ACTA ARITHMETICA
LVIII.4 (1991)

Chen's theorem in totally real algebraic number fields

by

JÜRGEN G. HINZ (Marburg)

1. Introduction. One of the oldest problems in the theory of numbers is the so-called binary Goldbach conjecture: Every sufficiently large even integer can be expressed as the sum of two primes. In 1948 A. Rényi proved that there exists a constant $m_0 \in N$ such that $2N = P_1 + P_{m_0}$ for all sufficiently large N. Here, P_m denotes an integer having at most m prime factors, equal or distinct. For $m = 1$, P_1 is simply a prime. Subsequent researches were aimed at the reduction of m_0 in Rényi's result. The best approximation up-to-date is due to Chen and enables us to take $m_0 = 2$. The proof combines Selberg's weighted sieve, Bombieri's theorem on the distribution of primes in arithmetic progressions, the large sieve and also some analytical methods. A very clear exposition of Chen's proof was given by H. Halberstam and H.-E. Richert in [1].

Several mathematicians have established versions of the sieve procedure for use in algebraic number fields. Specially, these investigations lead to approximations to Goldbach's conjecture in the language of a number field. We recall that an algebraic integer $\alpha \neq 0$ is said to be even if all prime ideals $\mathfrak{p}$ with $N\mathfrak{p} = 2$ divide α, and is said to be prime if the principal ideal (α) is a prime ideal. Let us write Π_m for a totally positive algebraic integer which has at most m prime ideal divisors, counted according to multiplicity. The first result on the Goldbach problem in number fields was obtained by H. Rademacher [9]. He was able to show that every totally positive even algebraic integer ξ with sufficiently large norm can be represented in the form $\xi = \Pi_7 + \Pi_7$. In this terminology, Rademacher's approach has been improved to $\xi = \Pi_2 + \Pi_3$ by A. I. Vinogradov [14] (see also [4] for a simpler proof) and to $\xi = \Pi_1 + \Pi_{m_0}$ with a fixed m_0 by the author [7].

In the special case of a totally real algebraic number field K of degree n over the rationals further improvements are possible. An application of the author's version of Bombieri's prime number theorem in K (see [7]) yields $\xi = \Pi_1 + \Pi_3$. The primary concern of the present paper is to generalize to K in an appropriate way the result given by Chen, namely to prove that $\xi = \Pi_1 + \Pi_2$.

In the sequel, let Z_K denote the ring of integers in K. For a given totally positive even number $\xi \in Z_K$ we consider the sets

(1) $$\mathfrak{R} = \mathfrak{R}(\xi^{(1)}, \ldots, \xi^{(n)}) = \{\alpha \in Z_K;\ 0 < \alpha^{(k)} < \xi^{(k)},\ k = 1, \ldots, n\}$$

and

(2) $$\mathfrak{P} = \mathfrak{P}(\xi^{(1)}, \ldots, \xi^{(n)}) = \{\omega \in \mathfrak{R};\ \omega \text{ prime}\}.$$

It is convenient to assume for the computations that

(3) $$c_1(N\xi)^{1/n} \leqslant \xi^{(k)} \leqslant c_2(N\xi)^{1/n}, \quad k = 1, \ldots, n,$$

where the positive constants c_1 and c_2 depend on the field K only. If the ξ do not obey the inequalities (3) one can restore (3) by multiplying the ξ by a suitable chosen totally positive unit of K.

Our object will be to derive an explicit lower bound for the number of primes $\omega \in \mathfrak{P}$ such that $(\xi - \omega)$ consists of at most two prime ideal factors.

THEOREM. *There exists a positive constant $N_0 = N_0(K)$ such that, if ξ is even and $N\xi > N_0$, then*

(4) $$|\{\omega \in \mathfrak{P};\ \xi - \omega = \Pi_2\}| > 0.34\sqrt{d}(2^{n-1}hR)^{-2} \prod_{N\mathfrak{p}=2} N\mathfrak{p} \prod_{N\mathfrak{p}>2}\left(1 - \frac{1}{(N\mathfrak{p}-1)^2}\right) \prod_{\substack{N\mathfrak{p}>2\\ \mathfrak{p}|\xi}} \frac{N\mathfrak{p}-1}{N\mathfrak{p}-2} \frac{N\xi}{\log^2 N\xi}.$$

Here, d denotes the discriminant, h the class number and R the regulator of the field K.

From a quantitative point of view, this result should be compared with the author's upper estimate (see [6], Theorem 1.2)

$$|\{\omega \in \mathfrak{P};\ \xi - \omega = \Pi_1\}| \leqslant 8\sqrt{d}(2^{n-1}hR)^{-2} \prod_{N\mathfrak{p}=2} N\mathfrak{p} \prod_{N\mathfrak{p}>2}\left(1 - \frac{1}{(N\mathfrak{p}-1)^2}\right) \prod_{\substack{N\mathfrak{p}>2\\ \mathfrak{p}|\xi}} \frac{N\mathfrak{p}-1}{N\mathfrak{p}-2} \frac{N\xi}{\log^2 N\xi} \times\left\{1 + O\left(\frac{\log\log N\xi}{\log N\xi}\right)\right\}.$$

We continue by making a few remarks about the proof of (4). The method is influenced by the elaboration of Chen's corresponding result in the rational case, given by Halberstam and Richert in [1]. It is possible to model some of their main arguments appropriate to our situation.

First Selberg's weighted sieve method and Bombieri's prime number theorem are essential ingredients in Chen's work. The author ([3], [7]) has succeeded in extending these concepts to K.

Chen's basic idea may be outlined very vaguely as follows: At first one sifts the sequences $\{2N - p;\ p < 2N\}$ (resp. $\{2N - p;\ p < 2N,\ 2N - p \equiv 0 \bmod p_1\}$) in a conventional manner by Selberg's weighted linear sieve method, getting good lower (resp. upper) bounds for the cardinality of those elements which survive the sifting process. Then one removes in a suitable way the contribution of unwanted representations of $2N$ in the form $2N = p + p_1p_2p_3$. To this end Chen introduces the novel procedure of sifting the sequence

(5) $$\{2N - (p_1p_2)p;\ p < 2N/(p_1p_2)\}$$

instead of the different set $\{2N - p;\ p < 2N,\ 2N - p \equiv 0 \bmod p_1p_2\}$.

It turns out that one of the main difficulties in K arises in connection with the transformation of

$$\{\xi - \omega;\ \omega \in \mathfrak{P},\ \xi - \omega \equiv 0 \bmod \mathfrak{p}_1\mathfrak{p}_2\}$$

in a sequence which is analogous to the one given in (5). Let us only mention that the product $\mathfrak{p}_1\mathfrak{p}_2$ need not be principal. Consequently, we have to carry out a reduction from ideals to algebraic integers; this amounts to using the finiteness of the class number h of K. These steps have no parallel in Chen's work.

Another complication in number fields comes from the fact that one needs some information about asymptotic estimates for sums of type

$$\sum_{\substack{\alpha \in \mathfrak{R}\\ \alpha \equiv 0 \bmod \mathfrak{a}}} \chi(\alpha)\Lambda\left(\frac{\alpha}{\mathfrak{a}}\right), \quad \chi \bmod \mathfrak{q}.$$

Here, the ideal $\mathfrak{a}$ must not be principal. This problem does not occur in the rational case, where one can simply substitute and appeal to the prime number theorem. Using the Siegel summation formula we succeed in obtaining suitable estimates.

Finally, our method represents even in the rational case a substantial simplification of the analytic part of known proofs of Chen's theorem. In connection with the application of the large sieve we avoid the technique of contour integration. The argument presented here utilizes ideas introduced already by the author in [7]. In our approach the basic inequality of the large sieve method is used in an extended form which gives a proof of Chen's result by essentially elementary means. This is the most novel feature of the paper.

The positive constants $c_3, \ldots, c_{25}$ coming up in the sequel and all the constants implied by the $\ll$ and O symbols depend at most on the field K; where other parameters occur and dependence on them is to be kept explicit, we use e.g. $\ll_t$ or O_t to indicate that the constants implied here depend also on t.

Throughout the paper, small German letters stand for integral ideals of K, particularly $\mathfrak{p}, \mathfrak{p}_1, \ldots$ always denote prime ideals. The letter ω will be used for primes in K only.

2. Fundamental lemmas. We begin by deriving a weak form of the author's version of Bombieri's prime number theorem in K which is most appropriate for our sieve applications.

Let $\mathfrak{q}$ be an integral ideal of K. We put, for sufficiently large $N\xi$,

$$I = I(\xi^{(1)}, \ldots, \xi^{(n)}) = (2^{n-1}hR)^{-1} \int_2^{\xi^{(1)}} \ldots \int_2^{\xi^{(n)}} \frac{du_1 \ldots du_n}{\log(u_1 \ldots u_n)}.$$

An evaluation of I can be obtained from [2], Lemma 6, in the form

$$I = (2^{n-1}hR)^{-1} \frac{N\xi}{\log N\xi} + O\left(\frac{N\xi}{\log^2 N\xi}\right). \tag{6}$$

LEMMA 1. *Let*

$$R(\xi; \mathfrak{q}) = \sum_{\substack{\omega \in \mathfrak{P} \\ \omega \equiv \xi \bmod \mathfrak{q}}} 1 - \frac{I \cdot L(\mathfrak{q})}{\Phi(\mathfrak{q})},$$

where Φ denotes Euler's function in K, and where L is defined by

$$L(\mathfrak{q}) = \begin{cases} 0 & \text{if } (\xi, \mathfrak{q}) \neq 1, \\ 1 & \text{if } (\xi, \mathfrak{q}) = 1. \end{cases} \tag{7}$$

Then there exists a positive constant $A_0 = A_0(n)$ such that, for $N\xi \geq N_0(K)$,

$$\sum_{N\mathfrak{q} \leq N\xi^{1/2}(\log N\xi)^{-A_0}} \mu^2(\mathfrak{q}) 3^{v(\mathfrak{q})} |R(\xi; \mathfrak{q})| \ll N\xi (\log N\xi)^{-3}.$$

Here, $v(\mathfrak{q})$ denotes the number of distinct prime ideal divisors of $\mathfrak{q}$.

Proof. First, we note that, for $A > 0$,

$$\sum_{\substack{N\mathfrak{q} \leq N\xi^{1/2}(\log N\xi)^{-A} \\ (\mathfrak{q},\xi) \neq 1}} \mu^2(\mathfrak{q}) 3^{v(\mathfrak{q})} |R(\xi; \mathfrak{q})| \ll \sum_{N\mathfrak{q} \leq N\xi^{1/2}(\log N\xi)^{-A}} \mu^2(\mathfrak{q}) 3^{v(\mathfrak{q})} \sum_{\substack{\omega \in \mathfrak{P} \\ \omega | \mathfrak{q}}} 1.$$

It is easy to check (cf. (9) in [2]) that

$$\sum_{\substack{\omega \in \mathfrak{P} \\ \omega | \mathfrak{q}}} 1 \ll (\log N\xi)^{n-1} \log N\mathfrak{q},$$

and an appeal to the simple result

$$\sum_{N\mathfrak{q} \leq Q} \frac{\mu^2(\mathfrak{q}) m^{v(\mathfrak{q})}}{N\mathfrak{q}} \ll_m (\log Q)^m, \quad m \in \mathbf{N},\ Q \geq 2, \tag{8}$$

leads at once to the stated $\ll$ term, if we choose $A \geq n+6$.

Let us now suppose that $(\mathfrak{q}, \xi) = 1$, so that $L(\mathfrak{q}) = 1$ is satisfied. We begin by noting the trivial estimate

$$R(\xi; \mathfrak{q}) \ll \frac{N\xi}{N\mathfrak{q}} + 1.$$

Then, by Cauchy's inequality,

$$\sum_{\substack{N\mathfrak{q} \leq N\xi^{1/2}(\log N\xi)^{-A} \\ (\mathfrak{q},\xi)=1}} \mu^2(\mathfrak{q}) 3^{v(\mathfrak{q})} |R(\xi; \mathfrak{q})| \ll (N\xi)^{1/2} \sum_{\substack{N\mathfrak{q} \leq N\xi^{1/2}(\log N\xi)^{-A} \\ (\mathfrak{q},\xi)=1}} \frac{\mu^2(\mathfrak{q}) 3^{v(\mathfrak{q})}}{N\mathfrak{q}^{1/2}} |R(\xi; \mathfrak{q})|^{1/2}$$

$$\ll (N\xi)^{1/2} \left\{ \sum_{N\mathfrak{q} \leq N\xi} \frac{\mu^2(\mathfrak{q}) 9^{v(\mathfrak{q})}}{N\mathfrak{q}} \right\}^{1/2} \left\{ \sum_{N\mathfrak{q} \leq N\xi^{1/2}(\log N\xi)^{-A}} \max_{(\gamma,\mathfrak{q})=1} \left| \sum_{\substack{\omega \in \mathfrak{P} \\ \omega \equiv \gamma \bmod \mathfrak{q}}} 1 - \frac{I}{\Phi(\mathfrak{q})} \right| \right\}^{1/2}.$$

To complete the proof it only remains to apply (8) and the author's version [7] of Bombieri's theorem in the setting of a totally real algebraic number field.

