A limit theorem for the Riemann zeta-function near the critical line in the complex space

by

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In honour of Professor J. Kubilius
on his 70th birthday

Let \( s = \sigma + it \) be a complex variable and let \( \zeta(s) \) denote the Riemann zeta-function. H. Bohr noted in [2] that asymptotically the behaviour of the \( \zeta \)-function in the half plane \( \sigma > 1/2 \) is governed by some probabilistic laws. This idea has been implemented in [3], [4], [7].

Let \( \text{meas}\{A\} \) be the Lebesgue measure of the set \( A \) and

\[
v_T(\ldots) = \frac{1}{T} \text{meas}\{t \in [0, T], \ldots\}
\]

where instead of dots we write the conditions which are satisfied by \( t \). Let \( C \) denote the complex space. About 1940 A. Selberg (unpublished) has shown the following theorem.

**Theorem A.** If a measurable set \( A \subseteq C \) has positive Jordan content then

\[
\lim_{T \to \infty} v_T \left( \frac{\ln \zeta(1/2 + it)}{\sqrt{\ln \ln t}} \in A \right) = \frac{1}{\pi} \int_A e^{-x^2 - y^2} dxdy.
\]

Note that Theorem A can be found in [8] where its proof is also sketched. It is easy to see that the sets in Theorem A constitute a convergence-determining class. Let \( \mathcal{B}(S) \) denote the class of Borel subsets of the space \( S \). Then it follows from Theorem A that the probability measure

\[
v_T \left( \frac{\ln \zeta(1/2 + it)}{2^{-1/2} \ln \ln T} \in A \right), \quad A \in \mathcal{B}(C),
\]

as \( T \to \infty \) converges weakly to the normal probability measure

\[
\frac{1}{2\pi} \int_A e^{-\frac{(x^2 + y^2)}{2}} dxdy.
\]

Here we use the norming factor \( \sqrt{2^{-1/2} \ln \ln T} \) to obtain the normal distribution with parameters 0 and 1.
Let \( P \) be a probability measure on \((C, \mathcal{B}(C))\). The function
\[
w(\tau, k) = \int_{C(0)} |z|^k e^{i\tau z} dP, \quad \tau \in \mathbb{R}, \quad k \in \mathbb{Z},
\]
is called the characteristic transform of the measure \( P \) [10].

The lognormal probability measure on \((C, \mathcal{B}(C))\) is defined by the characteristic transform
\[
w(\tau, k) = \exp \left\{ -\frac{\tau^2}{2} - \frac{k^2}{2} \right\}.
\]

The lognormal distribution function \( G(x) \) is defined by
\[
G(x) = \begin{cases} 
\Phi(\ln x), & x > 0, \\
0, & x \leq 0,
\end{cases}
\]
where \( \Phi(x) = \int_{-\infty}^{x} e^{-u^2/2} du \).

If \( 1/2 \leq \sigma < 1 \) and \( \zeta(s) \neq 0, a \in \mathbb{R} \), then \( \zeta(s) \) is understood as \( \exp \{ \ln \zeta(s) \} \) where \( \ln \zeta(s) \) is defined by continuous displacement from the point \( s = 2 \) along the path joining the points \( 2, 2 + it \) and \( \sigma + it \).

Since the function \( \eta: C \to C \) defined by the formula \( \eta(s) = e^s \) is continuous, we have from the weak convergence of the probability measure (1) that the probability measure
\[
v_T \left( \zeta^{1/2 - \frac{1}{2} \ln T} (1/2 + it) \in A \right), \quad A \in \mathcal{B}(C),
\]
converges weakly as \( T \to \infty \) to the lognormal probability measure.

The aim of our note is to extend the result of A. Selberg to the strip \( 1/2 \leq \sigma \leq 1 + 1/2 + 1/\ln T \).

**Theorem.** Let \( 1/2 \leq \sigma \leq 1/2 + 1/\ln T \). Then the probability measure
\[
v_T \left( \frac{\ln \zeta(s + it)}{\sqrt{2^{-1} \ln T}} \in A \right), \quad A \in \mathcal{B}(C),
\]
converges weakly as \( T \to \infty \) to the lognormal probability measure.

