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A limit theorem for the Riemann zeta-function near the critical line in the complex space

by

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In honour of Professor J. Kubilius on his 70th birthday

Let $s=\sigma+it$ be a complex variable and let $\zeta(s)$ denote the Riemann zeta-function. H. Bohr noted in [2] that asymptotically the behaviour of the ζ-function in the half plane $\sigma>1/2$ is governed by some probabilistic laws. This idea has been implemented in [3], [4], [7].

Let meas$\{A\}$ be the Lebesgue measure of the set A and

$$\nu_T(\ldots)=\frac{1}{T}\operatorname{meas}\{t\in[0,T],\ldots\}$$

where instead of dots we write the conditions which are satisfied by t. Let $\boldsymbol{C}$ denote the complex space. About 1940 A. Selberg (unpublished) has shown the following theorem.

THEOREM A. *If a measurable set $A\subseteq\boldsymbol{C}$ has positive Jordan content then*

$$\lim_{T\to\infty}\nu_T\left(\frac{\ln\zeta(1/2+it)}{\sqrt{\ln\ln t}}\in A\right)=\frac{1}{\pi}\iint_A e^{-x^2-y^2}\,dx\,dy.$$

Note that Theorem A can be found in [8] where its proof is also sketched. It is easy to see that the sets in Theorem A constitute a convergence-determining class. Let $\mathscr{B}(S)$ denote the class of Borel subsets of the space S. Then it follows from Theorem A that the probability measure

$$\nu_T\left(\frac{\ln\zeta(1/2+it)}{\sqrt{2^{-1}\ln\ln T}}\in A\right),\qquad A\in\mathscr{B}(\boldsymbol{C}), \tag{1}$$

as $T\to\infty$ converges weakly to the normal probability measure

$$\frac{1}{2\pi}\iint_A e^{-(x^2+y^2)/2}\,dx\,dy.$$

Here we use the norming factor $\sqrt{2^{-1}\ln\ln T}$ to obtain the normal distribution with parameters 0 and 1.

Let P be a probability measure on $(\boldsymbol{C}, \mathscr{B}(\boldsymbol{C}))$. The function

$$w(\tau, k) \stackrel{\text{def}}{=} \int_{\boldsymbol{C}\setminus\{0\}} |s|^{i\tau} e^{ik \arg s} dP, \qquad \tau \in \boldsymbol{R},\ k \in \boldsymbol{Z},$$

is called the *characteristic transform* of the measure P [10].

The *lognormal probability measure* on $(\boldsymbol{C}, \mathscr{B}(\boldsymbol{C}))$ is defined by the characteristic transform

$$w(\tau, k) = \exp\left\{-\frac{\tau^2}{2} - \frac{k^2}{2}\right\}.$$

The *lognormal distribution function* $G(x)$ is defined by

$$G(x) = \begin{cases} \Phi(\ln x), & x > 0, \\ 0, & x \leqslant 0, \end{cases} \qquad \Phi(x) = \int_{-\infty}^{x} e^{-u^2/2} du.$$

If $1/2 \leqslant \sigma < 1$ and $\zeta(s) \neq 0$, $a \in \boldsymbol{R}$, then $\zeta^a(s)$ is understood as $\exp\{a \ln \zeta(s)\}$ where $\ln \zeta(s)$ is defined by continuous displacement from the point $s = 2$ along the path joining the points 2, $2+it$ and $\sigma+it$.

Since the function $h\colon \boldsymbol{C} \to \boldsymbol{C}$ defined by the formula $h(s) = e^s$ is continuous, we have from the weak convergence of the probability measure (1) that the probability measure

$$\nu_T\left(\zeta^{1/\sqrt{2^{-1}\ln\ln T}}(1/2+it) \in A\right), \qquad A \in \mathscr{B}(\boldsymbol{C}), \tag{2}$$

converges weakly as $T \to \infty$ to the lognormal probability measure.

The aim of our note is to extend the result of A. Selberg to the strip $1/2 \leqslant \sigma \leqslant 1/2 + 1/\ln T$.

