ACTA ARITHMETICA
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# Note on a paper by H. L. Montgomery – II

by

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**1. Introduction.** In this paper we apply the method of H. L. Montgomery [2] to the Hurwitz zeta-function $\zeta(s, \beta)$ defined by $\sum_{n=0}^{\infty}(n+\beta)^{-s}$ (where $0 < \beta \leqslant 1$, $s = \sigma + it$) and its analytic continuations and more general Dirichlet series. The omega-theorems like $\Omega(t^{1/2-\sigma})$ in $\sigma < 1/2$ and $\Omega((\log t)^{1/2})$ for $\sigma = 1/2$ can be proved by mean-value considerations. The method of K. Ramachandra [3], developed by R. Balasubramanian and K. Ramachandra [1], yields better omega results in $1/2 \leqslant \sigma < 1$ for a dense set of $\beta$ (see [4]). But in the most general case, we can only prove

THEOREM 1. *Let $1/2 \leqslant \sigma_0 < 1$, $0 \leqslant \theta < 2\pi$, $\varepsilon > 0$. Let $y_0$ be the positive solution of $e^{y_0} = 2y_0 + 1$, let $l$ be an integer constant satisfying $l \geqslant 6$, $c_2 = 2y_0/(2y_0+1)^2$, $0 < c_1 < c_2$. Then for $T \geqslant T_0$ depending on these constants, we have*

$$\operatorname{Re}(e^{-i\theta}\zeta(\sigma_0 + it_0, \beta)) \geqslant \frac{1}{1-\sigma_0} c_0 c_1 (\log t_0)^{1-\sigma_0}$$

*for at least one $t_0$ in $\frac{1}{2}T^{\varepsilon} \leqslant t_0 \leqslant \frac{3}{2}T$, where $c_0 = \cos(2\pi/l)(\log l)^{\sigma_0 - 1}$.*

THEOREM 2. *Let $0 \leqslant \theta < 2\pi$, $\varepsilon > 0$, $\varepsilon_1 > 0$. Then for $T \geqslant T_0$, depending on these constants, we have*

$$\operatorname{Re}(e^{-i\theta}\zeta(1 + it_0, \beta)) \geqslant (\tfrac{1}{2}\cos^2(\theta/2) - \varepsilon_1)\log\log t_0$$

*for at least one $t_0$ in $\frac{1}{2}T^{\varepsilon} \leqslant t_0 \leqslant \frac{3}{2}T$.*

Remark 1. Proof of Theorem 1 uses the method of Montgomery [2] (see also [5]) while Theorem 2 requires a change in the kernel function, the method being the same.

Remark 2. In [4] we proved the following:

THEOREM. *Let $\exp_0 x = 2^x$, $\beta = \sum_{n=1}^{\infty} a_n/b_n$ (we can take $a_n = 1$ for all $n \geqslant 1$) where $b_1 = 2$ and $b_{n+1} = \exp_0\exp_0\exp_0\exp_0\exp_0 b_n$ for $n \geqslant 2$. Then we have*

$$|\zeta(1+it, \beta)| = \Omega\left\{\exp\left(\frac{\log\log\log t}{\log\log\log\log t}\right)\right\}.$$

Theorem 2 is an improvement of this theorem.

Remark 3. The investigations of this paper go through for very general Dirichlet series. For example we can take $\mathrm{Re}(e^{-i\theta}F(\sigma_0+it_0))$ where

$$F(s)=\sum_{n=1}^{\infty} d_n/\lambda_n^s$$

($d_n \geqslant 0$, $\sum_{X\leqslant n\leqslant 2X} d_n \asymp X$ and $0<\lambda_1<\lambda_2<\ldots$ with $\lambda_n \asymp n$). The only further condition that we need is analytic continuation and a growth condition of the type $|F(s)|<t^A$ or even $\exp\exp(t/100)$ for $t \geqslant c$ and $\sigma \geqslant \sigma_0$.