**Corollary 1.** Let \( 1/2 \leq \sigma \leq 1/2 + 1/\ln T \). Then the probability measure
\[
v_T \left( \frac{\ln \zeta(s + it)}{\sqrt{2^{-1} \ln T}} \in A \right), \quad A \in \mathcal{B}(C),
\]
converges weakly as \( T \to \infty \) to the probability measure
\[
\frac{1}{2\pi} \int_A \left\{ e^{-\frac{(x^2 + y^2)}{2}} \right\} dx dy.
\]

**Corollary 2.** Let \( 1/2 \leq \sigma \leq 1/2 + 1/\ln T \). Then the distribution function
\[
v_T \left( \left| \zeta(s + it) \right|^{1/2 - \frac{1}{2} \ln T} < x \right)
\]
converges pointwise as \( T \to \infty \) to the lognormal distribution function \( G(x) \).

**Corollary 3.** Let \( 1/2 \leq \sigma \leq 1/2 + 1/\ln T \). Then the distribution function
\[
v_T \left( \frac{\arg \zeta(s + it)}{\sqrt{2^{-1} \ln T}} \leq x \right)
\]
converges pointwise as \( T \to \infty \) to the normal distribution function \( \Phi(x) \).

Note that in [5] the limit theorems for \( \ln \zeta(s + it) \) and \( \ln \zeta(s + it) \) when \( \sigma = 1/2 \) or \( \sigma = 1/2 + 1/\ln T \) have been obtained but the strip \( 1/2 \leq \sigma \leq 1/2 + 1/\ln T \) was not considered.

For the proof of the theorem we shall use some properties of spaces of analytic functions. Let
\[
A_T = \left\{ s \in C, \frac{1}{2} \frac{1}{\ln T} < \sigma < 1 \right\}
\]
and let \( H(A_T) \) denote the space of analytic functions on \( A_T \) equipped with the topology of uniform convergence on compacta.

It is well known that there exists a sequence \( \{ K_T,n \} \) of compact subsets of \( A_T \) such that
\[
A_T = \bigcup_{n=1}^{\infty} K_T,n.
\]
Moreover, the sets \( K_T,n \) can be chosen to satisfy the following conditions:
(a) \( K_T,n \subset K_{T,n+1} \);
(b) \( K \subset A_T \) and \( K \) compact implies \( K \subset K_T,n \) for some \( n \).

For \( f, g \in H(A_T) \) let
\[
\varrho_T(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\varrho_{T,n}(f, g)}{1 + \varrho_{T,n}(f, g)}
\]
where
\[
\varrho_{T,n}(f, g) = \sup_{s \in K_T,n} |f(s) - g(s)|.
\]
Then \( \varrho_T \) is a metric on \( H(A_T) \) which induces the usual topology. Note that the theory of spaces of analytic functions is comprehensively presented in [5].

**Lemma 1.** Let \( K \) be a compact subset of \( A_T \). Then for every \( \varepsilon > 0 \) and \( \varepsilon_1 > 0 \)
\[
\lim_{T \to \infty} \sup_{s \in K} \{ \varrho_{T}(s + it) > \varepsilon \ln(T)^{5/4 + \varepsilon_1} \} = 0(T)
\]
as \( T \to \infty \).

**Proof.** In virtue of the Chebyshev inequality
\[
\frac{1}{\varepsilon \ln(T)^{5/4 + \varepsilon_1}} \int_0^{\infty} \sup_{s \in K} \varrho_{T}(s + it) dt.
\]
Let $L_T$ be a simple closed curve enclosing the set $K$. Then by Cauchy's theorem

$$
\zeta(s+it) = \frac{1}{2\pi i} \int_{L_T} \frac{\zeta(z+it)}{z-s} \, dz.
$$

Therefore

$$
\sup_{s \in K} |\zeta(s+it)| \leq \frac{1}{2\pi \delta_T} \int_{L_T} |\zeta(z+it)| \, |dz|
$$

where $\delta_T$ is the distance of $L_T$ from the set $K$. Thus for sufficiently large $T$

$$
\int_{0}^{T} \sup_{s \in K} |\zeta(s+it)| \, dt \leq \frac{1}{2\pi \delta_T} \int_{0}^{T} |dz| \int_{L_T} |\zeta(z+it)| \, dt
$$

$$
\leq \frac{1}{2\pi \delta_T} \int_{0}^{T} |dz| \int_{L_T} |\zeta(Re z+it)| \, dt
$$