THEOREM. *Let $1/2 \leqslant \sigma \leqslant 1/2 + 1/\ln T$. Then the probability measure*

$$\nu_T\left(\zeta^{1/\sqrt{2^{-1}\ln\ln T}}(\sigma+it) \in A\right), \qquad A \in \mathscr{B}(\boldsymbol{C}),$$

converges weakly as $T \to \infty$ to the lognormal probability measure.

COROLLARY 1. *Let $1/2 \leqslant \sigma \leqslant 1/2 + 1/\ln T$. Then the probability measure*

$$\nu_T\left(\frac{\ln \zeta(\sigma+it)}{\sqrt{2^{-1}\ln\ln T}} \in A\right), \qquad A \in \mathscr{B}(\boldsymbol{C}),$$

converges weakly as $T \to \infty$ to the probability measure

$$\frac{1}{2\pi} \iint_A e^{-(x^2+y^2)/2} dx\, dy.$$

COROLLARY 2. *Let $1/2 \leqslant \sigma \leqslant 1/2 + 1/\ln T$. Then the distribution function*

$$\nu_T\left(|\zeta(\sigma+it)|^{1/\sqrt{2^{-1}\ln\ln T}} < x\right)$$

converges pointwise as $T \to \infty$ to the lognormal distribution function $G(x)$.

COROLLARY 3. *Let $1/2 \leqslant \sigma \leqslant 1/2 + 1/\ln T$. Then the distribution function*

$$\nu_T\left(\frac{\arg \zeta(\sigma+it)}{\sqrt{2^{-1}\ln\ln T}} < x\right)$$

converges pointwise as $T \to \infty$ to the normal distribution function $\Phi(x)$.

Note that in [8] the limit theorems for $|\ln \zeta(\sigma+it)|$ and $|\ln|\zeta(\sigma+it)||$ when $\sigma = 1/2$ or $o(1) = \sigma - 1/2 \geqslant 1/\ln T$ have been obtained but the strip $1/2 \leqslant \sigma \leqslant 1/2 + 1/\ln T$ was not considered.

For the proof of the theorem we shall use some properties of spaces of analytic functions. Let

$$\Delta_T = \left\{s \in \boldsymbol{C},\ \frac{1}{2} - \frac{1}{\ln T} < \sigma < 1\right\}$$

and let $H(\Delta_T)$ denote the space of analytic functions on Δ_T equipped with the topology of uniform convergence on compacta.

It is well known that there exists a sequence $\{K_{T,n}\}$ of compact subsets of Δ_T such that

$$\Delta_T = \bigcup_{n=1}^{\infty} K_{T,n}.$$

Moreover, the sets $K_{T,n}$ can be chosen to satisfy the following conditions:

(a) $K_{T,n} \subset K_{T,n+1}$;

(b) $K \subset \Delta_T$ and K compact implies $K \subset K_{T,n}$ for some n.

For $f, g \in H(\Delta_T)$ let

$$\varrho_T(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\varrho_{T,n}(f, g)}{1 + \varrho_{T,n}(f, g)}$$

where

$$\varrho_{T,n}(f, g) = \sup_{s \in K_{T,n}} |f(s) - g(s)|.$$

Then ϱ_T is a metric on $H(\Delta_T)$ which induces the usual topology. Note that the theory of spaces of analytic functions is comprehensively presented in [5].

LEMMA 1. *Let K be a compact subset of Δ_T. Then for every $\varepsilon > 0$ and $\varepsilon_1 > 0$*

$$\operatorname{meas}\left\{\tau \in [0, T],\ \sup_{s \in K} |\zeta(s+i\tau)| \geqslant \varepsilon (\ln T)^{5/4+\varepsilon_1}\right\} = o(T)$$

as $T \to \infty$.