## 2. Notation.

(1) Let $x$ be real and positive, $s=\sigma+it$. We define

$$x^s = e^{s\log x}.$$

(2) Let $0 \leqslant \theta < 2\pi$ and $k$ be real. We denote

$$F_1(s, k\theta) = e^{ik\theta}\zeta(s, \beta),$$

$$F_2(s, \theta) = \left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2 (2+x^s e^{i\theta}+x^{-s}e^{-i\theta}),$$

$$F_3(s, \theta) = \left(\frac{e^{i\theta+\alpha s}-e^{-i\theta-\alpha s}-e^{i\theta}+e^{-i\theta}}{s}\right)^2.$$

(3) $[x]$ denotes the integral part of $x$.

(4) $\|\theta\| = \min_n|\theta-n|$, where $n$ runs through all integers.

(5) $f \ll g$ implies that $|f| \leqslant Ag$ where $A$ is some positive constant.

## 3. Some lemmas.

LEMMA 3.1. *Let $\theta_1, \theta_2, \ldots, \theta_M$ be distinct positive real numbers and suppose that $0<1/l \leqslant 1/6$, where $l$ is an integer constant. Then there exists a positive integer $r'$ such that $1 \leqslant r' \leqslant J = l^M$ and $\|r'\theta_m\| < 1/l$ for $1 \leqslant m \leqslant M$.*

Proof. See for example Section 8.2 of [6].

LEMMA 3.2. *Let $\theta_1, \theta_2, \ldots, \theta_M$ of Lemma 3.1 be $P\log(n+\beta)$ where $P$ is a fixed positive integer and $n$ runs through a set of integers which include those satisfying $|\log((n+\beta)/x)| \leqslant 2\alpha$ where $x \geqslant 1$, $\beta$ fixed and $\alpha$ is a positive quantity which will be fixed later. Let $l_1 = r'P$.*

*Then we have*

(i) $\cos(2\pi l_1\log(n+\beta)) \geqslant \cos(2\pi/l)$,

(ii) $P \leqslant l_1 \leqslant l^M P$.

Proof. See for example [5] or Section 8.2 of [6].



LEMMA 3.3. *For $1/2 \leqslant \sigma_0 < 1$, we have*

$$\sum_{|\log((n+\beta)/x)|\leqslant 2\alpha} \frac{1}{(n+\beta)^{\sigma_0}}\left(2\alpha-\left|\log\left(\frac{n+\beta}{x}\right)\right|\right) = \left(\frac{2\sinh(\alpha(1-\sigma_0))}{1-\sigma_0}\right)^2 x^{1-\sigma_0}+O(x^{-\sigma_0}).$$

Proof. Replacing $n+\beta$ by $n$ in the left-hand side of the lemma gives an error $O(x^{-\sigma_0})$ and now the lemma follows from the fact that

$$\sum_{n\leqslant x}\frac{1}{n^{\sigma_0}} = \frac{x^{1-\sigma_0}}{1-\sigma_0}+\zeta(\sigma_0)+O(x^{-\sigma_0})$$

and using Stieltjes integral and integrating by parts.

LEMMA 3.4. *Let $x>0$, $c>0$. Then we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s^2}ds = \begin{cases}\log x & \text{if } x \geqslant 1,\\ 0 & \text{if } 0<x\leqslant 1.\end{cases}$$

Proof. See for example [5].

LEMMA 3.5. *Let $\alpha>0$, $c>0$, $x>0$. Then we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2 x^s ds = \begin{cases}2\alpha-|\log x| & \text{if } |\log x| \leqslant 2\alpha,\\ 0 & \text{if } |\log x| \geqslant 2\alpha.\end{cases}$$

Proof. The proof follows from Lemma 3.4.