$$
= \frac{B}{\delta_T} \int_{0}^{T} |dz| \int_{L_T} |\zeta(Re z+it)| \, dt.
$$

Here $B$ denotes a number (not always the same) which is bounded by a constant. Let $\sigma_{0,T} = \inf \{|Re z|, z \in L_T\}$. We can choose the contour $L_T$ to satisfy the condition $\sigma_{0,T} = 1/2 - \ln T$. Then $\delta_T \geq 1/\ln T$, and by (4)

$$
\int_{0}^{T} \sup_{s \in K} |\zeta(s+it)| \, dt = B |L_T|/\ln T \sup_{\sigma \geq \sigma_{0,T}} \int_{0}^{T} |\zeta(\sigma + it)| \, dt
$$

where $|L_T|$ is the length of $L_T$. Since in virtue of the functional equation for the Riemann zeta-function (see, for example [15])

$$
\zeta(1-\sigma+it) = B (|t|+1)^{\sigma-1/2} |\zeta(\sigma + it)|,
$$

in view of the estimate ([14], [6])

$$
\int_{0}^{T} |(1/2 + it)| \, dt = BT (\ln T)^{1/4}
$$

we deduce that

$$
\sup_{\sigma \geq \sigma_{0,T}} \int_{0}^{T} |\zeta(\sigma + it)| \, dt = BT (\ln T)^{1/4}.
$$

Hence in view of the estimate (5) and (3), since $|L_T| = B$, the proof of Lemma 1 is complete.

Let

$$
\sigma_T = \frac{1}{2} + \frac{1}{(\ln T)^{1/3}}.
$$

**Lemma 2.** For every $\varepsilon > 0$

$$
v_T(|\zeta(\sigma_T+it)|) \geq \varepsilon (\ln T)^{23/24} = o(1) \quad \text{as} \quad T \to \infty.
$$

**Proof.** By Cauchy's theorem

$$
\zeta'(\sigma_T+it) = \frac{1}{2\pi i} \int_{L_T} \frac{\zeta(z+it)}{(z-\sigma_T)^2} \, dz
$$

where $L$ is the circle of radius $(\ln T)^{-1/3}$ with centre at $\sigma_T$. Hence

$$
\zeta'(\sigma_T+it) = B (\ln T)^{2/3} \int_{L_T} |\zeta(\sigma + it)| \, |dz|.
$$

Therefore the estimate (6) gives us

$$
v_T(|\zeta'(\sigma_T+it)|) \geq \varepsilon (\ln T)^{23/24}
$$

where

$$
\begin{align*}
B &= \frac{B}{\varepsilon (\ln T)^{23/24}} \int_{0}^{T} |\zeta(Re z+it)| \, dt \\
&= \frac{B}{\varepsilon (\ln T)^{23/24}} \int_{0}^{T} |\zeta(Re z+it)| \, dt
\end{align*}
$$

This proves the lemma.

**Lemma 3.** Let $1/2 \leq \alpha \leq \sigma_T$. Then for every $\varepsilon > 0$, $\varepsilon > 0$ and for $k = 1, 2, 3$

$$
v_T(|\zeta(k\alpha + it)|) \geq \varepsilon (\ln T)^{5/4 + \varepsilon/2} = o(1) \quad \text{as} \quad T \to \infty.
$$

**Proof.** First we shall prove that

$$
I_{T,k} = \frac{1}{T} \text{meas} \left\{ t \in [0, T], \zeta \left( \frac{t}{T} \right) \left( \frac{k\alpha + it}{\ln T} \right) \right\} = o(1).
$$

In fact, applying the Chebyshev inequality and Lemma 1 we obtain

$$
I_{T,k} \leq \frac{1}{\varepsilon (\ln T)^{5/4 + \varepsilon/2}} \int_{0}^{T} |\zeta(\sigma + it)| \, dt
$$

$$
= \frac{1}{\varepsilon (\ln T)^{5/4 + \varepsilon/2}} \int_{0}^{T} |\zeta(\sigma + it)| \, dt
$$

$$
= \frac{1}{\varepsilon (\ln T)^{5/4 + \varepsilon/2}} \int_{0}^{T} |\zeta(\sigma + it)| \, dt
$$

Let for $A \in \mathcal{B}(H(\Delta_T))$

$$
Q_T(A) = \frac{1}{T} \text{meas} \left\{ t \in [0, T], \zeta \left( \frac{t}{T} \right) \left( \frac{\alpha + it}{\ln T} \right) \in A \right\}.
$$