Proof. In virtue of the Chebyshev inequality

$$\operatorname{meas}\left\{\tau \in [0, T],\ \sup_{s \in K} |\zeta(s+i\tau)| \geqslant \varepsilon (\ln T)^{5/4+\varepsilon_1}\right\} \leqslant \frac{1}{\varepsilon (\ln T)^{5/4+\varepsilon_1}} \int_0^T \sup_{s \in K} |\zeta(s+i\tau)| d\tau. \tag{3}$$

Let L_T be a simple closed curve enclosing the set K. Then by Cauchy's theorem

$$\zeta(s+i\tau) = \frac{1}{2\pi i}\int_{L_T} \frac{\zeta(z+i\tau)}{z-s}dz.$$

Therefore

$$\sup_{s\in K}|\zeta(s+i\tau)| \leqslant \frac{1}{2\pi\delta_T}\int_{L_T}|\zeta(z+i\tau)|\,|dz|$$

where δ_T is the distance of L_T from the set K. Thus for sufficiently large T

$$(4)\qquad \int_0^T \sup_{s\in K}|\zeta(s+i\tau)|d\tau \leqslant \frac{1}{2\pi\delta_T}\int_{L_T}|dz|\int_0^T|\zeta(z+i\tau)|d\tau$$
$$\leqslant \frac{1}{2\pi\delta_T}\int_{L_T}|dz|\int_{-|\operatorname{Im} z|}^{T+|\operatorname{Im} z|}|\zeta(\operatorname{Re} z+i\tau)|d\tau$$
$$= \frac{B}{\delta_T}\int_{L_T}|dz|\int_0^{2T}|\zeta(\operatorname{Re} z+i\tau)|d\tau.$$

Here B denotes a number (not always the same) which is bounded by a constant. Let $\sigma_{0,T} = \inf\{\operatorname{Re} z,\ z\in L_T\}$. We can choose the contour L_T to satisfy the condition $\sigma_{0,T} = 1/2-2/\ln T$. Then $\delta_T \geqslant 1/\ln T$, and by (4)

$$(5)\qquad \int_0^T \sup_{s\in K}|\zeta(s+i\tau)|d\tau = B|L_T|\ln T \sup_{\sigma\geqslant\sigma_{0,T}}\int_0^T|\zeta(\sigma+it)|dt$$

where $|L_T|$ is the length of L_T. Since in virtue of the functional equation for the Riemann zeta-function (see, for example [15])

$$\zeta(1-\sigma+it) = B(|t|+1)^{\sigma-1/2}|\zeta(\sigma+it)|,$$

in view of the estimate ([14], [6])

$$(6)\qquad \int_0^T|\zeta(1/2+it)|dt = BT(\ln T)^{1/4}$$

we deduce that

$$\sup_{\sigma\geqslant\sigma_{0,T}}\int_0^{2T}|\zeta(\sigma+it)|dt = BT(\ln T)^{1/4}.$$

Hence in view of the estimate (5) and (3), since $|L_T| = B$, the proof of Lemma 1 is complete.

Let

$$\sigma_T = \frac{1}{2}+\frac{1}{(\ln T)^{1/3}}.$$

LEMMA 2. *For every* $\varepsilon > 0$

$$\nu_T\left(|\zeta'(\sigma_T+it)| \geqslant \varepsilon(\ln T)^{23/24}\right) = o(1) \quad \text{as } T\to\infty.$$

Proof. By Cauchy's theorem

$$\zeta'(\sigma_T+it) = \frac{1}{2\pi i}\int_L \frac{\zeta(z+it)}{(z-\sigma_T)^2}dz$$

where L is the circle of radius $(\ln T)^{-1/3}$ with centre at $s=\sigma_T$. Hence

$$\zeta'(\sigma_T+it) = B(\ln T)^{2/3}\int_L|\zeta(z+it)|\,|dz|.$$

Therefore the estimate (6) gives us

$$\nu_T\left(|\zeta'(\sigma_T+it)| \geqslant \varepsilon(\ln T)^{23/24}\right)$$
$$= \frac{B}{\varepsilon T(\ln T)^{7/24}}\int_L|dz|\int_0^{2T}|\zeta(\operatorname{Re} z+it)|dt$$
$$= \frac{B}{\varepsilon T(\ln T)^{7/24}}\sup_{\sigma\geqslant 1/2}\int_0^{2T}|\zeta(\sigma+it)|dt = \frac{B}{\varepsilon(\ln T)^{1/24}}.$$

This proves the lemma.