LEMMA 3.6. *Let $0 \leqslant \theta < 2\pi$, $\alpha>0$, $1/2 \leqslant \sigma_0 < 1$ be constants. Let $s=\sigma+it$, $s_0=\sigma_0+it_0$ where $t_0 \geqslant 1000$. Then we have*

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} F_1(s+s_0, -\theta)F_2(s, \theta)ds = \sum_{|\log((n+\beta)/x)|\leqslant 2\alpha}\frac{1}{(n+\beta)^{s_0}}\left(2\alpha-\left|\log\left(\frac{n+\beta}{x}\right)\right|\right)+O((\log x)^2).$$

Proof. This follows from Lemma 3.5.

LEMMA 3.7. *Let $\theta$, $\alpha$, $\sigma_0$ be as in Lemma 3.6. Then the contribution of $|t| \geqslant \tau = t_0/2$ to the integral in Lemma 3.6 is $O(xt_0^{-1/2})$. Also the contributions from the integrals over $[i\tau, 1+i\tau]$ and $[-i\tau, 1-i\tau]$ are $O(xt_0^{-1/2})$.*

Proof. The proof follows from the fact that

$$\zeta(s, \beta) = O(t^{1/2}) \quad \text{for } 1/2 \leqslant \sigma \leqslant 1.$$

LEMMA 3.8. *We have for $\tau = t_0/2$,*

$$\mathrm{Re}\left(\frac{1}{2\pi i}\int_{-i\tau}^{i\tau} F_1(s+s_0, -\theta)F_2(s, \theta)ds\right) = \sum_{|\log((n+\beta)/x)|\leqslant 2\alpha}\frac{\cos(t_0\log(n+\beta))}{(n+\beta)^{\sigma_0}}\left(2\alpha-\left|\log\left(\frac{n+\beta}{x}\right)\right|\right)+O((\log x)^2).$$

Proof. The proof follows from Lemmas 3.6 and 3.7.

LEMMA 3.9. *For* $\tau = t_0/2$, *we have*

$$\{\max_{\substack{|t|\leqslant\tau\\ \sigma=0}}(\operatorname{Re} F_1(s+s_0, -\theta))\}\left(\frac{1}{2\pi i}\int_{\substack{|t|\leqslant\tau\\ \sigma=0}} F_2(s,\theta)ds\right)$$

$$\geqslant \sum_{|\log((n+\beta)/x)|\leqslant 2\alpha} \frac{\cos(t_0\log(n+\beta))}{(n+\beta)^{\sigma_0}}\left(2\alpha - \left|\log\left(\frac{n+\beta}{x}\right)\right|\right) + O((\log x)^2).$$

Proof. The proof follows from Lemma 3.8.

LEMMA 3.10. *Let* $\tau = t_0/2$. *Then we have*

$$\frac{1}{2\pi i}\int_{\substack{|t|\leqslant\tau\\ \sigma=0}} F_2(s,\theta)ds = 4\alpha + O(1/\tau).$$

Proof. See for example [5].

LEMMA 3.11. *Let* $t_0 = 2\pi l_1$ *where* $l_1$ *is as in Lemma* 3.2. *Then we have*

$$4\alpha \max_{\substack{|t|\leqslant\tau\\ \sigma=0}}(\operatorname{Re} F_1(s+s_0, -\theta))$$

$$\geqslant \cos\left(\frac{2\pi}{l}\right)\left(\sum_{|\log((n+\beta)/x)|\leqslant 2\alpha}(n+\beta)^{-\sigma_0}\left(2\alpha - \left|\log\left(\frac{n+\beta}{x}\right)\right|\right) + O((\log x)^2)\right).$$

Proof. The proof follows from Lemmas 3.2 (for an explanation see § 4), 3.9 and 3.10.

**4. Proof of Theorem 1.** In Lemma 3.2, let $t_0 = 2\pi l_1$. We can make this choice because there is no singularity inside the rectangle $\sigma \geqslant \sigma_0$, $|t_0 - t| \leqslant \tau$ $(\tau = t_0/2)$. We fix $P = [T^\varepsilon]$ where $\varepsilon > 0$ is an arbitrary small positive constant. We choose $T$ such that $t_0 + \tau \leqslant T$ and $l^M P = T$, i.e. $M = [(1-\varepsilon)(\log T)/\log l]$.