Then from (8) we deduce that for every bounded continuous function $X$ on $H(\Delta_T)$

$$
\int_{\Omega(\Delta_T)} X(f) dQ_T = X(0) + o(1).
$$
In fact, let \( \delta > 0 \). Then
\[
\int_{H(\delta \tau)} X(f) dQ_T - X(0) = \int_{H(\delta \tau)} (X(f) - X(0)) dQ_T
\]
\[= \int_{\varepsilon < \delta} (X(f) - X(0)) dQ_T + \int_{\varepsilon > \delta} (X(f) - X(0)) dQ_T.
\]
From the properties of the space \( H(\delta \tau) \) and from the continuity of \( X \) it follows that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( T > 0 \)
\[
\int_{\varepsilon < \delta} (X(f) - X(0)) dQ_T < \varepsilon.
\]
Let us fix such a \( \delta \). Since \( X \) is bounded, from the estimate (8) we see that there exists \( T_0 \) such that for \( T \geq T_0 \)
\[
\int_{\varepsilon > \delta} (X(f) - X(0)) dQ_T < \varepsilon.
\]
From (10)–(12) we find that for \( T \geq T_0 \)
\[
\int_{H(\delta \tau)} X(f) dQ_T - X(0) < 2\varepsilon.
\]
This proves (9).

Since the differentiation operator \( D \) is continuous on \( H(\delta \tau) \) (this is a simple consequence of Cauchy's formula), the function \( X(D(f)) \) is continuous and bounded (see, for example [1], p. 29). Consequently, from (9) we find that
\[
\int_{H(\delta \tau)} X(D(f)) dQ_T = X(0) + o(1)
\]
and upon transformation of the integral (see [1], Appendix II, formula (1)) we obtain
\[
\int_{H(\delta \tau)} X(f) dQ_T \frac{D^{-1}}{D(f)} = X(0) + o(1).
\]
Since \( X(f) \) is any bounded continuous function, we find that for every \( \varepsilon > 0 \)
\[
\int_{\varepsilon > \delta} (X(f) - X(0)) dQ_T \frac{D^{-1}}{D(f)} = o(1),
\]
and thus (7) is valid for \( k = 1 \).

The cases \( k = 2, 3 \) can be proved similarly.

The estimate (7) implies the relation
\[
\nu_T \left( \frac{|\zeta''(\sigma + it)|}{(\ln T)^{5/4 + \epsilon_1}} \right) \geq \varepsilon = o(1).
\]
Hence
\[
\nu_T \left( \frac{|\zeta''(\sigma + it)|}{(\ln T)^{5/4 + \epsilon_1}} \right) \geq \varepsilon, \quad \nu_T \left( \frac{|\zeta''(\sigma + it)|}{(\ln T)^{5/4 + \epsilon_1}} \right) \leq \frac{1}{2} = o(1).
\]
It is easy to see that if
\[
\left| \frac{\zeta''(\sigma + it)}{(\ln T)^{5/4 + \epsilon_1}} \right| \geq \frac{1}{2},
\]
then
\[
\nu_T \left( \frac{|\zeta''(\sigma + it)|}{(\ln T)^{5/4 + \epsilon_1}} \right) \geq \frac{1}{3}.
\]
In virtue of (13)
\[
\nu_T \left( \frac{|\zeta''(\sigma + it)|}{(\ln T)^{5/4 + \epsilon_1}} \right) \geq \frac{1}{3} = o(1).
\]
Therefore
\[
\nu_T \left( \frac{|\zeta''(\sigma + it)|}{(\ln T)^{5/4 + \epsilon_1}} \right) \geq \frac{1}{2} = o(1).
\]
Hence and from (14) the assertion of the lemma follows.