LEMMA 3. *Let* $1/2 \leqslant \sigma \leqslant \sigma_T$. *Then for every* $\varepsilon > 0$, $\varepsilon_1 > 0$ *and for* $k = 1, 2, 3$

$$\nu_T\left(|\zeta^{(k)}(\sigma+it)| \geqslant \varepsilon(\ln T)^{5/4+\varepsilon_1}\right) = o(1) \quad \text{as} \quad T\to\infty.$$

Proof. First we shall prove that

$$(7)\qquad I_{T,k} \overset{\text{def}}{=} \frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T],\ \varrho_T\left(\frac{\zeta^{(k)}(s+i\tau)}{(\ln T)^{5/4+\varepsilon_1}}, 0\right)\geqslant\varepsilon\right\} = o(1).$$

In fact, applying the Chebyshev inequality and Lemma 1 we obtain

$$(8)\quad I_{T,0} \leqslant \frac{1}{\varepsilon T}\int_0^T\sum_{n=1}^{\infty}2^{-n}\frac{\sup_{s\in K_{T,n}}|\zeta(s+i\tau)|}{(\ln T)^{5/4+\varepsilon_1}+\sup_{s\in K_{T,n}}|\zeta(s+i\tau)|}d\tau$$
$$= \frac{1}{\varepsilon T}\sum_{n=1}^{\infty}2^{-n}\left(\int_{\substack{0\leqslant\tau\leqslant T\\ \sup_{s\in K_{T,n}}|\zeta(s+i\tau)|<(\ln T)^{5/4+\varepsilon_1/2}}}\frac{\sup_{s\in K_{T,n}}|\zeta(s+i\tau)|}{(\ln T)^{5/4+\varepsilon_1}+\sup_{s\in K_{T,n}}|\zeta(s+i\tau)|}d\tau\right.$$
$$\left.+\int_{\substack{0\leqslant\tau\leqslant T\\ \sup_{s\in K_{T,n}}|\zeta(s+i\tau)|\geqslant(\ln T)^{5/4+\varepsilon_1/2}}}\frac{\sup_{s\in K_{T,n}}|\zeta(s+i\tau)|}{(\ln T)^{5/4+\varepsilon_1}+\sup_{s\in K_{T,n}}|\zeta(s+i\tau)|}\right)d\tau = o(1).$$

Let for $A\in\mathscr{B}(H(\Delta_T))$

$$Q_T(A) = \frac{1}{T}\operatorname{meas}\left\{t\in[0,T],\ \frac{\zeta(s+i\tau)}{(\ln T)^{5/4+\varepsilon_1}}\in A\right\}.$$

Then from (8) we deduce that for every bounded continuous function X on $H(\Delta_T)$

$$(9)\qquad \int_{H(\Delta_T)} X(f)dQ_T = X(0)+o(1).$$

In fact, let $\delta>0$. Then

$$(10)\qquad \int_{H(\Delta_T)} X(f)dQ_T-X(0)=\int_{H(\Delta_T)}(X(f)-X(0))dQ_T$$
$$=\int_{\varrho_T(f,0)<\delta}(X(f)-X(0))dQ_T+\int_{\varrho_T(f,0)\geqslant\delta}(X(f)-X(0))dQ_T.$$

From the properties of the space $H(\Delta_T)$ and from the continuity of X it follows that for every $\varepsilon>0$ there exists $\delta>0$ such that for all $T>0$

$$(11)\qquad \Big|\int_{\varrho_T(f,0)<\delta}(X(f)-X(0))dQ_T\Big|<\varepsilon.$$

Let us fix such a δ. Since X is bounded, from the estimate (8) we see that there exists T_0 such that for $T\geqslant T_0$

$$(12)\qquad \Big|\int_{\varrho_T(f,0)\geqslant\delta}(X(f)-X(0))dQ_T\Big|<\varepsilon.$$

From (10)–(12) we find that for $T\geqslant T_0$

$$\Big|\int_{H(\Delta_T)} X(f)dQ_T-X(0)\Big|<2\varepsilon.$$

This proves (9).