Since $M = [xe^{2\alpha}]$, we choose $x = (1-\varepsilon)(\log t_0)/(e^{2\alpha}\log l)$, where $\alpha$ is a positive constant to be chosen later. We note that $x \leqslant t_0^{1/3}$ and so the error in Lemma 3.7 is $o(1)$.

Now from Lemmas 3.3, 3.9, 3.10 and 3.11, we have

$$4\alpha \max_{\substack{|t|\leqslant\tau\\ \sigma=0}}(\operatorname{Re} F_1(s+s_0, -\theta))$$

$$\geqslant \cos\left(\frac{2\pi}{l}\right)\left(\frac{2\sinh(\alpha(1-\sigma_0))}{1-\sigma_0}\right)^2 \frac{(1-\varepsilon)^{1-\sigma_0}}{e^{2\alpha(1-\sigma_0)}(\log l)^{1-\sigma_0}}(\log t_0)^{1-\sigma_0}.$$



Put $2\alpha(1-\sigma_0) = \delta$. Then

$$\max_{\substack{|t|\leqslant\tau\\ \sigma=0}}(\operatorname{Re} F_1(s+s_0, -\theta)) \geqslant \frac{1}{2}\frac{\cos(2\pi/l)}{(\log l)^{1-\sigma_0}}\frac{(1-\varepsilon)^{1-\sigma_0}}{1-\sigma_0}\left\{\frac{1-e^{-\delta}}{\sqrt{\delta}}\right\}^2(\log t_0)^{1-\sigma_0}$$

$$\geqslant \frac{\cos(2\pi/l)}{(\log l)^{1-\sigma_0}}\frac{1}{1-\sigma_0}c_1(\log t_0)^{1-\sigma_0}$$

(by choosing $\delta > 0$ such that $(1-e^{-\delta})/\sqrt{\delta}$ is maximum) where $c_1$ is any positive constant less than $c_2 = 2y_0/(2y_0+1)^2$, where $y_0$ is the root of the equation $e^{y_0} = 2y_0 + 1$. This proves the theorem (since $\varepsilon$ is arbitrary).

**5. Some lemmas on the line** $\sigma_0 = 1$. Throughout this section $\alpha$ will be a variable. We assume that $10 \leqslant \alpha \leqslant 10\log\log t_0$ where $t_0 \geqslant 1000$ and $\tau = t_0/2$. Let $s = \sigma + it$, $s_0 = 1 + it_0$.

LEMMA 5.1. *Let $k$ be any fixed real number and let* $0 \leqslant |k|\theta < 2\pi$. *Then we have*

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} F_1(s+s_0, k\theta)F_3(s,\theta)ds = e^{(k+2)i\theta}\sum_{n+\beta\leqslant e^{2\alpha}}\frac{1}{(n+\beta)^{s_0}}(2\alpha - \log(n+\beta))$$

$$-2e^{(k+1)i\theta}(e^{i\theta}-e^{-i\theta})\sum_{n+\beta\leqslant e^{\alpha}}\frac{1}{(n+\beta)^{s_0}}(\alpha - \log(n+\beta)).$$

Proof. The proof follows from Lemma 3.4.

LEMMA 5.2. *Let $k$ be any fixed real number and let* $0 \leqslant |k|\theta < 2\pi$. *Then we have*

$$\frac{1}{2\pi i}\int_{\substack{|t|\geqslant\tau\\ \sigma=1}} F_1(s+s_0, k\theta)F_3(s,\theta)ds = O(e^{2\alpha}(\log t_0)/\tau).$$

Proof. The proof follows from the fact that

$$\zeta(s,\beta) = O(\log t) \quad \text{for } \sigma \geqslant 1,\ t \geqslant 2.$$

LEMMA 5.3. *We have*

(i)
$$\sum_{n+\beta\leqslant e^{2\alpha}}\frac{1}{n+\beta}(2\alpha - \log(n+\beta)) = 2\alpha^2 + O(\alpha),$$

