1)\mu(\nu_2)\lambda(\nu_1)\lambda(\nu_2)\omega([\nu_1,\nu_2])}{[\nu_1,\nu_2]}\Big\} \tag{3.10}$$

$$= X V_1 V_2, \text{ say.}$$

Now, we use the following well-known result, the so-called "fundamental lemma" (see Iwaniec [9]):

$$V_1 = V(\zeta)\{1 + O(e^{-\log L/\log\zeta})\}. \tag{3.11}$$

What concerns V_2, we change back $\mu(\nu)\lambda(\nu)$ to original Selberg's λ_ν, according to (3.2). Hence

$$V_2 = \sum_{\substack{\nu_1,\nu_2|P(z,\zeta) \\ \nu_i<D}} \frac{\lambda(\nu_1)\lambda(\nu_2)\omega([\nu_1,\nu_2])}{[\nu_1,\nu_2]} + O\Big((\log D)^{-1} \sum_{\substack{\nu_1,\nu_2|P(z,\zeta) \\ \nu_i<D}} \frac{\omega([\nu_1,\nu_2])}{[\nu_1,\nu_2]}\Big) \tag{3.12}$$

$$= V_{2,1} + V_{2,2}, \text{ say.}$$

Using the results of Ch. 6 of the book of Halberstam and Richert [8] we obtain, for $\zeta = D^{\varepsilon^2}$,

$$V_{2,1} = \frac{V(z)}{V(\zeta)}\Big\{\frac{1}{\sigma_k(s)} + O(e^{-1/\varepsilon})\Big\}. \tag{3.13}$$

For $V_{2,2}$ we have

$$V_{2,2} \leqslant \frac{1}{\log D} \sum_{\nu\nu_1\nu_2|P(z,\zeta)} \frac{\omega(\nu\nu_1\nu_2)}{\nu\nu_1\nu_2} \leqslant \frac{1}{\log D}\Big(\sum_{\nu|P(z,\zeta)} \frac{\omega(\nu)}{\nu}\Big)^3 \tag{3.14}$$

$$\leqslant \frac{1}{\log D} \prod_{\zeta<p<z} \Big\{1 + \frac{\omega(p)}{p}\Big\}^3 \ll \frac{1}{\log D}\Big(\frac{\log z}{\log\zeta}\Big)^3.$$

By (3.10) to (3.14), we infer

$$S_1 = V(z)X\Big\{\frac{1}{\sigma_k(s)} + O\Big(\frac{1}{\log D}\Big(\frac{\log z}{\log\zeta}\Big)^{4k}\Big) + O\Big(\exp\Big(-\frac{1}{2\varepsilon}\Big)\Big)\Big\} \tag{3.15}$$

and we are left with the problem of estimating the error term of (3.9) coming from (1.1). We have

$$R(\mathcal{N}, \mathcal{P}, z) = \sum_{\substack{l|P(\zeta) \\ l<L}} p^+(l) + \sum_{\substack{\nu_1,\nu_2<D \\ \nu_1,\nu_2|P(z,\zeta)}} \mu(\nu_1)\mu(\nu_2)\lambda(\nu_1)\lambda(\nu_2)R(\mathcal{N}, l[\nu_1,\nu_2]) \tag{3.16}$$

$$= \sum_{\substack{l|P(\zeta) \\ l<L}} p^+(l) + \sum_{\substack{\nu_1\nu_2<D,\nu_2<D \\ \nu\nu_1\nu_2|P(z,\zeta)}} \mu(\nu_1)\mu(\nu_2)\lambda(\nu\nu_1)\lambda(\nu\nu_2)R(\mathcal{N}, l\nu\nu_1\nu_2).$$

We remark that in the last sum we can replace the condition $\nu\nu_2 < D$ with $\nu_2 < D$ by virtue of the presence of $\lambda(\nu\nu_2)$.

Using the Mellin transform, we have

$$(\log^+ x)^k = \frac{k!}{2\pi i}\int_{(\sigma)} x^s \frac{ds}{s^{k+1}} \quad \text{for } \sigma > 0. \tag{3.17}$$

We have indeed

$$f(x) = \frac{1}{2\pi i}\int_{(\sigma)} \tilde{f}(s)x^{-s}ds \tag{3.18}$$

where $\tilde{f}(s)$ denotes as usual the Mellin transform of f. Then (3.17) follows by applying (3.18) to the function $f(x) = (\log^+ x)^k/k!$ and observing that in this case $\tilde{f}(s) = 1/s^{k+1}$.