Since the differentiation operator D is continuous on $H(\Delta_T)$ (this is a simple consequence of Cauchy's formula), the function $X(D(f))$ is continuous and bounded (see, for example [1], p. 29). Consequently, from (9) we find that as $T\to\infty$

$$\int_{H(\Delta_T)} X(D(f))dQ_T=X(0)+o(1)$$

and upon transformation of the integral (see [1], Appendix II, formula (1)) we obtain

$$\int_{H(\Delta_T)} X(f)dQ_TD^{-1}=X(0)+o(1).$$

Since $X(f)$ is any bounded continuous function, we find that for every $\varepsilon>0$

$$\int_{\varrho_T(f,0)\geqslant\varepsilon}(X(f)-X(0))dQ_TD^{-1}=o(1),$$

and thus (7) is valid for $k=1$.

The cases $k=2,\ 3$ can be proved similarly.

The estimate (7) implies the relation

$$(13)\qquad \nu_T\left(\frac{|\zeta^{(k)}(\sigma+it)|}{(\ln T)^{5/4+\varepsilon_1}+|\zeta^{(k)}(\sigma+it)|}\geqslant\varepsilon\right)=o(1).$$

Hence

$$(14)\qquad \nu_T\left(\frac{|\zeta^{(k)}(\sigma+it)|}{(\ln T)^{5/4+\varepsilon_1}}\geqslant\varepsilon,\ \frac{|\zeta^{(k)}(\sigma+it)|}{(\ln T)^{5/4+\varepsilon_1}}<\frac{1}{2}\right)=o(1).$$

It is easy to see that if

$$\frac{|\zeta^{(k)}(\sigma+it)|}{(\ln T)^{5/4+\varepsilon_1}}\geqslant\frac{1}{2},$$

then

$$\frac{|\zeta^{(k)}(\sigma+it)|}{(\ln T)^{5/4+\varepsilon_1}+|\zeta^{(k)}(\sigma+it)|}\geqslant\frac{1}{3}.$$

In virtue of (13)

$$\nu_T\left(\frac{|\zeta^{(k)}(\sigma+it)|}{(\ln T)^{5/4+\varepsilon_1}+|\zeta^{(k)}(\sigma+it)|}\geqslant\frac{1}{3}\right)=o(1).$$

Therefore

$$\nu_T\left(\frac{|\zeta^{(k)}(\sigma+it)|}{(\ln T)^{5/4+\varepsilon_1}}\geqslant\frac{1}{2}\right)=o(1).$$

Hence and from (14) the assertion of the lemma follows.

Proof of the theorem. Let $\varkappa=(2^{-1}\ln\ln T)^{-1/2}$. For all $A\in\mathscr{B}(C)$ we have

$$(15)\qquad \nu_T(\zeta^\varkappa(\sigma+it)\in A)$$
$$=\nu_T\left(\zeta^\varkappa(1/2+it)\frac{\zeta^\varkappa(\sigma+it)}{\zeta^\varkappa(1/2+it)}\in A\right)+o(1)$$
$$=\nu_T\left(\zeta^\varkappa(1/2+it)\left(1+\frac{\zeta(\sigma+it)-\zeta(1/2+it)}{\zeta(1/2+it)}\right)^\varkappa\in A\right)+o(1)$$
$$=\nu_T\left(\zeta^\varkappa(1/2+it)\left(1+\frac{(\sigma-1/2)\zeta'(1/2+it)+(\sigma-1/2)^2\Phi_T(t)}{\zeta(1/2+it)}\right)^\varkappa\in A\right)+o(1)$$