(ii)
$$\sum_{n+\beta\leqslant e^{\alpha}}\frac{1}{n+\beta}(\alpha - \log(n+\beta)) = \alpha^2/2 + O(\alpha).$$

Proof. In (i), replacing $n+\beta$ by $n$ gives an error $O(\alpha)$ and the result follows from the fact that

$$\sum_{n\leqslant e^{2\alpha}}\frac{1}{n}(2\alpha - \log n) = \int_1^{e^{2\alpha}}\frac{1}{v}(2\alpha - \log v)dv + O(\alpha).$$

Now (ii) follows from (i) by replacing $\alpha$ by $\alpha/2$.

LEMMA 5.4. *We have for* $0 \leqslant \theta < 2\pi$

$$\frac{1}{2\pi i}\int_{\substack{|t|\leqslant\tau\\ \sigma=0}} F_3(s,\theta)ds = 2\alpha + O(1/\tau).$$

Proof. We have

$$\frac{1}{2\pi i}\int_{\substack{|t|\geqslant\tau\\ \sigma=0}} F_3(s,\theta)ds = O(1/\tau)$$

and

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F_3(s,\theta)ds = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} F_3(s,\theta)ds = 2\alpha \quad \text{(by Lemma 3.4)},$$

which proves the lemma.

LEMMA 5.5. *Let* $0 \leqslant \theta < 2\pi$. *Then we have*

$$\max_{\substack{|t|\leqslant\tau\\ \sigma=0}} (\operatorname{Re} F_1(s+s_0, -\theta)) \geqslant \alpha\cos^2(\theta/2) + O(\alpha/l) + O(e^{2\alpha}(\log t_0)/\tau).$$

Proof. From Lemmas 5.1 and 5.2, we have, as before,

$$\frac{1}{2\pi i}\int_{-i\tau}^{i\tau} F_1(s+s_0, k\theta)F_3(s,\theta)ds$$

$$= e^{(k+2)i\theta}\sum_{n+\beta\leqslant e^{2\alpha}} \frac{\cos(t_0\log(n+\beta)) - i\sin(t_0\log(n+\beta))}{n+\beta}(2\alpha - \log(n+\beta))$$

$$-2e^{(k+1)i\theta}(e^{i\theta}-e^{-i\theta})\sum_{n+\beta\leqslant e^{\alpha}} \frac{\cos(t_0\log(n+\beta)) - i\sin(t_0\log(n+\beta))}{n+\beta}(\alpha - \log(n+\beta))$$
$$+O(e^{2\alpha}(\log t_0)/\tau).$$

Now, from Lemma 5.4, we have

$$\{2\alpha + O(1/\tau)\}\max_{\substack{|t|\leqslant\tau\\ \sigma=0}} (\operatorname{Re} F_1(s+s_0, k\theta))$$

$$\geqslant \sum_{n+\beta\leqslant e^{2\alpha}} \frac{\cos(t_0\log(n+\beta))\cos((k+2)\theta) + \sin(t_0\log(n+\beta))\sin((k+2)\theta)}{n+\beta} \times \{2\alpha - \log(n+\beta)\}$$

$$+4\sin\theta \sum_{n+\beta\leqslant e^{\alpha}} \frac{\cos(t_0\log(n+\beta))\sin((k+1)\theta) - \sin(t_0\log(n+\beta))\cos((k+1)\theta)}{n+\beta} \times \{\alpha - \log(n+\beta)\}$$

$$+O(e^{2\alpha}(\log t_0)/\tau).$$



By choosing $t_0$ in an appropriate manner (for explanation see §6) with $\frac{1}{2}T^{\varepsilon} \leqslant t_0 < \frac{3}{2}T$, from Lemma 3.2, we have