We continue by stating the important inequalities for the weighted linear sieve in a form introduced by the author in [4].

LEMMA 2. *Let $\mathfrak{q}$ be a squarefree integral ideal of K having no prime ideal factor $\mathfrak{p}$ such that $N\mathfrak{p} < z$, where z is a real number satisfying $2 \leq z \leq \sqrt{N\xi}/N\mathfrak{q}$. For brevity we write*

$$S(\xi; \mathfrak{q}, z) = \left|\{\omega \in \mathfrak{P};\ \omega \equiv \xi \bmod \mathfrak{q},\ (\xi - \omega, V(z)) = 1\}\right|,$$

where $V(z)$ is the notation for the product over those prime ideals of K whose norms are less than z. Then, for sufficiently large $N\xi$,

$$S(\xi; \mathfrak{q}, z) \leq \frac{L(\mathfrak{q})}{\Phi(\mathfrak{q})} \cdot I \cdot \prod_{\substack{N\mathfrak{p} < z \\ \mathfrak{p} \nmid \xi}} \left(1 - \frac{1}{N\mathfrak{p} - 1}\right) \left\{ F\left(\frac{\log(\sqrt{N\xi}/N\mathfrak{q})}{\log z}\right) + O\left(\frac{(\log\log N\xi)^4}{(\log\log\sqrt{N\xi}/N\mathfrak{q})^7}\right) \right\} + R_0(\mathfrak{q}),$$

$$S(\xi; \mathfrak{q}, z) \geq \frac{L(\mathfrak{q})}{\Phi(\mathfrak{q})} \cdot I \cdot \prod_{\substack{N\mathfrak{p} < z \\ \mathfrak{p} \nmid \xi}} \left(1 - \frac{1}{N\mathfrak{p} - 1}\right) \left\{ f\left(\frac{\log(\sqrt{N\xi}/N\mathfrak{q})}{\log z}\right) + O\left(\frac{(\log\log N\xi)^4}{(\log\log\sqrt{N\xi}/N\mathfrak{q})^7}\right) \right\} - R_0(\mathfrak{q}).$$

Here, we have used the abbreviation

$$R_0(\mathfrak{q}) = \sum_{\substack{N\mathfrak{a} \leq M \\ \mathfrak{a} | V(z)}} 3^{v(\mathfrak{a})} |R(\xi; \mathfrak{a}\mathfrak{q})|, \quad \textit{where} \quad M = \max\left(1, \frac{\sqrt{N\xi}}{N\mathfrak{q}} \left(\log \frac{\sqrt{N\xi}}{N\mathfrak{q}}\right)^{-2A_0}\right).$$

The given estimates are also true if $1 < \sqrt{N\xi}/N\mathfrak{q} < z$ but $z \ll (\sqrt{N\xi}/N\mathfrak{q})^{c_3}$. The functions $f(t)$ and $F(t)$, which were introduced in [12], are defined by

$$f(t) = 0, \quad F(t) = 2e^{\gamma_0}/t, \quad 0 < t \leq 2,$$

$$(tf(t))' = F(t-1), \quad (tF(t))' = f(t-1), \quad t \geq 2.$$

Proof. This is a slightly modified form of Lemma 7 in [4]. The procedure is in all ways analogous to that at the corresponding stage in [4] (see also [3]). Apart from minor changes, there are no new difficulties. The main difference is that in our case, we use the sequence $\{\xi-\omega;\ \omega\in\mathfrak{P}(\xi^{(1)},\ldots,\xi^{(n)})\}$ and Lemma 1 instead of the sequence

$$\{\xi-\omega_1\omega_2;\ \omega_1,\omega_2\in\mathfrak{P}(\sqrt{\xi^{(1)}},\ldots,\sqrt{\xi^{(n)}})\}$$

and Lemma 2 in [4].

Our next object will be to quote from [7] an extended form of the large sieve estimate in K. Let $x_1,\ldots,x_n,\ y_1,\ldots,y_n,\ z_1,\ldots,z_n$ be positive real numbers satisfying (with $x=x_1\ldots x_n$, $y=y_1\ldots y_n$, $z=z_1\ldots z_n$) the conditions

$$x^{1/n}\ll x_k\ll x^{1/n},\quad y^{1/n}\ll y_k\ll y^{1/n},\quad z^{1/n}\ll z_k\ll z^{1/n},\quad k=1,\ldots,n,$$

and suppose that $1\ll z\ll xy$. Let $\mathfrak{a},\mathfrak{b}$ be integral ideals of K. We require a non-trivial upper bound for the expression

$$\sum_{N\mathfrak{q}\leqslant Q}\frac{N\mathfrak{q}}{\Phi(\mathfrak{q})}\sum_{\chi\bmod\mathfrak{q}}^{*}\max_{z_k}\Big|\sum_{\alpha,\beta}''c_1(\alpha)c_2(\beta)\chi(\alpha\beta)\Big|,$$

where the coefficients $c_1(\alpha)$, $c_2(\beta)$ are arbitrary complex numbers. The asterisk indicates that the sum runs over primitive characters χ modulo $\mathfrak{q}$, and $\sum''$ is used to mean summation over such integers $\alpha\in\mathfrak{R}(x_1,\ldots,x_n)$, $\beta\in\mathfrak{R}(y_1,\ldots,y_n)$ for which both

$$\alpha\equiv 0 \bmod \mathfrak{a},\quad \beta\equiv 0\bmod\mathfrak{b}$$

and

$$0<\alpha^{(k)}\beta^{(k)}<z_k,\quad k=1,\ldots,n,$$

are satisfied.

Lemma 3. *For $Q\geqslant 1$ we have*

$$\sum_{N\mathfrak{q}\leqslant Q}\frac{N\mathfrak{q}}{\Phi(\mathfrak{q})}\sum_{\chi\bmod\mathfrak{q}}^{*}\max_{z_k}\Big|\sum_{\alpha,\beta}''c_1(\alpha)c_2(\beta)\chi(\alpha\beta)\Big|$$
$$\ll\Big(Q^2+\frac{x}{N\mathfrak{a}}\Big)^{1/2}\Big(Q^2+\frac{y}{N\mathfrak{b}}\Big)^{1/2}(\log xy)^n\Big\{\sum_{\alpha}'|c_1(\alpha)|^2\Big\}^{1/2}\Big\{\sum_{\beta}'|c_2(\beta)|^2\Big\}^{1/2}$$
$$+Q^2(\log xy)^n\Big\{\sum\nolimits_1'|c_1(\alpha)|\,|c_2(\beta)|+\sum\nolimits_2'|c_1(\alpha)|\,|c_2(\beta)|\Big\}.$$

The dash at the sign of summation indicates that the sum is restricted to those $\alpha\in\mathfrak{R}(x_1,\ldots,x_n)$, $\beta\in\mathfrak{R}(y_1,\ldots,y_n)$ which are divisible by $\mathfrak{a}$ and $\mathfrak{b}$ respectively. The sum $\sum_1'$ contains all $\alpha\in\mathfrak{R}(x_1,\ldots,x_n)$, $\alpha\equiv 0\bmod\mathfrak{a}$ and $\beta\in\mathfrak{R}(y_1,\ldots,y_n)$, $\beta\equiv 0\bmod\mathfrak{b}$ for which there exists at least one integer $m\leqslant n$ such that $\alpha^{(m)}\beta^{(m)}<1$. In $\sum_2'$ we group together all α,β under consideration which satisfy, in addition,

$$1\leqslant\alpha^{(k)}\beta^{(k)}\leqslant z_k+1,\quad k=1,\ldots,n,$$

and

$$\alpha^{(m)}\beta^{(m)}>z_m-1$$

for at least one integer $m\leqslant n$.

Proof. This is (3.12) of [7].

Moreover, we need some information about asymptotic estimates for sums of type

$$\sum_{\substack{\alpha\in\mathfrak{R}(x_1,\ldots,x_n)\\ \alpha\equiv 0\bmod\mathfrak{a}}}\chi(\alpha)\Lambda\Big(\frac{\alpha}{\mathfrak{a}}\Big)=\sum_{\substack{\alpha\in\mathfrak{R}(x_1,\ldots,x_n)\\ (\alpha)/\mathfrak{a}=\mathfrak{p}^m}}\chi(\alpha)\log N\mathfrak{p},\qquad \chi\bmod\mathfrak{q}.$$

Our essential tool is an identity given by Siegel. We start by making the preparations necessary for the application of this summation formula.

Let $\mathfrak{q}$ be an integral ideal of K and $\eta_1,\ldots,\eta_{n-1}$ totally positive fundamental units mod $\mathfrak{q}$, i.e., with $\eta_m\equiv 1\bmod\mathfrak{q}$, $m=1,\ldots,n-1$. Consider the matrix

$$M(\mathfrak{q})=\begin{bmatrix}\frac{1}{n} & \log\eta_1^{(1)} & \ldots & \log\eta_{n-1}^{(1)}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{1}{n} & \log\eta_1^{(n)} & \ldots & \log\eta_{n-1}^{(n)}\end{bmatrix}$$

and the inverse matrix of $M(\mathfrak{q})$

$$M(\mathfrak{q})^{-1}=\begin{bmatrix}1 & \ldots & 1\\ e_1^{(1)} & \ldots & e_n^{(1)}\\ \cdots & \cdots & \cdots\\ e_1^{(n-1)} & \ldots & e_n^{(n-1)}\end{bmatrix}.$$

A Grössencharacter λ modulo $\mathfrak{q}$ for ideal numbers $\hat{\gamma}$ is defined as

$$\lambda(\hat{\gamma})=\prod_{k=1}^{n}|\hat{\gamma}^{(k)}|^{2\pi i\sum_{q=1}^{n-1}m_q e_k^{(q)}},$$

where $m_1,\ldots,m_{n-1}$ are rational integers. Let us introduce the abbreviation

$$E_k(m)=E_k(m_1,\ldots,m_{n-1})=2\pi\sum_{q=1}^{n-1}m_q e_k^{(q)},\qquad k=1,\ldots,n.$$

By $\mathfrak{a}=(\hat{\alpha}_0)$ we denote an integral ideal of K coprime to $\mathfrak{q}$, which will be kept fixed in the sequel. We are now in a position to introduce the author's extended version of the Siegel summation formula (see (9) in [5]):

For $\sigma > 1$ and for positive real numbers $y_1, \ldots, y_n$ we have

$$\int_0^{y_1}\cdots\int_0^{y_n}\Big\{\sum_{\substack{\alpha\in\Re(x_1+u_1,\ldots,x_n+u_n)\\ \alpha\equiv 0 \bmod \mathfrak{a}}} \chi(\alpha)\Lambda\Big(\frac{\alpha}{\mathfrak{a}}\Big)\Big\}\, du_1\ldots du_n \tag{9}$$

$$= \frac{1}{2\pi i}(2^{n-1}hR)^{-1}\sum_{\psi}\ \sideset{}{^0}\sum_{m_1,\ldots,m_{n-1}=-\infty}^{\infty} \lambda\chi\psi(\hat{\alpha}_0)\int_{\sigma-i\infty}^{\sigma+i\infty}(N\mathfrak{a})^{-s}\Big(-\frac{\zeta_K'(s,\lambda\chi\psi)}{\zeta_K(s,\lambda\chi\psi)}\Big)$$

$$\times\prod_{k=1}^{n}\frac{(x_k+y_k)^{s+1-iE_k(m)}-x_k^{s+1-iE_k(m)}}{(s-iE_k(m))(s+1-iE_k(m))}\,ds,$$

where the mark at the sign of summation indicates that the sum runs only over such numbers $m_1, \ldots, m_{n-1}$ which determine a Grössencharacter $\lambda\chi\psi$ for ideals. $\sum_\psi$ is used to mean summation over all characters of the Abelian class group of order $2^n h$; the unit element of this class group being the class of all totally positive integers of the field K. Finally, $\zeta_K(s, \lambda\chi\psi)$, $s = \sigma+it$, denotes Hecke's well-known zeta-function.

LEMMA 4. *Let χ be a primitive character* mod $\mathfrak{q}$. *Suppose that $\mathfrak{a}$ satisfies $N\mathfrak{a} \leqslant x$ and $\mathfrak{q}$ satisfies $N\mathfrak{q} \leqslant (\log(2x/N\mathfrak{a}))^A$ for a fixed constant $A > 0$. Then we have, in an obvious notation,*

$$\sum_{\substack{\alpha\in\Re(x_1,\ldots,x_n)\\ \alpha\equiv 0 \bmod \mathfrak{a}}} \chi(\alpha)\Lambda\Big(\frac{\alpha}{\mathfrak{a}}\Big)$$

$$= E_0(\mathfrak{q})(2^{n-1}hR)^{-1}\frac{x}{N\mathfrak{a}} + O_A\Big(\frac{x}{N\mathfrak{a}}\exp\Big(-c_4\Big(\log\frac{2x}{N\mathfrak{a}}\Big)^{1/2}\Big)\Big),$$

where $E_0(\mathfrak{q}) = 1$ if $\mathfrak{q} = (1)$ and $E_0(\mathfrak{q}) = 0$ if $\mathfrak{q} \neq (1)$.