Hence

$$R(\mathcal{N}, \mathcal{P}, z) = \Big(\frac{k!}{2\pi i}\Big)^2 \int_{(\sigma)}\int_{(\sigma)} \frac{R(s_1,s_2)}{s_1^{k+1}s_2^{k+1}}ds_1ds_2 \tag{3.19}$$

where

$$R(s_1, s_2) = (\log D)^{-2k} \sum_{\substack{l<L \\ l|P(\zeta)}} p^+(l)$$

$$+ \sum_{\substack{\nu\nu_1 < D \\ \nu\nu_1\nu_2|P(z,\zeta)}} \sum_{\nu_2 < D} \mu(\nu_1)\mu(\nu_2)\left(\frac{D}{\nu\nu_1}\right)^{s_1}\left(\frac{D}{\nu\nu_2}\right)^{s_2} R(\mathcal{N}, \nu l\nu_1\nu_2).$$

Choosing $\sigma = (\log D)^{-1}$, we see that any $R(s_1, s_2)$ can be written as a bilinear form

$$R(s_1, s_2) = \sum_{\substack{m<DL \\ mn|P(z)}} \sum_{n<D} \alpha(m)\beta(n)R(\mathcal{N}, mn) \tag{3.20}$$

with

$$\alpha(m) \ll \tau(m), \qquad \beta(n) \ll 1. \tag{3.21}$$

Setting

$$R_1 = \sup_{s_1, s_2} |R(s_1, s_2)|$$

we have by (3.18)

$$R(\mathcal{N}, \mathcal{P}, z) \ll R_1\sigma^{-2k} \ll L^c R_1. \tag{3.22}$$

Collecting together (3.6), (3.7), (3.9), (3.15), (3.22) and letting $\varepsilon \to 0$, we get our theorem.

4. Proof of Theorems 2 and 3. Since the proof of these theorems is very similar to the one given for the corresponding results in the book of Halberstam and Richert [8], we shall be brief, making only some remarks about the expression for the error terms, in order to get the bilinear form (2.9), (2.13).

For Theorem 2, we refer to Theorem 7.3 of [8]. The starting point is Buchstab's identity (2.14). On the right of (2.14), one takes z_1 small enough that a fundamental lemma of the type (3.11) holds, whilst an upper bound for the sum is given by means of Theorem 1. Then, as in [8], the main terms, collected together, produce the main term of (2.8).

Now, the resulting error term is

$$\sum_{\substack{m<\zeta L \\ mn|P(z_1)}} \sum_{n<D} \alpha(m)\beta(n)R(\mathcal{N}, mn) \tag{4.1}$$

$$+ \sum_{z_1 \leqslant p < z} \sum_{\substack{m<\zeta D/\sqrt{p} \\ mn|P(p)}} \sum_{n<D/\sqrt{p}} \alpha(m)\beta(n)R(\mathcal{N}, mn) = \sum\nolimits_1 + \sum\nolimits_2, \text{ say.}$$

Note that in writing $\sum_1$ we have already used the fact that the error term in the "fundamental lemma" has bilinear form. This is easily seen using (2.14) with z_1 and applying Theorem 1.

In $\sum_2$, clearly

$$\mu = pm < \zeta D\sqrt{p} < \zeta D\sqrt{z}.$$

Setting $\varrho(n)$ = greatest prime factor of n, we have

$$\sum\nolimits_2 = \sum_{\mu<\zeta D/\sqrt{z}} \sum_{\substack{n<D/\sqrt{\varrho(\mu)} \\ \mu/\sqrt{\varrho(\mu)}<\zeta D,\, \mu n|P(z) \\ z_1<\varrho(m)<z}} \alpha\left(\frac{\mu}{\varrho(\mu)}\right)\beta(n)R(\mathcal{N}, \mu n) \tag{4.2}$$

and from this the expression (2.9) for the error term is clear.

Finally, Theorem 3 is obtained by considering

$$W = \sum_{\substack{n\in\mathcal{N} \\ (n,P(X^{1/v}))=1}} \left\{1 - \lambda \sum_{\substack{X^{1/v}<p<X^{1/u} \\ p|n,\, p\in\mathcal{P}}} \left(1 - u\frac{\log p}{\log y}\right)\right\} \tag{4.3}$$

and observing that

$$W = S(\mathcal{N}, \mathcal{P}, z) - \lambda \sum_{z \leqslant p < y} \left(1 - \frac{\log p}{\log y}\right) S(\mathcal{A}_p, \mathcal{P}, z), \tag{4.4}$$

with $z = x^{1/v}$ and $y = x^{1/u}$. Here, one applies Theorem 2 for a lower bound for $S(\mathcal{N}, \mathcal{P}, z)$ and Theorem 1 for an upper bound for the sum on the right of (4.4). Then, one concludes as in Theorem 10.1 of [8] with arguments similar to the previous ones for the bilinear form (2.13) of the error.