$$1 \geqslant \cos(t_0\log(n+\beta)) \geqslant \cos(2\pi/l) = 1 + O(1/l^2)$$

and so

$$\sin(t_0\log(n+\beta)) = O(1/l).$$

From Lemmas 5.3 and 5.4, we have

$$\{2\alpha + O(1/\tau)\}\max_{\substack{|t|\leqslant\tau\\ \sigma=0}} (\operatorname{Re} F_1(s+s_0, k\theta))$$
$$\geqslant \{\cos((k+2)\theta) + O(1/l)\}(2\alpha^2 + O(\alpha))$$
$$+\{4\sin\theta\sin((k+1)\theta) + O(1/l)\}(\alpha^2/2 + O(\alpha)) + O(e^{2\alpha}(\log t_0)/\tau).$$

Therefore

$$\max_{\substack{|t|\leqslant\tau\\ \sigma=0}} (\operatorname{Re} F_1(s+s_0, k\theta)) \geqslant \alpha\cos\theta\cos((k+1)\theta) + O(\alpha/l) + O(e^{2\alpha}(\log t_0)/\tau).$$

One can check that $k = -2$ is the optimal value, which gives

$$\max_{\substack{|t|\leqslant\tau\\ \sigma=0}} (\operatorname{Re} F_1(s+s_0, -2\theta)) \geqslant \alpha\cos^2\theta + O(\alpha/l) + O(e^{2\alpha}(\log t_0)/\tau).$$

Now the lemma follows by replacing $\theta$ by $\theta/2$.

**6. Proof of Theorem 2.** As before in the proof of Theorem 1, we put $t_0 = 2\pi l_1$ and $R = 1$. We fix $P = [T^{\varepsilon}]$ where $\varepsilon > 0$ is a small positive constant. We choose $T$ such that $t_0 + \tau \leqslant T$ and $l^M P = T$, i.e.

$$M = [(1-\varepsilon)(\log T)/\log l].$$

We fix $M \geqslant [e^{2\alpha}]$. Now we choose $\alpha$ such that $e^{2\alpha} = (1-\varepsilon)(\log t_0)/\log l$. We have

$$\alpha = (\tfrac{1}{2} - \varepsilon_1)\log\log t_0.$$

We note that $\alpha$ satisfies our condition on $\alpha$ and now the theorem follows from Lemma 5.5.

#### References

[1] R. Balasubramanian and K. Ramachandra, *On the frequency of Titchmarsh's phenomenon for* $\zeta(s)$. *III*, Proc. Indian Acad. Sci., A 86 (4) (1977), 341–351.

[2] H. L. Montgomery, *Extreme values of the Riemann zeta function*, Comment. Math. Helv. 52 (1977), 511–518.

[3] K. Ramachandra, *On the frequency of Titchmarsh's phenomenon for* $\zeta(s)$. *I*, J. London Math. Soc. (2) 8 (1974), 683–690.

[4] K. Ramachandra and A. Sankaranarayanan, *Omega theorems for the Hurwitz zeta-function*, Arch. Math. 53 (1989), 469–481.
[5] —, —, *Note on a paper by H. L. Montgomery – I*, Publ. Inst. Math. (Beograd) (1991).
[6] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd ed., revised by D. R. Heath-Brown, Oxford Univ. Press, Oxford 1986.

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## APPENDIX BY THE REFEREE

In response to a question by the authors whether $\operatorname{Re}\zeta(1+it) = \Omega_-(\log\log t)$, the referee points out the following:

THEOREM. *Let $r, \theta, \phi$ be constants satisfying $r > 0, 0 \leqslant \theta < 2\pi, 0 \leqslant \phi < 2\pi$, $z = re^{i\phi}$, and in $\sigma \geqslant 1$, $G(s) = e^{i\theta}(\zeta(s))^z$ where as usual $s = \sigma + it$. Then as $t \to \infty$, we have*

$$\operatorname{Re} G(1+it) = \Omega_+((\log\log t)^r).$$

The method of proof is as follows. Let $A$ be a large positive constant and $N = N(t)$ be a large positive integer. We prove that if $\sigma = 1 + A(\log N)^{-1}$ then $\operatorname{Re} G(\sigma+it) = \Omega_+((\log\log t)^r)$. From this the theorem follows by a simple application of the maximum modulus principle. To prove this statement about $G(\sigma+it)$ we choose a large (but fixed) integer $M < N$ and write