Proof. In connection with the evaluation of Siegel's formula we shall make use of the following results (see [8]):

If λ is non-principal, i.e., not all $m_1, \ldots, m_{n-1}$ in the definition of λ are zero, then there exists a constant $c_5 > 0$ such that $\zeta_K(\sigma+it, \lambda\chi\psi)$ has no zero in the region

$$\sigma \geqslant 1 - c_5\Big\{\log\Big(N\mathfrak{q}\prod_{k=1}^{n}(2+|t-E_k(m)|)\Big)\Big\}^{-1}.$$

Moreover, we have, in this range of σ, the estimate

$$\frac{\zeta_K'}{\zeta_K}(\sigma+it, \lambda\chi\psi) \ll \log\Big(N\mathfrak{q}\prod_{k=1}^{n}(2+|t-E_k(m)|)\Big). \tag{10}$$

The stated results are also true for $\lambda = 1$ but $|t| \geqslant 5$. If $m_1 = \ldots = m_{n-1} = 0$, then for any $a > 0$ there exists a constant $c_6 = c_6(a, K) > 0$ such that, in the region

$$|t| \leqslant 5, \quad \sigma \geqslant 1 - c_6(N\mathfrak{q})^{-a},$$

the following formulae hold:

$$\zeta_K(s, \chi\psi)(s-1)^{E_0(\mathfrak{q})} \neq 0, \quad \frac{\zeta_K'}{\zeta_K}(s, \chi\psi) + \frac{E_0(\mathfrak{q})}{s-1} \ll N\mathfrak{q}^a.$$

Suppose now first that λ is non-principal. Then we replace the line of integration on the right of (9) by the curve

$$C_m\colon\ s_m(t) = \sigma_m(t) + it, \quad -\infty < t < \infty,$$

where

$$\sigma_m(t) = 1 - c_5\{\log(N\mathfrak{q}P_m(t))\}^{-1}, \quad P_m(t) = \prod_{k=1}^{n}(2+|t-E_k(m)|).$$

It is easily seen that this transformation is justified. Using (10) we obtain a contribution of order

$$\ll \sideset{}{^0}\sum_{m\neq 0}\int_{C_m}\frac{\log(N\mathfrak{q}P_m(t))}{P_m(t)^2}(N\mathfrak{a})^{-\sigma_m(t)}\prod_{k=1}^{n}(x_k+y_k)^{\sigma_m(t)+1}\,ds_m(t)$$

$$\ll (N\mathfrak{a})^{-1}\prod_{k=1}^{n}(x_k+y_k)^2\ \sideset{}{^0}\sum_{m\neq 0}\int_{-\infty}^{\infty}\frac{\log(N\mathfrak{q}P_m(t))}{P_m(t)^2}\prod_{k=1}^{n}\Big(\frac{x_k+y_k}{N\mathfrak{a}^{1/n}}\Big)^{-c_5(\log(N\mathfrak{q}P_m(t)))^{-1}}dt.$$

Let us now introduce a formula for $E_k(m)$ given by Rademacher in [10], p. 347:

$$E_k(m) = E_k'(m') = 2\pi\sum_{q=1}^{n-1} f_k^{(q)}(m_q'+z_q), \quad k = 1, \ldots, n,$$

where

$$\begin{bmatrix} 1 & \ldots & 1 \\ f_1^{(1)} & \ldots & f_n^{(1)} \\ \ldots & \ldots & \ldots \\ f_1^{(n-1)} & \ldots & f_n^{(n-1)} \end{bmatrix} = \begin{bmatrix} \frac{1}{n} & \log\varepsilon_1^{(1)} & \ldots & \log\varepsilon_{n-1}^{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{1}{n} & \log\varepsilon_1^{(n)} & \ldots & \log\varepsilon_{n-1}^{(n)} \end{bmatrix}^{-1}.$$

The numbers $\varepsilon_1, \ldots, \varepsilon_{n-1}$ denote the totally positive fundamental units mod(1). Changing the variables of summation from $m_1, \ldots, m_{n-1}$ to $m_1', \ldots, m_{n-1}'$ and changing the variable of integration from t to u given by $u = t - E_n'(m')$, we find that the expression under consideration is less than

$$\frac{\log 2N\mathfrak{q}}{N\mathfrak{a}}\prod_{k=1}^{n}(x_k+y_k)^2$$

$$\times\sum_{\substack{m'_1,\dots,m'_{n-1}=-\infty\\ \lambda\neq 1}}^{\infty}\int_{-\infty}^{\infty}\frac{\log\{(2+|u|)\prod_{k=1}^{n-1}(2+|u-(E'_k(m')-E'_n(m'))|)\}}{\{(2+|u|)\prod_{k=1}^{n-1}(2+|u-(E'_k(m')-E'_n(m'))|)\}^2}$$

$$\times\exp\left\{-\frac{c_5\log(2x/N\mathfrak{a})}{\log\{N\mathfrak{q}(2+|u|)\prod_{k=1}^{n-1}(2+|u-(E'_k(m')-E'_n(m'))|)\}}\right\}du.$$

The sums over $m'_1,\dots,m'_{n-1}$ can be estimated if one observes that

$$u-(E'_k(m')-E'_n(m')) = u-2\pi\sum_{q=1}^{n-1}(f_k^{(q)}-f_n^{(q)})(m'_q+z_q),\qquad k=1,\dots,n-1,$$

and that the determinant

$$\det\begin{bmatrix} f_1^{(1)}-f_n^{(1)} & \dots & f_1^{(n-1)}-f_n^{(n-1)}\\ \dots & \dots & \dots\\ f_{n-1}^{(1)}-f_n^{(1)} & \dots & f_{n-1}^{(n-1)}-f_n^{(n-1)}\end{bmatrix} = \pm\det\begin{bmatrix}\log\varepsilon_1^{(1)} & \dots & \log\varepsilon_{n-1}^{(1)}\\ \dots & \dots & \dots\\ \log\varepsilon_1^{(n-1)} & \dots & \log\varepsilon_{n-1}^{(n-1)}\end{bmatrix}^{-1}$$

does not vanish. Thus, the vectors

$$2\pi(f_1^{(1)}-f_n^{(1)},\dots,f_{n-1}^{(1)}-f_n^{(1)}),\dots,2\pi(f_1^{(n-1)}-f_n^{(n-1)},\dots,f_{n-1}^{(n-1)}-f_n^{(n-1)})$$

generate a $(n-1)$-dimensional lattice, depending only on the field K. Hence the expression we wish to estimate is at most

$$\ll\frac{\log 2N\mathfrak{q}}{N\mathfrak{a}}\prod_{k=1}^{n}(x_k+y_k)^2\sum_{l_1,\dots,l_{n-1}=0}^{\infty}\int_0^{\infty}\exp\left\{-\frac{c_5\log(2x/N\mathfrak{a})}{\log\{N\mathfrak{q}(2+|u|)\prod_{k=1}^{n-1}(3+l_k)\}}\right\}$$

$$\times\{(2+|u|)\prod_{k=1}^{n-1}(2+l_k)\}^{-3/2}du.$$

It remains to deal with the sums over $l_1,\dots,l_{n-1}$. Using the abbreviation

(11) $$T = T\left(\frac{x}{N\mathfrak{a}}\right) = \exp\left\{\left(\log\frac{2x}{N\mathfrak{a}}\right)^{1/2}\right\}$$

we dissect the sums under consideration into two parts $\sum_1$ and $\sum_2$ corresponding to $l_k \leqslant T$ for all k and $l_j > T$ for at least one j, respectively. The term $\sum_1$ is easily estimated by distinguishing the cases $0\leqslant u\leqslant T$ and $u > T$ respecting the

integration. This leads us to the bound

$$\sum\nolimits_1 \ll \exp\left\{-\frac{c_6\log(2x/N\mathfrak{a})}{\log N\mathfrak{q}+c_7(\log(2x/N\mathfrak{a}))^{1/2}}\right\}\int_0^{\infty}\frac{du}{(\log u)^{3/2}}\sum_{l_1,\dots,l_{n-1}=0}^{\infty}\prod_{k=1}^{n-1}(2+l_k)^{-3/2}$$

$$+\int_T^{\infty}\frac{du}{(2+u)^{3/2}}\sum_{l_1,\dots,l_{n-1}=0}^{\infty}\prod_{k=1}^{n-1}(2+l_k)^{-3/2}$$

$$\ll \exp\left\{-c_8\left(\log\frac{2x}{N\mathfrak{a}}\right)^{1/2}\right\}.$$

For $\sum_2$ we obtain at once

$$\sum\nolimits_2 \ll T^{-1/4}\int_0^{\infty}\frac{du}{(2+u)^{3/2}}\sum_{l_1,\dots,l_{n-1}=0}^{\infty}\prod_{k=1}^{n-1}(2+l_k)^{-5/4} \ll \exp\left\{-\frac{1}{4}\left(\log\frac{2x}{N\mathfrak{a}}\right)^{1/2}\right\}$$

so that finally

$$\sum_{m\neq 0}^{0}\int_{C_m}\frac{\log(N\mathfrak{q}P_m(t))}{P_m(t)^2}(N\mathfrak{a})^{-\sigma_m(t)}\prod_{k=1}^{n}(x_k+y_k)^{\sigma_m(t)+1}ds_m(t)$$

$$\ll (N\mathfrak{a})^{-1}\prod_{k=1}^{n}(x_k+y_k)^2\exp\left\{-c_9\left(\log\frac{2x}{N\mathfrak{a}}\right)^{1/2}\right\}.$$

Let us now consider the remaining case of $\lambda = 1$. Here, a segment C_0 and two curves C are defined as follows:

$$C_0\colon\ s(t) = \sigma_0+it,\quad \sigma_0 = 1-c_{10}(N\mathfrak{q})^{-a},\ |t|\leqslant 5,$$

$$C\colon\ s(t) = \sigma(t)+it,\quad \sigma(t) = 1-c_{10}\{\log N\mathfrak{q}|t|\}^{-1},\ |t|\geqslant 5.$$

We combine the two pairs of end points of C_0 and C by two segments parallel to the real axis. In this path we shall denote by W_0, W the sum of three segments and two curves respectively. Taking the constant c_{10} suitably for $a>0$ Hecke's zeta-function $\zeta_K(s,\chi\psi)$ has no zeros on the right-hand side of this path and satisfies the inequalities

$$\frac{\zeta'_K}{\zeta_K}(s,\chi\psi)+\frac{E_0(\mathfrak{q})}{s-1} \ll (N\mathfrak{q})^a \quad\text{if}\quad |t|\leqslant 5,$$

$$\frac{\zeta'_K}{\zeta_K}(s,\chi\psi) \ll \log(N\mathfrak{q}|t|) \quad\text{if}\quad |t|\geqslant 5.$$

Replacing the line of integration in (9) by $W_0\cup W$ and taking, for $\mathfrak{q} = (1)$ and $\psi = 1$, account of the simple pole of the integrand at $s = 1$ with the residue

$$\frac{1}{N\mathfrak{a}}\prod_{k=1}^{n}\frac{(x_k+y_k)^2-x_k^2}{2}$$

we find that

$$\frac{1}{2\pi i}(2^{n-1}hR)^{-1}\sum_{\psi}\chi\psi(\hat{\alpha}_0)\int_{\sigma-i\infty}^{\sigma+i\infty}(N\mathfrak{a})^{-s}\left(-\frac{\zeta_K'}{\zeta_K}(s,\chi\psi)\right)\prod_{k=1}^{n}\frac{(x_k+y_k)^{s+1}-x_k^{s+1}}{s(s+1)}ds$$

$$=\frac{E_0(\mathfrak{q})}{N\mathfrak{a}}(2^{n-1}hR)^{-1}\prod_{k=1}^{n}\frac{(x_k+y_k)^2-x_k^2}{2}+O\left\{(N\mathfrak{q})^a\int_{W_0}(N\mathfrak{a})^{-\sigma}\prod_{k=1}^{n}\frac{(x_k+y_k)^{\sigma+1}}{|s(s+1)|}ds\right\}$$

$$+O\left\{\int_W(N\mathfrak{a})^{-\sigma(t)}\frac{\log(N\mathfrak{q}|t|)}{|t|^{2n}}\prod_{k=1}^{n}(x_k+y_k)^{\sigma(t)+1}ds(t)\right\}.$$

The first remainder term gives a contribution of order

$$\ll(N\mathfrak{q})^a(N\mathfrak{a})^{-1}\prod_{k=1}^{n}(x_k+y_k)^2\exp\left\{-\frac{c_{10}\log(2x/N\mathfrak{a})}{N\mathfrak{q}^a}\right\}$$

$$\ll(N\mathfrak{a})^{-1}\prod_{k=1}^{n}(x_k+y_k)^2\exp\left\{-c_{11}\left(\log\frac{2x}{N\mathfrak{a}}\right)^{1-Aa}\right\},$$

which is admissible provided that $a\leqslant(2A)^{-1}$. For the second error term we divide the integral at T given in (11). This leads us to the upper estimate

$$\ll(N\mathfrak{a})^{-1}\prod_{k=1}^{n}(x_k+y_k)^2\int_5^{\infty}\exp\left\{-\frac{c_{11}\log(2x/N\mathfrak{a})}{\log(tN\mathfrak{q})}\right\}\frac{\log(tN\mathfrak{q})}{t^{2n}}dt$$

$$\ll\frac{\log 2N\mathfrak{q}}{N\mathfrak{a}}\prod_{k=1}^{n}(x_k+y_k)^2\left\{\exp\left\{-\frac{c_{11}\log(2x/N\mathfrak{a})}{\log N\mathfrak{q}+(\log(2x/N\mathfrak{a}))^{1/2}}\right\}\int_5^T\frac{\log t}{t^{2n}}dt+\int_T^{\infty}\frac{\log t}{t^{1/2}t^{2n-1/2}}dt\right\}$$

$$\ll(N\mathfrak{a})^{-1}\prod_{k=1}^{n}(x_k+y_k)^2\exp\left\{-c_{12}\left(\log\frac{2x}{N\mathfrak{a}}\right)^{1/2}\right\}.$$