Proof of the theorem. Let \( \sigma = (2^{-1} \ln \ln T)^{-1/2} \). For all \( A \in \mathcal{B}(C) \) we have
\[
\nu_T \left( \frac{\zeta''(\sigma + it) \zeta''(\sigma + it)}{(\ln T)^{5/4 + \epsilon_1}} \right) \in A
\]
\[
= \nu_T \left( \frac{\zeta''(1/2 + it) \zeta''(1/2 + it)}{(\ln T)^{5/4 + \epsilon_1}} \right) \in A + o(1)
\]
\[
= \nu_T \left( \frac{\zeta''(1/2 + it) \zeta''(1/2 + it)}{(\ln T)^{5/4 + \epsilon_1}} \right) \in A + o(1)
\]
\[
= \nu_T \left( \frac{\zeta''(1/2 + it) \zeta''(1/2 + it)}{(\ln T)^{5/4 + \epsilon_1}} \right) \in A + o(1)
\]
where
\[
\Phi_T(t) = \int \left( 1 - u \right) \zeta''(1/2 + it + u(\sigma - 1/2)) du.
\]
Similarly for all \( A \in \mathcal{B}(C) \)
\[
\nu_T \left( \frac{\zeta''(1/2 + it)}{(\ln T)^{5/4 + \epsilon_1}} \right) \in A
\]
\[
= \nu_T \left( \frac{\zeta''(1/2 + it) \zeta''(1/2 + it)}{(\ln T)^{5/4 + \epsilon_1}} \right) \in A
\]
where
\[
\Phi_T(t) = \int \left( 1 - u \right) \zeta''(1/2 + it + u(\sigma - 1/2)) du.
\]
Since $\sigma_T - 1/2 = (\ln T)^{-1/3}$, from Lemma 3 we deduce that
\[
v_T\left(\zeta^{(1/2 + it)} / (\ln T)^{3/2}\right) = v_T\left(\zeta^{(\sigma + it)} / (\ln T)^{3/2} + o(1)\right) + o(1).
\]
Thus taking into account Lemma 2 we find that for every $\epsilon > 0$
\[
v_T\left(\zeta^{(1/2 + it)} / (\ln T)^{3/2}\right) > \epsilon (\ln T)^{3/4} = o(1).
\]
From the properties of the probability measure (2) it follows that the distribution
\[
v_T\left(\left|\zeta^{(1/2 + it)}\right|^{1/\sqrt{2 - \ln \ln T}} < x\right)
\]
converges as $T \to \infty$ to the function $G(x)$. (See also [11]–[13].) Hence for every $\delta > 0$
\[
v_T\left(\left|\zeta^{(1/2 + it)}\right| < (\ln T)^{-\delta}\right)
\]
\[
= v_T\left(\left|\zeta^{(1/2 + it)}\right|^{1/\sqrt{2 - \ln \ln T}} < \exp\left\{-\delta(\ln \ln T)^{1/2}\right\}\right)
\]
\[
= G(\exp\left\{-\delta(\ln \ln T)^{1/2}\right\}) + o(1) = o(1).
\]
Now from the estimates (16) and (17) we obtain
\[
v_T\left(\left|\zeta^{(1/2 + it)}\right| > 1 / \ln \ln T\right) = o(1).
\]
Similarly Lemma 3 and the estimate (17) give us
\[
v_T\left(\left|\zeta^{(1/2 + it)}\right| > 1 / \ln \ln T\right) = o(1).
\]
Consequently from (15), (18) and (19) we find that
\[
v_T\left(\zeta^{(1/2 + it)} / (\ln T)^{3/2} \in A\right) = v_T\left(\zeta^{(1/2 + it)} / (\ln T)^{3/2} + o(1)\right) + o(1)
\]
\[
= v_T\left(\zeta^{(1/2 + it)} / (\ln T)^{3/2} + o(1)\right) + o(1).
\]
Since by the results of [9]
\[
\int_0^T \left|\zeta^{(1/2 + it)}\right|^{2s} dt < T + \int_0^T \left|\zeta^{(1/2 + it)}\right|^{2s} \ln \ln T dt = BT,
\]
we conclude that for every $E_T \to \infty$
\[
v_T\left(\left|\zeta^{(1/2 + it)}\right| > E_T\right) \leq \frac{1}{TE_T} \int_0^T \left|\zeta^{(1/2 + it)}\right|^{2s} dt = o(1).
\]
Hence and from (20) it follows that
\[
v_T\left(\zeta^{(\sigma + it)} / (\ln T)^{3/2} \in A\right) = v_T\left(\zeta^{(\sigma + it)} / (\ln T)^{3/2} + o(1)\right) + o(1).
\]
Thus the properties of probability measure (2) prove the theorem.

Corollary 1 is an obvious consequence of the theorem.

Corollaries 2 and 3 are consequences of Corollary 1.