(1) $$\log G(\sigma+it) = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3,$$

where

$$\sum\nolimits_1 = \sum_{n \leqslant M} z\log(1-p_n^{-s})^{-1} + i\theta,$$

$$\sum\nolimits_2 = \sum_{M<n\leqslant N} z\log(1-p_n^{-s})^{-1} \quad \text{and} \quad \sum\nolimits_3 = \sum_{n>N} z\log(1-p_n^{-s})^{-1}.$$

It is not hard to prove that $\sum_3 = O(e^{-A}A^{-1})$. Also

$$\sum\nolimits_1 = \sum\nolimits_4 + O\left(\frac{A\log M}{\log N}\right)$$

where

$$\sum\nolimits_4 = \sum_{n\leqslant M} z\log(1-p_n^{-1-it})^{-1} + i\theta.$$



In $\sum_4$ and $\sum_2$ we can (for suitable $t = t_\nu \to \infty$) replace $p_n^{-it}$ by $e^{-i\theta_n}$ with a total error $O(N^{-1/4})$ where $\theta_n$ $(1 \leqslant n \leqslant N)$ are arbitrary real numbers by the following:

LEMMA. *For all sufficiently large $N$ and any real numbers $\theta_n$ $(1 \leqslant n \leqslant N)$ there exists a $t$ in the range $N \leqslant t \leqslant \operatorname{Exp}(N^6)$ for which*

$$\cos(t\log p_n - \theta_n) \geqslant 1 - 4/N$$

*holds for all $n$ $(1 \leqslant n \leqslant N)$.*

Proof. (See Lemma $\delta$ on page 162 of Titchmarsh's book.) His version is the case when $\theta_n = \pi$ for all $n$. The proof of this lemma is the same as that of Lemma $\delta$, starting with

$$F(t) = 1 + \sum_{n\leqslant N} \operatorname{Exp}(it\log p_n - i\theta_n).$$

The rest of the proof of this lemma is very similar.

By this lemma, we have

$$\sum\nolimits_1 = \{\sum_{n\leqslant M} z\log(1-p_n^{-1}e^{-i\theta_n})^{-1} + i\theta\} + O\left(\frac{A\log M}{\log N} + N^{-1/4}\right),$$

$$\sum\nolimits_2 = \{\sum_{M<n\leqslant N} z\log(1-p_n^{-\sigma}e^{-i\theta_n})^{-1}\} + O(N^{-1/4}).$$

We put $\theta_n = \phi - \beta$, for $n \leqslant M$, where $\beta$ is chosen as follows. Note that

$$f(\beta) = \operatorname{Im}\{i\theta + \sum_{n\leqslant M} re^{i\phi}\log(1-p_n^{-1}e^{-i\phi+i\beta})^{-1}\}$$

equals $O(\sum_{n\leqslant M} rp_n^{-2}) = O(1)$ where $\beta = 0$ and when $\beta = \pi/2$ it is $r\sum_{n\leqslant M} p_n^{-1} + O(1)$. Thus for a large constant $M$ it is possible to choose $\beta$ in such a way that $f(\beta) = 2\pi l$ where $l$ is an integer. Clearly

$$\operatorname{Re}\sum_{n\leqslant M} re^{i\phi}\log(1-p_n^{-1}e^{-i\phi+i\beta})^{-1} = O(\log\log M).$$

Now we put $\theta_n = \phi$ for $M < n \leqslant N$ and

$$g = \operatorname{Re}\sum_{M<n\leqslant N} re^{i\phi}\log(1-p_n^{-\sigma}e^{-i\phi})^{-1}.$$