Summarizing, we have shown that

$$\int_0^{y_1}\ldots\int_0^{y_n}\left\{\sum_{\substack{\alpha\in\mathfrak{R}(x_1+u_1,\ldots,x_n+u_n)\\ \alpha\equiv 0\bmod\mathfrak{a}}}\chi(\alpha)\Lambda\left(\frac{\alpha}{\mathfrak{a}}\right)\right\}du_1\ldots du_n$$

$$=\frac{E_0(\mathfrak{q})}{N\mathfrak{a}}(2^{n-1}hR)^{-1}\prod_{k=1}^{n}\frac{(x_k+y_k)^2-x_k^2}{2}$$

$$+O\left\{\frac{1}{N\mathfrak{a}}\prod_{k=1}^{n}(x_k+y_k)^2\exp\left\{-c_{13}\left(\log\frac{2x}{N\mathfrak{a}}\right)^{1/2}\right\}\right\}.$$

Now it is easy to investigate the asymptotic behaviour of the sum

$$S(x_1,\ldots,x_n)=\sum_{\substack{\alpha\in\mathfrak{R}(x_1,\ldots,x_n)\\ \alpha\equiv 0\bmod\mathfrak{a}}}\chi(\alpha)\Lambda\left(\frac{\alpha}{\mathfrak{a}}\right).$$

Let us first deal with the special case of $\mathfrak{q}=(1)$. For $0<y_k<x_k$, $k=1,\ldots,n$, we have the inequalities

$$\int_0^{y_1}\ldots\int_0^{y_n}S(x_1-y_1+u_1,\ldots,x_n-y_n+u_n)du_1\ldots du_n$$

$$\leqslant y_1\ldots y_nS(x_1,\ldots,x_n)\leqslant\int_0^{y_1}\ldots\int_0^{y_n}S(x_1+u_1,\ldots,x_n+u_n)du_1\ldots du_n.$$

To complete the proof in this case it only remains to choose $y_1,\ldots,y_n$ suitably, namely

$$y_k=x_k\exp\left\{-\frac{c_{13}}{2n}\left(\log\frac{2x}{N\mathfrak{a}}\right)^{1/2}\right\},\qquad k=1,\ldots,n.$$

If $\mathfrak{q}\neq(1)$ we write $S(x_1,\ldots,x_n)$ in the form

$$S(x_1,\ldots,x_n)=\frac{1}{y_1\ldots y_n}\int_0^{y_1}\ldots\int_0^{y_n}(S(x_1,\ldots,x_n)-S(x_1+u_1,\ldots,x_n+u_n))du_1\ldots du_n$$

$$+\frac{1}{y_1\ldots y_n}\int_0^{y_1}\ldots\int_0^{y_n}S(x_1+u_1,\ldots,x_n+u_n)du_1\ldots du_n.$$

As for the first integral, we have, with the above choice of $y_1,\ldots,y_n$,

$$|S(x_1,\ldots,x_n)-S(x_1+u_1,\ldots,x_n+u_n)|$$

$$\leqslant\sum_{\substack{\alpha\in\mathfrak{R}(x_1+u_1,\ldots,x_n+u_n)\\ \alpha\equiv 0\bmod\mathfrak{a}}}\Lambda\left(\frac{\alpha}{\mathfrak{a}}\right)-\sum_{\substack{\alpha\in\mathfrak{R}(x_1,\ldots,x_n)\\ \alpha\equiv 0\bmod\mathfrak{a}}}\Lambda\left(\frac{\alpha}{\mathfrak{a}}\right)$$

$$=(2^{n-1}hR)^{-1}(N\mathfrak{a})^{-1}\left\{\prod_{k=1}^{n}(x_k+u_k)-\prod_{k=1}^{n}x_k\right\}+O\left\{\frac{x}{N\mathfrak{a}}\exp\left\{-c_{14}\left(\log\frac{2x}{N\mathfrak{a}}\right)^{1/2}\right\}\right\}$$

$$\ll\frac{x}{N\mathfrak{a}}\exp\left\{-c_{15}\left(\log\frac{2x}{N\mathfrak{a}}\right)^{1/2}\right\}.$$

This completes the proof of Lemma 4.

Finally, we shall make use of the following result.

LEMMA 5. *If* $N\xi\geqslant N_0$, *then*

$$\sum_{(N\xi)^{1/10}\leqslant N\mathfrak{p}_1<(N\xi)^{1/3}}\ \sum_{(N\xi)^{1/3}\leqslant N\mathfrak{p}_2<(N\xi/N\mathfrak{p}_1)^{1/2}}\frac{1}{N\mathfrak{p}_1N\mathfrak{p}_2\log(N\xi/(N\mathfrak{p}_1N\mathfrak{p}_2))}<\frac{0.493}{\log N\xi}.$$

Proof. The proof runs analogously to the one given in [1], Lemma 11.1.

3. Proof of the Theorem. Let $\xi \in Z_K$ be a totally positive even number with sufficiently large norm. Our method starts from the inequality

$$|\{\omega \in \mathfrak{P};\ \xi-\omega = \Pi_2\}|$$

$$\geqslant \sum_{\substack{\omega\in\mathfrak{P}\\ (\xi-\omega, V(N\xi^{1/10}))=1}}' \Bigg\{1-\frac{1}{2}\cdot \sum_{\substack{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi^{1/3}\\ \mathfrak{p}_1|\xi-\omega,\,\mathfrak{p}_1\nmid\xi}} 1$$

$$-\frac{1}{2}\cdot \sum_{\substack{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi^{1/3}\\ \mathfrak{p}_1|\xi-\omega,\,\mathfrak{p}_1\nmid\xi}} \ \sum_{\substack{N\xi^{1/3}\leqslant N\mathfrak{p}_2<(N\xi/N\mathfrak{p}_1)^{1/2}\\ \mathfrak{p}_2|\xi-\omega,\,\mathfrak{p}_2\nmid\xi\\ (\xi-\omega)=\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3}} 1\Bigg\},$$

where the dash at the sign of summation here indicates that, in addition, $\omega\nmid\xi$ and $\xi-\omega$ is square-free relative to the prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ occurring in the inner sums on the right. To prove this, we have to show that the only positive contribution of the weight (the expression in parentheses) can arise from those elements $\omega \in \mathfrak{P}$ for which $\xi-\omega$ are Π_2's. If the weight is equal to 1, we have

$$(\xi-\omega,\ V(N\xi^{1/3})) = 1;$$

and in this case $\xi-\omega$ can plainly be only a Π_2. It remains to consider a weight of size 1/2. Here, we observe first that $\xi-\omega = \mathfrak{p}_1\cdot\mathfrak{a}$, where $N\mathfrak{p}_1 < N\xi^{1/3}$ and $(\mathfrak{a},\ V(N\xi^{1/3})) = 1$, so that $\mathfrak{a}$ is at most a Π_2. If

$$\mathfrak{a} = \mathfrak{p}_2\mathfrak{p}_3, \quad N\xi^{1/3} \leqslant N\mathfrak{p}_2 \leqslant N\mathfrak{p}_3,$$

we must have $N\mathfrak{p}_2 < (N\xi)^{1/2}(N\mathfrak{p}_1)^{-1/2}$, since otherwise $N(\xi-\omega) \geqslant N\mathfrak{p}_1(N\xi/N\mathfrak{p}_1) = N\xi$, which is impossible, in view of (2). This contradiction implies that

$$(\xi-\omega) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3, \quad N\xi^{1/10} \leqslant N\mathfrak{p}_1 < N\xi^{1/3}, \quad N\xi^{1/3} \leqslant N\mathfrak{p}_2 < (N\xi/N\mathfrak{p}_1)^{1/2}.$$

But then the weight is reduced to 0. Hence, there is only a contribution of order 1/2 if $\mathfrak{a}$ has at most one prime ideal factor, so that $\xi-\omega$ is again at worst a Π_2.

Next, we may set aside from further consideration the restrictions implied by $\sum'$. To see this we have only to observe that their contribution is at most

$$\ll \sum_{\substack{\omega\in\mathfrak{P}\\ \omega|\xi}} 1 + \sum_{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi^{1/3}} \left(\frac{N\xi}{N\mathfrak{p}_1^2}+1\right)$$

$$+ \sum_{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi^{1/3}}\ \sum_{N\xi^{1/3}\leqslant N\mathfrak{p}_2<(N\xi/N\mathfrak{p}_1)^{1/2}} \left(\frac{N\xi}{N\mathfrak{p}_1 N\mathfrak{p}_2^2}+1\right)$$

$$\ll (\log N\xi)^n + N\xi^{9/10} + N\xi^{1/3} + N\xi^{2/3}\log N\xi \ll N\xi^{9/10}.$$

It therefore follows that

$$|\{\omega \in \mathfrak{P};\ \xi-\omega = \Pi_2\}| \geqslant S(\xi;(1),N\xi^{1/10}) - \frac{1}{2}\cdot \sum_{\substack{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi^{1/3}\\ \mathfrak{p}_1\nmid\xi}} S(\xi;\mathfrak{p}_1,N\xi^{1/10})$$

$$-\frac{1}{2}\cdot \sum_{\substack{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi^{1/3}\\ \mathfrak{p}_1\nmid\xi}}\ \sum_{\substack{N\xi^{1/3}\leqslant N\mathfrak{p}_2<(N\xi/N\mathfrak{p}_1)^{1/2}\\ \mathfrak{p}_2\nmid\xi}}\ \sum_{\substack{\omega\in\mathfrak{P}\\ (\xi-\omega)=\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3}} 1 + O(N\xi^{9/10}).$$

Now Lemma 2 enables us to bound the first two terms on the right. We find that

$$S(\xi;(1),N\xi^{1/10}) - \frac{1}{2}\cdot \sum_{\substack{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi^{1/3}\\ \mathfrak{p}_1\nmid\xi}} S(\xi;\mathfrak{p}_1,N\xi^{1/10})$$

$$\geqslant I\cdot \prod_{\substack{N\mathfrak{p}<N\xi^{1/10}\\ \mathfrak{p}\nmid\xi}} \left(1-\frac{1}{N\mathfrak{p}-1}\right)\Bigg\{f(5)$$

$$-\frac{1}{2}\cdot \sum_{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi^{1/3}} \frac{1}{N\mathfrak{p}_1-1} F\left(10\frac{\log(\sqrt{N\xi}/N\mathfrak{p}_1)}{\log N\xi}\right) - O((\log\log N\xi)^{-3})\Bigg\}$$

$$-R_0(1) - \sum_{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi^{1/3}} R_0(\mathfrak{p}_1).$$

Respecting $R_0(1)$ and $R_0(\mathfrak{p}_1)$ we have at our disposal Lemma 1, according to which

$$R_0(1) \leqslant \sum_{N\mathfrak{a}\leqslant\sqrt{N\xi}(\log N\xi)^{-A_0}} \mu^2(\mathfrak{a})3^{\nu(\mathfrak{a})}|R(\xi;\mathfrak{a})| \ll N\xi(\log N\xi)^{-3},$$

$$\sum_{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi^{1/3}} R_0(\mathfrak{p}_1) \leqslant \sum_{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi^{1/3}}\ \sum_{\substack{N\mathfrak{a}\leqslant(\sqrt{N\xi}/N\mathfrak{p}_1)(\log N\xi)^{-A_0}\\ \mathfrak{a}|V(N\xi^{1/10})}} 3^{\nu(\mathfrak{a})}|R(\xi;\mathfrak{a}\mathfrak{p}_1)|$$

$$\leqslant \sum_{N\mathfrak{q}\leqslant\sqrt{N\xi}(\log N\xi)^{-A_0}} \mu^2(\mathfrak{q})3^{\nu(\mathfrak{q})}|R(\xi;\mathfrak{q})| \ll N\xi(\log N\xi)^{-3}.$$

Let us still approximate the sum

$$\sum\nolimits_0 = \sum_{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi^{1/3}} \frac{1}{N\mathfrak{p}_1-1} F\left(10\frac{\log(\sqrt{N\xi}/N\mathfrak{p}_1)}{\log N\xi}\right)$$

by an integral. Arguing in the same way as at the corresponding stage in Chapter 9 of [1] we find that $\sum_0$ can be written in the form

$$\sum\nolimits_0 = \int_3^{10} F\left(5-\frac{10}{t}\right)\frac{dt}{t} + O\left(\frac{\log\log N\xi}{\log N\xi}\right).$$