where

$$\Phi_T(t)=\int_0^1(1-u)\zeta''(1/2+it+u(\sigma-1/2))du.$$

Similarly for all $A\in\mathscr{B}(C)$

$$\nu_T\left(\frac{\zeta'(1/2+it)}{(\ln T)^{23/24}}\in A\right)$$
$$=\nu_T\left(\frac{\zeta'(\sigma_T+it)}{(\ln T)^{23/24}}-\frac{(\sigma_T-1/2)\zeta''(1/2+it)}{(\ln T)^{23/24}}-\frac{(\sigma_T-1/2)^2\Phi_{1T}(t)}{(\ln T)^{23/24}}\in A\right)$$

where

$$\Phi_{1T}(t)=\int_0^1(1-u)\zeta'''(1/2+it+u(\sigma_T-1/2))du.$$

Since $\sigma_T - 1/2 = (\ln T)^{-1/3}$, from Lemma 3 we deduce that

$$\nu_T\left(\frac{\zeta'(1/2+it)}{(\ln T)^{23/24}} \in A\right) = \nu_T\left(\frac{\zeta'(\sigma_T+it)}{(\ln T)^{23/24}} + o(1) \in A\right) + o(1).$$

Thus taking into account Lemma 2 we find that for every $\varepsilon > 0$

$$\nu_T\left(|\zeta'(1/2+it)| \geqslant \varepsilon(\ln T)^{23/24}\right) = o(1). \tag{16}$$

From the properties of the probability measure (2) it follows that the distribution function

$$\nu_T\left(|\zeta(1/2+it)|^{1/\sqrt{2^{-1}\ln\ln T}} < x\right)$$

converges as $T \to \infty$ to the function $G(x)$. (See also [11]–[13].) Hence for every $\delta > 0$

$$\begin{aligned} (17)\quad \nu_T\left(|\zeta(1/2+it)| < (\ln T)^{-\delta}\right) &= \nu_T\left(|\zeta(1/2+it)|^{1/\sqrt{2^{-1}\ln\ln T}} < \exp\{-\delta(2\ln\ln T)^{1/2}\}\right) \\ &= G\left(\exp\{-\delta(2\ln\ln T)^{1/2}\}\right) + o(1) = o(1). \end{aligned}$$

Now from the estimates (16) and (17) we obtain

$$\nu_T\left(\frac{(\sigma-1/2)|\zeta'(1/2+it)|}{|\zeta(1/2+it)|} \geqslant \frac{1}{\ln\ln T}\right) = o(1). \tag{18}$$

Similarly Lemma 3 and the estimate (17) give us

$$\nu_T\left(\frac{(\sigma-1/2)^2\,|\Phi_T(t)|}{|\zeta(1/2+it)|} \geqslant \frac{1}{\ln\ln T}\right) = o(1). \tag{19}$$

Consequently from (15), (18) and (19) we find that

$$\begin{aligned} (20)\quad \nu_T\left(\zeta^{\varkappa}(\sigma+it) \in A\right) &= \nu_T\left(\zeta^{\varkappa}(1/2+it)\left(1+o(1)\right)^{\varkappa} \in A\right) + o(1) \\ &= \nu_T\left(\zeta^{\varkappa}(1/2+it)\left(1+o(1)\right) \in A\right) + o(1). \end{aligned}$$

Since by the results of [9]

$$\int_0^T |\zeta(1/2+it)|^{2\varkappa}\,dt \leqslant T + \int_0^T |\zeta(1/2+it)|^{2/[\sqrt{2^{-1}\ln\ln T}]}\,dt = BT,$$

we conclude that for every $E_T \underset{T\to\infty}{\to} \infty$

$$\nu_T\left(|\zeta(1/2+it)|^{\varkappa} > E_T\right) \leqslant \frac{1}{TE_T^2}\int_0^T |\zeta(1/2+it)|^{2\varkappa}\,dt = o(1).$$

Hence and from (20) it follows that

$$\nu_T\left(\zeta^{\varkappa}(\sigma+it) \in A\right) = \nu_T\left(\zeta^{\varkappa}(1/2+it) + o(1) \in A\right) + o(1).$$

Thus the properties of probability measure (2) prove the theorem.

Corollary 1 is an obvious consequence of the theorem.

Corollaries 2 and 3 are consequences of Corollary 1.

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