References

[1] E. Bombieri, *Le grand crible dans la théorie analytique des nombres*, Astérisque 10 (1974).
[2] —, *The asymptotic sieve*, Rend. Accad. Naz. XL (5) 1/2 (1975–76), 243–269.
[3] E. Bombieri, J. B. Friedlander and H. Iwaniec, *Primes in arithmetic progressions to large moduli*, Acta Math. 156 (1986), 203–251.
[4] —, —, —, *Primes in arithmetic progressions to large moduli. II*, Math. Ann. 277 (1987), 361–393.
[5] —, —, —, *Primes in arithmetic progressions to large moduli. III*, J. Amer. Math. Soc. 2 (2) (1989), 215–224.
[6] P. Elliott and H. Halberstam, *A conjecture in prime number theory*, Symp. Math. 4 (INDAM Rome, 1968–69), 59–72.
[7] E. Fouvry and F. Grupp, *Weighted sieves and twin prime factors*, Duke Math J. 58 (3) (1989), 731–748.
[8] H. Halberstam and H. E. Richert, *Sieve Methods*, Academic Press, London 1974.
[9] H. Iwaniec, *Rosser's sieve*, Acta Arith. 36 (1980), 171–202.
[10] —, *A new form of the error term in the linear sieve*, ibid. 37 (1980), 307–320.
[11] —, *Almost primes represented by quadratic polynomials*, Invent. Math. 47 (1978), 171–188.
[12] —, *Sieving limits*, Sém. Delange–Pisot–Poitou 1979/80, Birkhäuser.

[13] H. Iwaniec, I. van de Lune and H. te Riele, *The limits of Buchstab's iteration sieve*, Report 2W 129/79, Mathematisch Centrum, Amsterdam 1979.
[14] S. Salerno, *A note on Selberg sieve*, Acta Arith. 45 (1986), 279–288.
[15] A. Selberg, *Sieve methods*, Proc. Sympos. Pure Math. 20 (1971), 311–351.

ISTITUTO DI FISICA, MATEMATICA ED INFORMATICA
UNIVERSITÀ DI SALERNO
84081 Fisciano (SA)
Italy

Received on 28.9.1989
and in revised form on 30.7.1990 (1973)

ACTA ARITHMETICA
LIX.1 (1991)

Recouvrement optimal du cercle par les multiples d'un intervalle

par

M. Deleglise (Villeurbanne)

Introduction. Une partie A de N est une *base asymptotique d'ordre* h si tout entier assez grand est une somme d'au plus h éléments de A et si h est le plus petit entier tel que cette propriété soit vérifiée.

Erdős et Graham [E-G] ont défini la notion d'ordre exact de la manière suivante: Une partie A de N est une *base asymptotique d'ordre exact* h si tout entier assez grand est une somme d'exactement h éléments de A, et si h est le plus petit entier vérifiant cette propriété.

Il existe des bases asymptotiques d'ordre h qui n'admettent pas d'ordre exact, par exemple l'ensemble des entiers impairs. Erdős et Graham donnent une condition nécessaire et suffisante très simple pour qu'une base asymptotique possède un ordre exact et étudient le problème de l'évaluation de l'ordre exact maximal d'une base d'ordre h. Plus précisément, ils définissent la fonction $g(h)$ qui est le maximum des ordres exacts des parties A qui sont des bases d'ordre h et qui admettent un ordre exact.

On sait actuellement que

$$h^2/3-2h \leqslant g(h) \leqslant h^2/2+3h/2 \quad \text{pour tout } h \geqslant 2. \tag{0.1}$$

La majoration est due à Nash [N] (cf. [G, Bx] pour une preuve simplifiée). La minoration est due à G. Grekos (voir [G, th] ou [G, Bx]). Celui-ci définit la fonction $L(h)$ égale à la longueur du plus petit intervalle fermé I de $T = R/Z$ tel que $I, 2I, \ldots, hI$ recouvrent T (l'intervalle kI étant défini par récurrence par l'égalité $kI = I+(k-1)I$, ou encore $kI = (k\alpha, k\beta)$ si $I = (\alpha, \beta)$), et il démontre que:

$$g(h) \geqslant 1/L(h) \quad \text{et} \quad L(h) \leqslant 3/h^2+18/h^3,$$

ce qui donne la minoration de (0.1).

Dans cet article nous démontrons les deux théorèmes suivants:

Théorème A. *Soit $I = [\alpha, \alpha+L]$ un intervalle fermé de longueur minimum tel que $I, 2I, \ldots, hI$ recouvrent T; alors, si $h \geqslant 3$:*

$$L = \begin{cases} 3/(h(h+2)) & si\ h \equiv 0\ ou\ 1 \pmod 3, \\ 3/(h(h+2)-2) & si\ h \equiv 2 \pmod 3. \end{cases}$$