These results now yield

$$S(\xi; (1), N\xi^{1/10}) - \frac{1}{2} \cdot \sum_{\substack{N\xi^{1/10} \leqslant N\mathfrak{p}_1 < N\xi^{1/3} \\ \mathfrak{p}_1 \nmid \xi}} S(\xi; \mathfrak{p}_1, N\xi^{1/10})$$

$$\geqslant I \cdot \prod_{\substack{N\mathfrak{p} < N\xi^{1/10} \\ \mathfrak{p} \nmid \xi}} \left(1 - \frac{1}{N\mathfrak{p}-1}\right) \left\{ f(5) - \frac{1}{2} \int_3^{10} F\left(5 - \frac{10}{t}\right) \frac{dt}{t} \right\} \{1 - ((\log\log N\xi)^{-3})\}.$$

Our next step will be to cite from the literature ([4], Lemma 3; [1], p. 323) the following two estimates:

$$\prod_{\substack{N\mathfrak{p} < z \\ \mathfrak{p} \nmid \xi}} \left(1 - \frac{1}{N\mathfrak{p}-1}\right)$$

$$= \frac{e^{-\gamma_0}}{a_K} \prod_{N\mathfrak{p}=2} N\mathfrak{p} \prod_{N\mathfrak{p}>2} \left(1 - \frac{1}{(N\mathfrak{p}-1)^2}\right) \prod_{\substack{N\mathfrak{p}>2 \\ \mathfrak{p} \mid \xi}} \frac{N\mathfrak{p}-1}{N\mathfrak{p}-2} \frac{1}{\log z} \left\{ 1 + O\left(\frac{\log\log N\xi}{\log z}\right) \right\},$$

where γ_0 denotes Euler's constant and where a_K is the residue of Dedekind's zeta-function in K,

$$10e^{\gamma_0} \left\{ f(5) - \frac{1}{2} \int_3^{10} F\left(5 - \frac{10}{t}\right) \frac{dt}{t} \right\} > 1.32028.$$

From these we deduce readily

$$|\{\omega \in \mathfrak{P}; \xi - \omega = \Pi_2\}|$$
$$> \frac{1.32\sqrt{d}}{(2^{n-1}hR)^2} \prod_{N\mathfrak{p}=2} N\mathfrak{p} \prod_{N\mathfrak{p}>2} \left(1 - \frac{1}{(N\mathfrak{p}-1)^2}\right) \prod_{\substack{N\mathfrak{p}>2 \\ \mathfrak{p} \mid \xi}} \frac{N\mathfrak{p}-1}{N\mathfrak{p}-2} \frac{N\xi}{\log^2 N\xi} - \frac{1}{2} \sum\nolimits_1,$$

where

$$\sum\nolimits_1 = \sum_{\substack{N\xi^{1/10} \leqslant N\mathfrak{p}_1 < N\xi^{1/3} \\ \mathfrak{p}_1 \nmid \xi}} \sum_{\substack{N\xi^{1/3} \leqslant N\mathfrak{p}_2 < (N\xi/N\mathfrak{p}_1)^{1/2} \\ \mathfrak{p}_2 \nmid \xi}} \sum_{\substack{\omega \in \mathfrak{P} \\ (\xi-\omega) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3}} 1.$$

Now our problem has been reduced to that of establishing a suitable bound for $\sum_1$. To this end we transform $\sum_1$ into an expression in preparation for the application of Selberg's upper sieve in K.

Let us introduce the abbreviation

$$\mathfrak{M} = \mathfrak{M}(\xi) = \{\mathfrak{p}_1\mathfrak{p}_2; N\xi^{1/10} \leqslant N\mathfrak{p}_1 < N\xi^{1/3} \leqslant N\mathfrak{p}_2 < (N\xi/N\mathfrak{p}_1)^{1/2}\},$$

so that

$$|\mathfrak{M}| \ll N\xi^{2/3}.$$

From now on, let

$$z^2 = (N\xi)^{(1/2)-a},$$

where $a > 0$ will be chosen later. The counting number

$$|\{\omega \in \mathfrak{P}; (\xi - \omega) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\}|$$

is at most

$$|\{\omega \in \mathfrak{P}; N\omega > z, (\xi - \omega) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\}| + |\{\alpha \in \mathfrak{R}; N\alpha \leqslant z\}|.$$

It is easy to confirm that

$$|\{\alpha \in \mathfrak{R}; N\alpha \leqslant z\}| \ll z(\log N\xi)^{n-1}.$$

Hence

$$\sum\nolimits_1 \leqslant \sum_{\substack{\mathfrak{p}_1\mathfrak{p}_2 \in \mathfrak{M} \\ (\mathfrak{p}_1\mathfrak{p}_2, \xi) = 1}} |\{\omega \in \mathfrak{P}; N\omega > z, (\xi - \omega) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\}| + O(N\xi^{11/12}).$$

From our point of view, this representation is of little use as the ideal $\mathfrak{p}_1\mathfrak{p}_2$ need not be principal. We may, however, use the following technique to carry out the reduction from ideals to algebraic integers. Let $\mathfrak{p}_1\mathfrak{p}_2 \in \mathfrak{M}$ belong to a given narrow ideal class $\mathfrak{C}$. There exist ideals $\mathfrak{a}_0, \mathfrak{b}_0$ in the classes $\mathfrak{C}^{-1}$, $\mathfrak{C}$ respectively having $N\mathfrak{a}_0, N\mathfrak{b}_0 \leqslant \sqrt{d}$. We write $\mathfrak{p}_1\mathfrak{p}_2\mathfrak{a}_0 = (\varkappa)$, $\mathfrak{a}_0\mathfrak{b}_0 = (\theta_0)$, where

$$\text{(12)} \quad \begin{aligned} c_{16}(N\mathfrak{p}_1 N\mathfrak{p}_2 N\mathfrak{a}_0)^{1/n} &\leqslant \varkappa^{(k)} \leqslant c_{17}(N\mathfrak{p}_1 N\mathfrak{p}_2 N\mathfrak{a}_0)^{1/n}, \\ c_{18}(N\mathfrak{a}_0 N\mathfrak{b}_0)^{1/n} &\leqslant \theta_0^{(k)} \leqslant c_{19}(N\mathfrak{a}_0 N\mathfrak{b}_0)^{1/n}, \end{aligned} \quad k = 1, \ldots, n.$$

Hence

$$(\theta_0(\xi - \omega)) = (\mathfrak{p}_1\mathfrak{p}_2\mathfrak{a}_0)(\mathfrak{p}_3\mathfrak{b}_0) = (\varkappa)(\pi'), \quad \text{say}.$$

Returning to the estimation of $\sum_1$ we infer at once from these transformations that

$$|\{\omega \in \mathfrak{P}; N\omega > z, (\xi - \omega) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\}|$$
$$= \left| \left\{ \pi \in \mathfrak{R}\left(\frac{\theta_0^{(1)}\xi^{(1)}}{\varkappa^{(1)}}, \ldots, \frac{\theta_0^{(n)}\xi^{(n)}}{\varkappa^{(n)}}\right); \pi \equiv 0 \bmod \mathfrak{b}_0, (\pi)\mathfrak{b}_0^{-1} = \mathfrak{p}_3, \xi - \frac{\varkappa\pi}{\theta_0} \text{ prime}, N\left(\xi - \frac{\varkappa\pi}{\theta_0}\right) > z \right\} \right|$$
$$\leqslant \left| \left\{ \pi \in \mathfrak{R}\left(\frac{\theta_0^{(1)}\xi^{(1)}}{\varkappa^{(1)}}, \ldots, \frac{\theta_0^{(n)}\xi^{(n)}}{\varkappa^{(n)}}\right); \pi \equiv 0 \bmod \mathfrak{b}_0, (\pi)\mathfrak{b}_0^{-1} = \mathfrak{p}_3, (\theta_0\xi - \varkappa\pi, V_0(z)) = 1 \right\} \right|,$$

where

$$V_0(z) = \prod_{\substack{N\mathfrak{p} < z \\ \mathfrak{p} \nmid \theta_0\xi}} \mathfrak{p}.$$

It is convenient to introduce the classical device of "weighting" the integers π with von Mangoldt's function Λ. Writing

$$\mathfrak{R}_1 = \mathfrak{R}\left(\frac{\theta_0^{(1)}\xi^{(1)}}{\varkappa^{(1)}}, \ldots, \frac{\theta_0^{(n)}\xi^{(n)}}{\varkappa^{(n)}}\right)$$

and letting $a_0 = (\log N\xi)^{-1/2}$ we have

$$\sum_{\substack{\alpha\in\mathfrak{R}_1,\alpha\equiv 0 \bmod \mathfrak{b}_0\\(\theta_0\xi-\varkappa\alpha, V_0(z))=1}} \Lambda\left(\frac{\alpha}{\mathfrak{b}_0}\right) \geqslant \sum_{\substack{\pi\in\mathfrak{R}_1,\pi\equiv 0 \bmod \mathfrak{b}_0\\(\theta_0\xi-\varkappa\pi, V_0(z))=1\\(\pi)\mathfrak{b}_0^{-1}=\mathfrak{p},\, N\mathfrak{p}>(N\xi/N\mathfrak{p}_1N\mathfrak{p}_2)^{1-a_0}}} \log N\mathfrak{p}$$

$$> (1-a_0)\log\frac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}\Big\{\sum_{\substack{\pi\in\mathfrak{R}_1,\pi\equiv 0 \bmod \mathfrak{b}_0\\(\theta_0\xi-\varkappa\pi, V_0(z))=1\\(\pi)\mathfrak{b}_0^{-1}=\mathfrak{p}}} 1 - \sum_{\substack{\pi\in\mathfrak{R}_1,\pi\equiv 0 \bmod \mathfrak{b}_0\\(\pi)\mathfrak{b}_0^{-1}=\mathfrak{p},\, N\mathfrak{p}\leqslant(N\xi/N\mathfrak{p}_1N\mathfrak{p}_2)^{1-a_0}}} 1\Big\}.$$

It is easy to check that

$$\left|\left\{\pi\in\mathfrak{R}_1;\ \pi\equiv 0 \bmod \mathfrak{b}_0,\ (\pi)\mathfrak{b}_0^{-1}=\mathfrak{p},\ N\mathfrak{p}\leqslant\left(\frac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}\right)^{1-a_0}\right\}\right| \ll \left(\frac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}\right)^{1-a_0}(\log N\xi)^{n-1}.$$

Thus

$$\sum\nolimits_1 < \frac{1}{1-a_0}\sum_{\mathfrak{C}}\sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M}\cap\mathfrak{C}} \frac{1}{\log\dfrac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}} \sum_{\substack{\alpha\in\mathfrak{R}_1,\alpha\equiv 0 \bmod \mathfrak{b}_0\\(\theta_0\xi-\varkappa\alpha, V_0(z))=1}} \Lambda\left(\frac{\alpha}{\mathfrak{b}_0}\right)$$

$$+O\left((\log N\xi)^{n-1}\sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M}}\left(\frac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}\right)^{1-a_0}+N\xi^{11/12}\right)$$

$$\leqslant (1+2a_0)\sum\nolimits_2+((N\xi)^{1-a_0/4}),$$

where

$$\sum\nolimits_2 = \sum_{\mathfrak{C}}\sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M}\cap\mathfrak{C}} \frac{1}{\log\dfrac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}} \sum_{\substack{\alpha\in\mathfrak{R}_1,\alpha\equiv 0 \bmod \mathfrak{b}_0\\(\theta_0\xi-\varkappa\alpha, V_0(z))=1}} \Lambda\left(\frac{\alpha}{\mathfrak{b}_0}\right).$$

On interchanging the order of summation this becomes

$$\sum\nolimits_2 = \sum_{\substack{\beta\in\mathfrak{R}(\xi^{(1)},\ldots,\xi^{(n)})\\(\xi-\beta, V_0(z))=1}} \sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M},\,\mathfrak{p}_1\mathfrak{p}_2|\beta} \Lambda\left(\frac{\beta}{\mathfrak{p}_1\mathfrak{p}_2}\right)\Big/\log\frac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2} = \sum_{\substack{\beta\in\mathfrak{R}(\xi^{(1)},\ldots,\xi^{(n)})\\(\xi-\beta, V_0(z))=1}} \Lambda_0(\beta),$$

where

$$\Lambda_0(\beta) = \sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M},\,\mathfrak{p}_1\mathfrak{p}_2|\beta} \Lambda\left(\frac{\beta}{\mathfrak{p}_1\mathfrak{p}_2}\right)\Big/\log\frac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}.$$

We are now in a position to apply Selberg's upper sieve method in K, which has been derived in [12] (see also [3] and [4]). It will be convenient to introduce, for $\mu(\mathfrak{d})\neq 0$, the multiplicative function

$$g(\mathfrak{d}) = L(\mathfrak{d})\prod_{\mathfrak{p}|\mathfrak{d}}(N\mathfrak{p}-1-L(\mathfrak{p}))^{-1},$$

where L is given by (7). Using, for positive real x, the sum

$$G(\mathfrak{d}, x) = \sum_{\substack{N\mathfrak{a}\leqslant x\\(\mathfrak{a},\mathfrak{d})=1}} \mu^2(\mathfrak{a})g(\mathfrak{a}),$$

we now define

$$\lambda_{\mathfrak{d}} = \mu(\mathfrak{d})\prod_{\substack{\mathfrak{p}|\mathfrak{d}\\\mathfrak{p}\nmid\xi}}\left(1-\frac{1}{N\mathfrak{p}-1}\right)^{-1}\frac{G(\mathfrak{d}, z/N\mathfrak{d})}{G((1), z)}$$

and observe immediately that this choice implies, as usual,

$$\lambda_{(1)} = 1, \quad \lambda_{\mathfrak{d}} = 0 \text{ if } N\mathfrak{d} > z, \quad |\lambda_{\mathfrak{d}}| \leqslant 1 \text{ if } \mathfrak{d}\,|\,V_0(z).$$