Clearly $g = r\log\log N + O(A + \log\log M)$. Also putting

$$h = \operatorname{Im}\sum_{M<n\leqslant N} re^{i\phi}\log(1-p_n^{-\sigma}e^{-i\phi})^{-1},$$

we have $h = O(\sum_{n>M} p_n^{-2}) = O(M^{-1})$. Collecting our results and taking exponentials in (1) we obtain

(2) $$G(\sigma+it) = \operatorname{Exp}(\sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3) = \operatorname{Exp}(r\log\log N + k)\operatorname{Exp}(J)$$

where $k$ is real and is $O(A+\log\log M)$ and

$$J = O\left(\frac{A\log M}{\log N}+N^{-1/4}+M^{-1}+e^{-A}A^{-1}\right).$$

We now take real parts in (2) and observe that

$$\operatorname{Exp}(r\log\log N) = (\log N)^r \gg (\log\log t)^r$$

by the inequality $\operatorname{Exp}(N^6) \geq t$. Also $t = t_\nu \to \infty$ since $t \geq N$.

This completes the proof of the statement made at the beginning of the proof and hence that of the theorem.

**Note added in proof.** By a modification of Lemma $\delta$ it is possible to prove by the method of the appendix that if $C\log\log T \leq H \leq T$, then

$$\max_{T\leq t\leq T+H} (\operatorname{Re} G(1+it))$$

exceeds a positive constant times $(\log\log H)^r$, where $C = C(r, \theta, \phi) > 0$. Moreover on Riemann hypothesis (or quasi-Riemann hypothesis), $C\log\log T$ can be replaced by $C\log\log\log T$.

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# Zum Ellipsoidproblem in algebraischen Zahlkörpern

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**1. Einleitung.** In [7] habe ich gezeigt, daß in einem total reellen algebraischen Zahlkörper vom Grade $n$ für die Anzahl $A_k(x; \mathfrak{a})$ der $k$-Tupel $(\nu_1, \ldots, \nu_k)$ von Zahlen eines Ideals $\mathfrak{a}$, deren Konjugierte dem Ungleichungssystem

$$(1.1) \qquad (\nu_1^{(p)})^2+\ldots+(\nu_k^{(p)})^2 \leq x_p \quad (x_p > 0;\ p = 1, \ldots, n)$$

genügen, folgende Asymptotik gilt:

$$(1.2) \qquad A_k(x; \mathfrak{a}) = \omega_k^n\left(\frac{X}{dN(\mathfrak{a})^2}\right)^{k/2} + O(X^{\frac{k}{2}-\frac{k}{n(k-1)+2}+\delta})$$

für $X = x_1 \ldots x_n \geq 1$ und jedes $\delta > 0$; hier bezeichnet

$$(1.3) \qquad \omega_k = \pi^{k/2}/\Gamma(\tfrac{1}{2}k+1)$$

das Volumen der $k$-dimensionalen Einheitskugel, $d$ ist die Diskriminante des Körpers und $N(\mathfrak{a})$ die Norm von $\mathfrak{a}$.

Für frühere Ergebnisse im Fall $k = 2$ (Kreisproblem) siehe Schaal [9], [10].

Die vorliegende Arbeit(1) erweitert obige Problemstellung in dreierlei Hinsicht:

- Der zugrundeliegende Zahlkörper braucht nicht total reell zu sein.
- An die Stelle der $k$-dimensionalen Kugeln (1.1) treten Ellipsoide von beliebiger Gestalt und Lage, deren Mittelpunkte insbesondere keine Gitterpunkte zu sein brauchen.
- Jede der Zahlen $\nu_j$ durchläuft ein eigenes Ideal $\mathfrak{a}_j$ $(j = 1, \ldots, k)$.

Es wird eine obere Abschätzung des Gitterrests erzielt, die im eingangs erwähnten Spezialfall die Gleichung (1.2) dahingehend verschärft, daß dort auch $\delta = 0$ zugelassen ist, und die sich im Falle des rationalen Zahlkörpers auf

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