Furthermore, following the steps described in detail in [3], p. 239, we obtain

$$\sum_{\mathfrak{d}_1|V_0(z)}\sum_{\mathfrak{d}_2|V_0(z)} \lambda_{\mathfrak{d}_1}\lambda_{\mathfrak{d}_2}\frac{1}{\Phi([\mathfrak{d}_1, \mathfrak{d}_2])} = G^{-1}((1), z).$$

This leads us to the upper bound

$$\sum\nolimits_2 \leqslant \sum_{\beta\in\mathfrak{R}(\xi^{(1)},\ldots,\xi^{(n)})} \Lambda_0(\beta)\Big\{\sum_{\substack{\mathfrak{d}|V_0(z)\\\mathfrak{d}|\xi-\beta}}\lambda_{\mathfrak{d}}\Big\}^2$$

$$= \sum_{\mathfrak{d}_1|V_0(z)}\sum_{\mathfrak{d}_2|V_0(z)} \lambda_{\mathfrak{d}_1}\lambda_{\mathfrak{d}_2} \sum_{\substack{\beta\in\mathfrak{R}(\xi^{(1)},\ldots,\xi^{(n)})\\\beta\equiv\xi \bmod [\mathfrak{d}_1,\mathfrak{d}_2]}} \Lambda_0(\beta)$$

$$= \sum_{\mathfrak{d}_1,\mathfrak{d}_2|V_0(z)} \frac{\lambda_{\mathfrak{d}_1}\lambda_{\mathfrak{d}_2}}{\Phi([\mathfrak{d}_1, \mathfrak{d}_2])} \sum_{\chi \bmod [\mathfrak{d}_1,\mathfrak{d}_2]} \bar\chi(\xi) \sum_{\beta\in\mathfrak{R}(\xi^{(1)},\ldots,\xi^{(n)})} \chi(\beta)\Lambda_0(\beta),$$

where the inner summation is over all characters χ mod $[\mathfrak{d}_1, \mathfrak{d}_2]$. The principal character χ_0 provides the term

$$G^{-1}((1), z)\sum_{\beta\in\mathfrak{R}(\xi^{(1)},\ldots,\xi^{(n)})} \Lambda_0(\beta) - \sum_{\mathfrak{d}_1,\mathfrak{d}_2|V_0(z)} \frac{\lambda_{\mathfrak{d}_1}\lambda_{\mathfrak{d}_2}}{\Phi([\mathfrak{d}_1, \mathfrak{d}_2])} \sum_{\substack{\beta\in\mathfrak{R}(\xi^{(1)},\ldots,\xi^{(n)})\\(\beta,[\mathfrak{d}_1,\mathfrak{d}_2])\neq 1}} \Lambda_0(\beta)$$

so that

(13) $$\sum\nolimits_2 \leqslant G^{-1}((1), z)\sum_{\beta\in\mathfrak{R}(\xi^{(1)},\ldots,\xi^{(n)})} \Lambda_0(\beta) + \sum\nolimits_3 + \sum\nolimits_4,$$

where

$$\sum_3 = \sum_{\substack{\mathfrak{d}|V_0(z)\\ N\mathfrak{d}\leqslant z^2}} \frac{\mu^2(\mathfrak{d})3^{v(\mathfrak{d})}}{\Phi(\mathfrak{d})} \sum_{\substack{\beta\in\mathfrak{N}(\xi^{(1)},\ldots,\xi^{(n)})\\ (\beta,\mathfrak{d})\neq 1}} \Lambda_0(\beta)$$

and

$$\sum_4 = \sum_{\substack{\mathfrak{d}|V_0(z)\\ N\mathfrak{d}\leqslant z^2}} \frac{\mu^2(\mathfrak{d})3^{v(\mathfrak{d})}}{\Phi(\mathfrak{d})} \left| \sum_{\substack{\chi \bmod \mathfrak{d}\\ \chi\neq\chi_0}} \bar{\chi}(\xi) \sum_{\beta\in\mathfrak{N}(\xi^{(1)},\ldots,\xi^{(n)})} \chi(\beta)\Lambda_0(\beta)\right|.$$

We shall deal separately with each of the expressions on the right of (13). For the first we have at our disposal Lemma 2.5 in [6], according to which

$$G^{-1}((1), z)$$
$$= \frac{\sqrt{d}}{2^{n-1}hR} \prod_{N\mathfrak{p}=2} N\mathfrak{p} \prod_{N\mathfrak{p}>2}\left(1-\frac{1}{(N\mathfrak{p}-1)^2}\right) \prod_{\substack{N\mathfrak{p}>2\\ \mathfrak{p}|\xi}} \frac{N\mathfrak{p}-1}{N\mathfrak{p}-2}\frac{1}{\log z}\left\{1+O\left(\frac{\log\log N\xi}{\log z}\right)\right\}.$$

Let us now estimate the sum

$$\sum_{\beta\in\mathfrak{N}(\xi^{(1)},\ldots,\xi^{(n)})} \sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M},\mathfrak{p}_1\mathfrak{p}_2|\beta} \Lambda\left(\frac{\beta}{\mathfrak{p}_1\mathfrak{p}_2}\right)\Big/\log\frac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}.$$

We see that this sum is equal to

$$\sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M}} \frac{1}{\log\dfrac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}} \sum_{\substack{\beta\in\mathfrak{N}(\xi^{(1)},\ldots,\xi^{(n)})\\ \beta\equiv 0 \bmod \mathfrak{p}_1\mathfrak{p}_2}} \Lambda\left(\frac{\beta}{\mathfrak{p}_1\mathfrak{p}_2}\right)$$
$$= (2^{n-1}hR)^{-1}N\xi \sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M}} \frac{1}{N\mathfrak{p}_1N\mathfrak{p}_2\log\dfrac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}}\left(1+O\left(\frac{1}{\log N\xi}\right)\right),$$

using Lemma 4 in the last step. An appeal to Lemma 5 now shows that

$$G^{-1}((1), z) \sum_{\beta\in\mathfrak{N}(\xi^{(1)},\ldots,\xi^{(n)})} \Lambda_0(\beta)$$
$$\leqslant \frac{1.972\sqrt{d}}{(2^{n-1}hR)^2} \prod_{N\mathfrak{p}=2} N\mathfrak{p} \prod_{N\mathfrak{p}>2}\left(1-\frac{1}{(N\mathfrak{p}-1)^2}\right) \prod_{\substack{N\mathfrak{p}>2\\ \mathfrak{p}|\xi}} \frac{N\mathfrak{p}-1}{N\mathfrak{p}-2}\frac{N\xi}{\log^2 N\xi}$$
$$\times\left(1+\left(\frac{\log\log N\xi}{\log N\xi}\right)\right).$$

We now turn our attention to the sum

$$\sum_3 = \sum_{\substack{\mathfrak{d}|V_0(z)\\ N\mathfrak{d}\leqslant z^2}} \frac{\mu^2(\mathfrak{d})3^{v(\mathfrak{d})}}{\Phi(\mathfrak{d})} \sum_{\substack{\beta\in\mathfrak{N}(\xi^{(1)},\ldots,\xi^{(n)})\\ (\beta,\mathfrak{d})\neq 1}} \Lambda_0(\beta).$$

Recalling that if $\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M}$, then $N\mathfrak{p}_2 \geqslant N\xi^{1/3} > z$, we obtain

$$\sum_{\substack{\beta\in\mathfrak{N}(\xi^{(1)},\ldots,\xi^{(n)})\\ (\beta,\mathfrak{d})\neq 1}} \Lambda_0(\beta) = \sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M}} \frac{1}{\log\dfrac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}} \sum_{\substack{\beta\in\mathfrak{N}(\xi^{(1)},\ldots,\xi^{(n)})\\ \beta\equiv 0 \bmod \mathfrak{p}_1\mathfrak{p}_2\\ (\beta,\mathfrak{d})\neq 1}} \Lambda\left(\frac{\beta}{\mathfrak{p}_1\mathfrak{p}_2}\right)$$
$$\ll (\log N\xi)^{n-1} \sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M}} \frac{1}{\log\dfrac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}} \sum_{\substack{N\mathfrak{a}\leqslant N\xi/N\mathfrak{p}_1N\mathfrak{p}_2\\ (\mathfrak{a},\mathfrak{d})\neq 1}} \Lambda(\mathfrak{a})$$
$$+(\log N\xi)^{n-1} \sum_{N\mathfrak{a}\leqslant N\xi} \Lambda(\mathfrak{a}) \sum_{\substack{N\xi^{1/10}\leqslant N\mathfrak{p}_1<N\xi/N\mathfrak{a}\\ \mathfrak{p}_1|\mathfrak{d}}} \sum_{N\mathfrak{p}_2\leqslant N\xi/N\mathfrak{p}_1N\mathfrak{a}} \frac{1}{\log\dfrac{N\xi}{N\mathfrak{p}_1N\mathfrak{p}_2}}.$$

But

$$\sum_{\substack{N\mathfrak{a}\leqslant x\\ (\mathfrak{a},\mathfrak{d})\neq 1}} \Lambda(\mathfrak{a}) = \sum_{\mathfrak{p}|\mathfrak{d}} \log N\mathfrak{p} \sum_{N\mathfrak{p}^m\leqslant x} 1 \ll v(\mathfrak{d})\log x,$$

whence

$$\sum_{\substack{\beta\in\mathfrak{N}(\xi^{(1)},\ldots,\xi^{(n)})\\ (\beta,\mathfrak{d})\neq 1}} \Lambda_0(\beta) \ll v(\mathfrak{d})|\mathfrak{M}|(\log N\xi)^{n-1}+(\log N\xi)^{n-2}\sum_{N\mathfrak{a}\leqslant N\xi}\frac{\Lambda(\mathfrak{a})}{N\mathfrak{a}} \sum_{\substack{\mathfrak{p}_1|\mathfrak{d}\\ N\mathfrak{p}_1\geqslant N\xi^{1/10}}} \frac{1}{N\mathfrak{p}_1}$$
$$\ll (|\mathfrak{M}|+N\xi^{9/10})v(\mathfrak{d})(\log N\xi)^{n-1}.$$

Substituting in $\sum_3$ we arrive at

$$\sum_3 \ll N\xi^{9/10}(\log N\xi)^{n-1} \sum_{N\mathfrak{d}\leqslant z^2} \frac{\mu^2(\mathfrak{d})3^{v(\mathfrak{d})}v(\mathfrak{d})}{\Phi(\mathfrak{d})}$$
$$\ll N\xi^{9/10}(\log N\xi)^{n-1} \sum_{N\mathfrak{d}\leqslant z^2} \frac{\mu^2(\mathfrak{d})3^{v(\mathfrak{d})}}{N\mathfrak{d}}\log N\mathfrak{d}.$$

An application of Lemma 2.3 in [3] now leads us to the estimate

$$\sum_3 \ll N\xi^{9/10}(\log N\xi)^{n+3}.$$

Summarizing, we have shown that

$$\sum_2 \leqslant \frac{1.972\sqrt{d}}{(2^{n-1}hR)^2} \prod_{N\mathfrak{p}=2} N\mathfrak{p} \prod_{N\mathfrak{p}>2}\left(1-\frac{1}{(N\mathfrak{p}-1)^2}\right) \prod_{\substack{N\mathfrak{p}>2\\ \mathfrak{p}|\xi}} \frac{N\mathfrak{p}-1}{N\mathfrak{p}-2}\frac{N\xi}{\log^2 N\xi}$$
$$\times\left(1+\left(\frac{\log\log N\xi}{\log N\xi}\right)\right)+\sum_4$$

and it remains to estimate $\sum_4$ given by

$$\sum_4 = \sum_{\substack{\mathfrak{d}|V_0(z)\\ N\mathfrak{d}\leqslant z^2}} \frac{\mu^2(\mathfrak{d})3^{v(\mathfrak{d})}}{\Phi(\mathfrak{d})} \left| \sum_{\substack{\chi \bmod \mathfrak{d}\\ \chi\neq\chi_0}} \bar{\chi}(\xi) \sum_{\beta\in\mathfrak{N}(\xi^{(1)},\ldots,\xi^{(n)})} \chi(\beta)\Lambda_0(\beta)\right|.$$

To this end we first carry out the reduction to primitive characters. Each character $\chi \neq \chi_0$ occurring in $\sum_4$ is induced by a unique primitive character χ^* to a modulus $\mathfrak{d}^*$ satisfying $\mathfrak{d}^* \neq 1$ and $\mathfrak{d}^* \mid \mathfrak{d}$. Since $\chi(\beta) = \chi^*(\beta)$ whenever $(\beta, \mathfrak{d}) = 1$ we have

$$\sum\nolimits_4 \leqslant \sum_{\substack{\mathfrak{d} \mid V_0(z)\\ N\mathfrak{d} \leqslant z^2}} \frac{\mu^2(\mathfrak{d})3^{v(\mathfrak{d})}}{\Phi(\mathfrak{d})} \sum_{\substack{N\mathfrak{d}^*>1\\ \mathfrak{d}^* \mid \mathfrak{d}}} \Big| \sum_{\chi^* \bmod \mathfrak{d}^*} \bar{\chi}^*(\xi) \sum_{\beta \in \Re(\xi^{(1)},\dots,\xi^{(n)})} \chi^*(\beta)\Lambda_0(\beta)\Big|$$

$$+ \sum_{\substack{\mathfrak{d} \mid V_0(z)\\ N\mathfrak{d} \leqslant z^2}} \frac{\mu^2(\mathfrak{d})3^{v(\mathfrak{d})}}{\Phi(\mathfrak{d})} \Big| \sum_{\substack{N\mathfrak{d}^*>1\\ \mathfrak{d}^* \mid \mathfrak{d}}} \sum_{\chi^* \bmod \mathfrak{d}^*} \bar{\chi}^*(\xi) \sum_{\substack{\beta \in \Re(\xi^{(1)},\dots,\xi^{(n)})\\ (\beta, \mathfrak{d}/\mathfrak{d}^*) \neq 1}} \chi^*(\beta)\Lambda_0(\beta)\Big|.$$

In each summation over $\mathfrak{d}$ we write $\mathfrak{d} = \mathfrak{d}^*\mathfrak{d}'$ and use the fact that then $\Phi(\mathfrak{d}) = \Phi(\mathfrak{d}^*)\Phi(\mathfrak{d}')$. It follows that

$$\text{(14)} \quad \sum\nolimits_4 \leqslant \Bigg\{ \sum_{\substack{\mathfrak{d}^* \mid V_0(z)\\ 1 < N\mathfrak{d}^* \leqslant z^2}} \frac{\mu^2(\mathfrak{d}^*)3^{v(\mathfrak{d}^*)}}{\Phi(\mathfrak{d}^*)} \Big| \sum_{\chi^* \bmod \mathfrak{d}^*} \bar{\chi}^*(\xi) \sum_{\beta \in \Re(\xi^{(1)},\dots,\xi^{(n)})} \chi^*(\beta)\Lambda_0(\beta)\Big| \Bigg\}$$

$$\times \sum_{N\mathfrak{d}' \leqslant z^2} \frac{\mu^2(\mathfrak{d}')3^{v(\mathfrak{d}')}}{\Phi(\mathfrak{d}')}$$

$$+ \sum_{\substack{\mathfrak{d}^*\mathfrak{d}' \mid V_0(z)\\ N\mathfrak{d}^*\mathfrak{d}' \leqslant z^2}} \frac{\mu^2(\mathfrak{d}^*\mathfrak{d}')3^{v(\mathfrak{d}^*\mathfrak{d}')}}{\Phi(\mathfrak{d}^*\mathfrak{d}')} \sum_{\substack{\beta \in \mathfrak{M}\\ (\beta, \mathfrak{d}') \neq 1}} \Lambda_0(\beta) \Big| \sum_{\chi^* \bmod \mathfrak{d}^*} \bar{\chi}^*(\xi)\chi^*(\beta)\Big|$$

$$\ll (\log N\xi)^4 \sum_{\substack{\mathfrak{q} \mid V_0(z)\\ 1 < N\mathfrak{q} \leqslant z^2}} \frac{\mu^2(\mathfrak{q})3^{v(\mathfrak{q})}}{\Phi(\mathfrak{q})} \Big| \sideset{}{^*}\sum_{\chi \bmod \mathfrak{q}} \bar{\chi}(\xi) \sum_{\beta \in \Re(\xi^{(1)},\dots,\xi^{(n)})} \chi(\beta)\Lambda_0(\beta)\Big|$$

$$+ \sum_{\substack{\mathfrak{q} \mid V_0(z)\\ N\mathfrak{q} \leqslant z^2}} \frac{\mu^2(\mathfrak{q})3^{v(\mathfrak{q})}}{\Phi(\mathfrak{q})} \sum_{\substack{N\mathfrak{q}' \leqslant z^2\\ \mathfrak{q}' \mid V_0(z)\\ (\mathfrak{q},\mathfrak{q}')=1}} \frac{\mu^2(\mathfrak{q}')3^{v(\mathfrak{q}')}}{\Phi(\mathfrak{q}')} \sum_{\substack{\beta \in \Re\\ (\beta, \mathfrak{q}') \neq 1}} \Lambda_0(\beta) N((\xi - \beta, \mathfrak{q})),$$

where the asterisk indicates summation over primitive characters mod $\mathfrak{q}$. For the last step we have used the fact that

$$\Big| \sideset{}{^*}\sum_{\chi \bmod \mathfrak{q}} \bar{\chi}(\xi)\chi(\beta)\Big| = \prod_{\mathfrak{p} \mid \mathfrak{q}} \Big| \sum_{\substack{\chi \bmod \mathfrak{p}\\ \chi \neq \chi_0}} \bar{\chi}(\xi)\chi(\beta)\Big| \leqslant \prod_{\substack{\mathfrak{p} \mid \mathfrak{q}\\ \mathfrak{p} \mid \xi - \beta}} (N\mathfrak{p} - 2) \leqslant \prod_{\mathfrak{p} \mid (\xi - \beta, \mathfrak{q})} N\mathfrak{p}.$$

By standard calculations it follows at once that, for $\beta \in \Re(\xi^{(1)}, \dots, \xi^{(n)})$,

$$\sum_{N\mathfrak{q} \leqslant z^2} \frac{\mu^2(\mathfrak{q})3^{v(\mathfrak{q})}}{\Phi(\mathfrak{q})} N((\xi - \beta, \mathfrak{q})) = \sum_{N\mathfrak{q} \leqslant z^2} \frac{\mu^2(\mathfrak{q})3^{v(\mathfrak{q})}}{\Phi(\mathfrak{q})} \sum_{\substack{\mathfrak{a} \mid \mathfrak{q}\\ \mathfrak{a} \mid \xi - \beta}} \Phi(\mathfrak{a})$$

$$= \sum_{\mathfrak{a} \mid \xi - \beta} \mu^2(\mathfrak{a})\Phi(\mathfrak{a}) \sum_{\substack{N\mathfrak{q} \leqslant z^2\\ \mathfrak{a} \mid \mathfrak{q}}} \frac{\mu^2(\mathfrak{q})3^{v(\mathfrak{q})}}{\Phi(\mathfrak{q})}$$

$$\ll (\log N\xi)^4 \sum_{\mathfrak{a} \mid \xi - \beta} \mu^2(\mathfrak{a})3^{v(\mathfrak{a})} \ll (N\xi)^b,$$

for any $b > 0$. Thus the contribution of the second term on the right of (14) to the sum $\sum_4$ is

$$\ll (N\xi)^b \sum_{N\mathfrak{q}' \leqslant z^2} \frac{\mu^2(\mathfrak{q}')3^{v(\mathfrak{q}')}}{\Phi(\mathfrak{q}')} \sum_{\substack{\beta \in \Re\\ (\beta, \mathfrak{q}') \neq 1}} \Lambda_0(\beta)$$

$$\ll (N\xi)^b \sum_{N\mathfrak{q} \leqslant z^2} \frac{\mu^2(\mathfrak{q})3^{v(\mathfrak{q})}}{\Phi(\mathfrak{q})} (|\mathfrak{M}| + N\xi^{9/10}) v(\mathfrak{q})(\log N\xi)^{n-1} \ll N\xi^{9/10+b}(\log N\xi)^{n+3}$$

by the argument that was used in the estimation of $\sum_3$. Returning to $\sum_4$ we obtain

$$\sum\nolimits_4 \ll (\log N\xi)^4 \sum\nolimits_5 + N\xi^{19/20},$$

where

$$\sum\nolimits_5 = \sum_{\substack{\mathfrak{q} \mid V_0(z)\\ 1 < N\mathfrak{q} \leqslant z^2}} \frac{\mu^2(\mathfrak{q})3^{v(\mathfrak{q})}}{\Phi(\mathfrak{q})} \Big| \sideset{}{^*}\sum_{\chi \bmod \mathfrak{q}} \bar{\chi}(\xi) \sum_{\beta \in \Re} \chi(\beta)\Lambda_0(\beta)\Big|.$$

We may therefore concentrate on $\sum_5$. An application of Cauchy's inequality yields

$$\sum\nolimits_5 \leqslant (\sum\nolimits_6)^{1/2} \cdot (\sum\nolimits_7)^{1/2},$$

where

$$\sum\nolimits_6 = \sum_{\substack{\mathfrak{q} \mid V_0(z)\\ 1 < N\mathfrak{q} \leqslant z^2}} \frac{\mu^2(\mathfrak{q})}{\Phi(\mathfrak{q})} \sideset{}{^*}\sum_{\chi \bmod \mathfrak{q}} \Big| \sum_{\beta \in \Re} \chi(\beta)\Lambda_0(\beta)\Big|$$

and

$$\sum\nolimits_7 = \sum_{\substack{\mathfrak{q} \mid V_0(z)\\ 1 < N\mathfrak{q} \leqslant z^2}} \frac{\mu^2(\mathfrak{q})9^{v(\mathfrak{q})}}{\Phi(\mathfrak{q})} \Big| \sideset{}{^*}\sum_{\chi \bmod \mathfrak{q}} \bar{\chi}(\xi) \sum_{\beta \in \Re} \chi(\beta)\Lambda_0(\beta)\Big|.$$

A straightforward verification shows that

$$\sum\nolimits_7 < \sum_{\substack{\mathfrak{q} \mid V_0(z)\\ 1 < N\mathfrak{q} \leqslant z^2}} \frac{\mu^2(\mathfrak{q})9^{v(\mathfrak{q})}}{\Phi(\mathfrak{q})} \sum_{\beta \in \Re} \Lambda_0(\beta) N((\xi - \beta, \mathfrak{q})) \leqslant \sum_{\beta \in \Re} \Lambda_0(\beta) \sum_{\mathfrak{a} \mid \xi - \beta} \Phi(\mathfrak{a}) \sum_{\substack{1 < N\mathfrak{q} \leqslant z^2\\ \mathfrak{a} \mid \mathfrak{q}}} \frac{\mu^2(\mathfrak{q})9^{v(\mathfrak{q})}}{\Phi(\mathfrak{q})}$$

$$\ll (\log N\xi)^{10} \sum_{\beta \in \Re} \Lambda_0(\beta) \sum_{\substack{N\mathfrak{a} \leqslant z^2\\ \mathfrak{a} \mid \xi - \beta}} \mu^2(\mathfrak{a})9^{v(\mathfrak{a})} \ll (\log N\xi)^{10} \sum_{N\mathfrak{a} \leqslant z^2} \mu^2(\mathfrak{a})9^{v(\mathfrak{a})} \sum_{\substack{\beta \in \Re\\ \beta \equiv \xi \bmod \mathfrak{a}}} \Lambda_0(\beta)$$

$$= (\log N\xi)^{10} \sum_{N\mathfrak{a} \leqslant z^2} \mu^2(\mathfrak{a})9^{v(\mathfrak{a})} \sum_{\substack{\beta \in \Re\\ \beta \equiv \xi \bmod \mathfrak{a}}} \sum_{\substack{\mathfrak{p}_1\mathfrak{p}_2 \in \mathfrak{M}\\ \mathfrak{p}_1\mathfrak{p}_2 \mid \beta}} \Lambda\Big(\frac{\beta}{\mathfrak{p}_1\mathfrak{p}_2}\Big) \Big/ \log \frac{N\xi}{N\mathfrak{p}_1 N\mathfrak{p}_2}$$

$$\ll (\log N\xi)^9 \sum_{N\mathfrak{a} \leqslant z^2} \mu^2(\mathfrak{a})9^{v(\mathfrak{a})} \sum_{\substack{\beta \in \Re\\ \beta \equiv \xi \bmod \mathfrak{a}}} \sum_{\mathfrak{b} \mid \beta} \Lambda\Big(\frac{\beta}{\mathfrak{b}}\Big) \ll N\xi(\log N\xi)^{19}.$$

Hence

$$\sum\nolimits_5 \ll N\xi^{1/2}(\log N\xi)^{10}\sum\nolimits_6^{1/2}.$$

To attack the sum $\sum_6$ we consider small and large values of $N\mathfrak{q}$ separately. If $1 < N\mathfrak{q} \leqslant Q_1$, where $Q_1 = (\log N\xi)^C$ with a positive constant C that will be chosen later, Lemma 4 enables us to bound the sum under consideration. Since $N\xi/N(\mathfrak{p}_1\mathfrak{p}_2) \geqslant N\xi^{1/3}$ for all elements $\mathfrak{p}_1\mathfrak{p}_2 \in \mathfrak{M}$, we may infer at once (in view of Lemma 5) that

$$\sum_{\beta\in\mathfrak{R}}\chi(\beta)\Lambda_0(\beta) = \sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M}}\frac{1}{\log\dfrac{N\xi}{N\mathfrak{p}_1 N\mathfrak{p}_2}}\sum_{\substack{\beta\in\mathfrak{R}(\xi^{(1)},\ldots,\xi^{(n)})\\ \beta\equiv 0 \bmod \mathfrak{p}_1\mathfrak{p}_2}}\chi(\beta)\Lambda\left(\frac{\beta}{\mathfrak{p}_1\mathfrak{p}_2}\right)$$

$$\ll N\xi\exp\left(-c_{20}(\log N\xi)^{1/2}\right)\sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M}}\frac{1}{N\mathfrak{p}_1 N\mathfrak{p}_2\log\dfrac{N\xi}{N\mathfrak{p}_1 N\mathfrak{p}_2}}$$

$$\ll N\xi\exp\left(-c_{20}(\log N\xi)^{1/2}\right),$$

so that

$$\sum\nolimits_6 \leqslant \sum_{Q_1<N\mathfrak{q}\leqslant z^2}\frac{1}{\Phi(\mathfrak{q})}\sum_{\chi \bmod \mathfrak{q}}^{*}\Big|\sum_{\beta\in\mathfrak{R}}\chi(\beta)\Lambda_0(\beta)\Big| + O\left(N\xi\exp\left(-c_{20}(\log N\xi)^{1/2}\right)\right).$$

The estimation of the first term on the right is one of the main difficulties to be overcome. We transform this sum into an expression in preparation for the application of the large sieve inequality given in Lemma 3. First we note that the above sum over $N\mathfrak{q}$ can be included in a sum of $\ll \log N\xi$ expressions, each of the form

$$\frac{2}{P}\sum_{P/2<N\mathfrak{q}\leqslant P}\frac{N\mathfrak{q}}{\Phi(\mathfrak{q})}\sum_{\chi \bmod \mathfrak{q}}^{*}\Big|\sum_{\beta\in\mathfrak{R}(\xi^{(1)},\ldots,\xi^{(n)})}\chi(\beta)\Lambda_0(\beta)\Big|,$$

where $Q_1 < P \leqslant z^2$. Our main problem is now to develop

$$\sum\nolimits_8 = \sum_{\beta\in\mathfrak{R}(\xi^{(1)},\ldots,\xi^{(n)})}\chi(\beta)\Lambda_0(\beta) = \sum_{\mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M}}\frac{1}{\log\dfrac{N\xi}{N\mathfrak{p}_1 N\mathfrak{p}_2}}\sum_{\substack{\beta\in\mathfrak{R}(\xi^{(1)},\ldots,\xi^{(n)})\\ \beta\equiv 0 \bmod \mathfrak{p}_1\mathfrak{p}_2}}\chi(\beta)\Lambda\left(\frac{\beta}{\mathfrak{p}_1\mathfrak{p}_2}\right)$$

into a more convenient form. To this end it is necessary to decompose the sum over $\mathfrak{p}_1\mathfrak{p}_2$ into $\ll \log N\xi$ sums of the type

$$\sum_{\substack{M/2<N(\mathfrak{p}_1\mathfrak{p}_2)\leqslant M\\ \mathfrak{p}_1\mathfrak{p}_2\in\mathfrak{M}}}, \quad \text{where } N\xi^{13/30} < M = 2^m \leqslant N\xi^{2/3}.$$

Let $\mathfrak{p}_1\mathfrak{p}_2 \in \mathfrak{M}$ satisfying $M/2 < N(\mathfrak{p}_1\mathfrak{p}_2) \leqslant M$ belong to a given narrow ideal class $\mathfrak{C}$. Then $(\beta) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{c}$ with $\mathfrak{c}\in\mathfrak{C}^{-1}$. Next, we choose fixed prime ideals $\mathfrak{a}_0$, $\mathfrak{b}_0$ satisfying

$$P^n < N\mathfrak{a}_0,\ N\mathfrak{b}_0 \leqslant 2P^n \tag{15}$$

in the classes $\mathfrak{C}^{-1}$, $\mathfrak{C}$ respectively. This can clearly be done in view of the prime ideal theorem for ideal classes modulo (1). We note that

$$(\mathfrak{a}_0, \mathfrak{q}) = (\mathfrak{b}_0, \mathfrak{q}) = 1 \quad \text{for all ideals } \mathfrak{q} \text{ with } P/2 < N\mathfrak{q} \leqslant P.$$

By the theory of units there exists a totally positive generator θ_0 of $\mathfrak{a}_0\mathfrak{b}_0$ such that

$$c_{21}N(\mathfrak{a}_0\mathfrak{b}_0)^{1/n} \leqslant \theta_0^{(k)} \leqslant c_{22}N(\mathfrak{a}_0\mathfrak{b}_0)^{1/n}, \quad k = 1, \ldots, n.$$

Hence

$$(\theta_0\beta) = (\mathfrak{p}_1\mathfrak{p}_2\mathfrak{a}_0)(\mathfrak{c}\mathfrak{b}_0) = (\alpha)(\gamma'), \quad \text{say}.$$

Again, we may assume that α, a representative totally positive generator of $\mathfrak{p}_1\mathfrak{p}_2$ satisfies

$$c_{23}N\alpha^{1/n} \leqslant \alpha^{(k)} \leqslant c_{24}N\alpha^{1/n}, \quad k = 1, \ldots, n. \tag{16}$$

Returning to $\sum_8$ we infer at once from these transformations that

$$\sum\nolimits_8 = \sum_m\sum_{\mathfrak{C}}\bar{\chi}(\theta_0)\sum_{\alpha\in\mathfrak{M}^*,\gamma\in\mathfrak{R}_2}^{''}\chi(\alpha)\chi(\gamma)\frac{\Lambda\left(\dfrac{\gamma}{\mathfrak{b}_0}\right)}{\log\dfrac{N\xi N\mathfrak{a}_0}{N\alpha}},$$

where $\mathfrak{R}_2$ is defined by

$$\mathfrak{R}_2 = \mathfrak{R}\left(c_{25}\left(\frac{2N\xi N\mathfrak{b}_0}{M}\right)^{1/n}, \ldots, c_{25}\left(\frac{2N\xi N\mathfrak{b}_0}{M}\right)^{1/n}\right), \quad c_{25} = \frac{c_2 c_{22}}{c_{23}},$$

and where $\mathfrak{M}^*$ denotes a set of mod (1) nonassociated number $\alpha \in Z_K$ satisfying (16) and

$$\frac{M}{2}N\mathfrak{a}_0 < N\alpha < MN\mathfrak{a}_0, \quad M = 2^m, \quad \frac{(\alpha)}{\mathfrak{a}_0}\in\mathfrak{M}.$$

$\sum''$ again signifies (cf. Lemma 3) that the conditions $\alpha \equiv 0 \bmod \mathfrak{a}_0$, $\gamma \equiv 0 \bmod \mathfrak{b}_0$ and

$$0 < \alpha^{(k)}\gamma^{(k)} < \theta_0^{(k)}\xi^{(k)}, \quad k = 1, \ldots, n,$$

are satisfied. Using the abreviated notation

$$\mathfrak{R}_1 = \mathfrak{R}\left(c_{24}(MN\mathfrak{a}_0)^{1/n}, \ldots, c_{24}(MN\mathfrak{a}_0)^{1/n}\right)$$

we put for $\alpha\in\mathfrak{R}_1$, $\alpha \equiv 0 \bmod \mathfrak{a}_0$,

$$c(\alpha) = \begin{cases}\left(\log\dfrac{N\xi N\mathfrak{a}_0}{N\alpha}\right)^{-1} & \text{if } \alpha\in\mathfrak{M}^*,\\ 0 & \text{if } \alpha\in\mathfrak{R}_1\backslash\mathfrak{M}^*.\end{cases}$$

This leads us to

$$\Sigma_8 = \sum_m \sum_{\mathfrak{C}} \bar{\chi}(\theta_0) \sum''_{\alpha\in\mathfrak{R}_1,\,\beta\in\mathfrak{R}_2} c(\alpha)\,\Lambda\left(\frac{\beta}{\mathfrak{b}_0}\right)\chi(\alpha)\chi(\beta).$$

We are now in a position to apply Lemma 3, and we obtain

$$(17)\qquad \frac{1}{P}\sum_{P/2<N\mathfrak{q}\le P}\frac{N\mathfrak{q}}{\Phi(\mathfrak{q})}\sum^*_{\chi \bmod \mathfrak{q}}|\Sigma_8|$$

$$\ll \frac{1}{P}(\log N\xi)^n\sum_m\sum_{\mathfrak{C}}(P^2+M)^{1/2}\left(P^2+\frac{N\xi}{M}\right)^{1/2}\Big\{\sum'_{\alpha\in\mathfrak{R}_1}|c(\alpha)|^2\Big\}^{1/2}\left\{\sum'_{\beta\in\mathfrak{R}_2}\Lambda^2\left(\frac{\beta}{\mathfrak{b}_0}\right)\right\}^{1/2}$$

$$+P(\log N\xi)^n\sum_m\sum_{\mathfrak{C}}\left\{\sum'_1|c(\alpha)|\,\Lambda\left(\frac{\beta}{\mathfrak{b}_0}\right)+\sum'_2|c(\alpha)|\,\Lambda\left(\frac{\beta}{\mathfrak{b}_0}\right)\right\}.$$

By standard calculations it follows at once that

$$\sum'_{\beta\in\mathfrak{R}_2}\Lambda^2\left(\frac{\beta}{\mathfrak{b}_0}\right)\ll\frac{N\xi}{M}(\log N\xi)^2;\qquad \sum'_{\alpha\in\mathfrak{R}_1}|c(\alpha)|^2\ll M(\log N\xi)^{-2}.$$

Thus we find that the first error term on the right-hand side of (17) is

$$\ll\left(PN\xi^{1/2}+N\xi^{5/6}+N\xi^{1-(13/60)}+\frac{N\xi}{P}\right)(\log N\xi)^{n+1}$$

$$\ll\left(N\xi^{1-a}+\frac{N\xi}{Q_1}\right)(\log N\xi)^{n+1}$$

for a suitable small value of a. This is acceptable if $Q_1=(\log N\xi)^{n+36}$. It remains to deal with the sums $\sum'_1$ and $\sum'_2$ in (17). As to $\sum'_1$ we put

$$v=c_{24}c_{25}(2N\xi N\mathfrak{a}_0N\mathfrak{b}_0)^{1/n}$$

so that

$$\sum'_1|c(\alpha)|\,\Lambda\left(\frac{\beta}{\mathfrak{b}_0}\right)\le\sum_{l=1}^{n}\sum_{\substack{\alpha\in\mathfrak{M}^*\\ \alpha\equiv 0 \bmod \mathfrak{a}_0}}\ \sum_{\substack{\beta\in\mathfrak{R}_2\\ \beta\equiv 0 \bmod \mathfrak{b}_0\\ 0<\alpha^{(l)}\beta^{(l)}<1}}\Lambda\left(\frac{\beta}{\mathfrak{b}_0}\right)\Big/\log\frac{N\xi N\mathfrak{a}_0}{N\alpha}$$

$$\ll(\log N\xi)^{-1}\sum_{l=1}^{n}\sum_{\substack{\gamma\in\mathfrak{R}(v,\dots,v)\\ \gamma\equiv 0 \bmod \mathfrak{a}_0\mathfrak{b}_0\\ 0<\gamma^{(l)}<1}}\ \sum_{\substack{N\mathfrak{a}\le M\\ \mathfrak{a}|\frac{\gamma}{\mathfrak{a}_0\mathfrak{b}_0}}}\Lambda\left(\frac{\gamma}{\mathfrak{a}_0\mathfrak{b}_0\mathfrak{a}}\right)$$

$$\ll\sum_{l=1}^{n}\sum_{\substack{\gamma\in\mathfrak{R}(v,\dots,v)\\ \gamma\equiv 0 \bmod \mathfrak{a}_0\mathfrak{b}_0\\ 0<\gamma^{(l)}<1}}1$$

$$\ll\frac{(N\xi N\mathfrak{a}_0N\mathfrak{b}_0)^{1-1/n}}{N\mathfrak{a}_0N\mathfrak{b}_0}+1\ll P^{-2}N\xi^{1-1/n}+1,$$

on using the condition (15) in the last step. This leads us again to an estimate which is negligible. Let us now turn to $\sum'_2$ as defined in Lemma 3. Putting

$$z_k=\theta_0^{(k)}\xi^{(k)},\qquad k=1,\dots,n,$$

we find that

$$\sum'_2|c(\alpha)|\,\Lambda\left(\frac{\beta}{\mathfrak{b}_0}\right)\ll\sum_{l=1}^{n}\sum_{\substack{\alpha\in\mathfrak{R}(z_1+1,\dots,z_n+1)\\ \alpha\equiv 0 \bmod \mathfrak{a}_0\mathfrak{b}_0\\ \alpha^{(l)}>z_l-1}}1\ll\frac{(z_1\dots z_n)^{1-1/n}}{N\mathfrak{a}_0N\mathfrak{b}_0}+1\ll P^{-2}N\xi^{1-1/n}+1.$$

Retracing our steps, we arrive at

$$\Sigma_6\ll N\xi(\log N\xi)^{-34}.$$

This completes the proof of Chen's result, in the form stated in the introduction.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARBURG
Hans-Meerwein-Straße, Lahnberge
W-3550 Marburg/Lahn
Germany

Received on 4.4.1990 (2025)