A Dirichlet series
for Hermitian modular forms of degree 2

by

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Introduction. In a recent paper [5], Kohnen and Skoruppa, using the
Rankin-Selberg method, investigated a novel Dirichlet series $D_{F,G}(s)$ associated
with a pair $F, G$ of Siegel cusp forms of integral weight for the modular group
$\text{Sp}_2(\mathbb{Z})$ of degree 2; prompted by the form of a functional equation it satisfies,
they established its “proportionality” with the “spinor” zeta function $Z_F(s)$
attached to $F$ by Andrianov, whenever $F$ is a Hecke eigenform and $G$ is in the
Maass space.

An analogous zeta function $Z_F(s)$ (but with “Euler factors of degree 6”) for
eigen cusp forms $F$ of weight $k$ with respect to the Hermitian modular group $\Gamma_2$
of degree 2 over $\mathcal{O} := \mathbb{Z}[\sqrt{-1}]$ has been studied by Gritsenko [2], [3]. Using
the results of Kojima [6] on the “Saito–Kurokawa descent” $F \mapsto f$ for $F$ in the
relevant Maass space, Gritsenko [2] proved (for $k$ divisible by 4) the identity

$$Z_F(s) = \zeta(s-k+1) \cdot L\left(s-k+2, \left(\frac{-4}{k}\right)\right) \cdot \zeta(s-k+3) \cdot R_F(s)$$

where $R_F$ is the (Rankin) symmetric square zeta function associated to $f$ and
$L(\cdot, \left(\frac{-4}{k}\right))$ is a Dirichlet $L$-series for the character $\left(\frac{-4}{k}\right)$.

This article is concerned with an analogue of $D_{F,G}$ for Hermitian cusp
forms of weight $k$ (divisible by 4) for $\Gamma_2$ and its possible relationship with $Z_F$.
Even when $F$ and $G$ are both in the Maass space, we find that $D_{F,G}$ and $Z_F$ are
no longer “proportional”, unlike in [5]. Although both $D_{F,G}(s)$ and $Z_F(s)$ admit
functional equations under $s \mapsto 2k - 3 - s$, one faces at the same time a disturbing
difference in the respective $F$-factors involved therein. Nevertheless, we
can prove that

$$D_{F,G}(s) = \text{constant} \times \frac{\zeta(s-k+2)}{L(s-k+2, \left(\frac{-4}{k}\right))} Z_F(s).$$

For a fundamental operator identity needed in the arguments, §4 provides
a proof (quite algebraic and) different from the rather sketchy proof in [5] for
the corresponding identity there; of course, both these identities have the same
structure. We also need to work out anew, for the “Saito–Kurokawa descent,” detailed arguments in regard to the aspect of its Hecke-equivariance which is somewhat slurred over in [6] and barely mentioned in [2].

1. Notation and terminology. For any complex matrix \( P = (p_{ij}) \), let \( \overline{P} := (\overline{p_{ij}}) \) where the bar denotes complex conjugation in \( \mathbb{C} \); let \( \mathcal{P} \) denote the transpose of \( P \). For a square matrix \( Q \) over \( \mathbb{C} \), let \( \det(Q) \) and \( |Q| \) stand respectively for the trace of \( Q \), determinant of \( P \) and absolute value of det \( P \).

For any \( (2, 2) \) matrix \( P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) we define \( P^* = \begin{pmatrix} \overline{c} & -\overline{b} \\ -\overline{a} & \overline{d} \end{pmatrix} \). Diag \( (a_1, \ldots, a_n) \) means \( (a, n) \) diagonal matrix with \( a_1, \ldots, a_n \) as its entries in that order. For any (commutative) ring \( R \), let \( \mathcal{M}_n(R) \) denote the ring of all \( n \times n \) matrices with entries from \( R \). We denote the \( (n, n) \) identity matrix in \( \mathcal{M}_n(\mathbb{C}) \) by \( E_n \); by 0, we mean a matrix of the appropriate size with all entries equal to 0. Whenever the product \( G \cdot H \) of complex matrices \( G, H \) is defined, we write \( GH \) for the same. For \( t \in \mathbb{N} \), let \( \mathcal{H}_n^{(2)}(\mathbb{Z}) := \{ M \in \mathcal{M}_n(\mathbb{Z}) \mid \det M = t \} \) and \( \Gamma_1 := \mathcal{H}_1^{(2)}(\mathbb{Z}) = \text{SL}(2, \mathbb{Z}) \), the elliptic modular group. For any matrix \( M \) with entries from \( \mathbb{Z} \), let \( \text{gcd}(M) \) denote the greatest common divisor of the entries of \( M \). If \( n \in \mathbb{N} \) and \( n = m^2 \) for \( m \in \mathbb{Z} \), we write \( n = 0 \).

The ring \( \mathbb{Z}[\sqrt{-1}] \) of algebraic integers in \( K := \mathbb{Q}(\sqrt{-1}) \) is denoted by \( \mathbb{Z} \); by \( \sqrt{-1} \), we always mean that square root with argument \( \pi/2 \). For \( a \in \mathbb{K} \) its “trace” \( a + \overline{a} \) is denoted by \( \text{tr}(a) \). Moreover, for \( z = (z_1, z_2) \in \mathbb{C} \times \mathbb{C} \) and \( a \in \mathbb{K} \), we shall simply write \( \text{tr}(a z) \) for \( a z_1 + \overline{a} z_2 \), without risk of confusion. By a positive-definite matrix \( H \), we mean a complex matrix \( H = H^* \) with all its eigenvalues positive and we then write \( H > 0 \). For any \( \gamma \in \mathcal{E} \), \( e(\gamma) \) stands for \( 2\pi i \text{tr}(\gamma) \).

Let \( \mathcal{H} \) denote the upper half-plane \( \{ \tau = u + \sqrt{-1} v \mid u, v \in \mathbb{R}, v > 0 \} \). For \( k \in \mathbb{N} \) and any Dirichlet character \( \chi \) (on \( \mathbb{Z} \)) modulo \( N \), the space of holomorphic cusp forms of weight \( k \) and character \( \chi \) for the congruence subgroup \( \Gamma_0(N) := \{ \gamma \in \Gamma_1 \mid \chi(\gamma) = 1 \} \) is denoted by \( S_k(\Gamma_0(N)) \).

The generalized upper half-plane \( \{ Z \in \mathcal{H} \mid -1 \leq \text{Re}(Z) < 0 \} \) is denoted by \( \mathcal{H} \). The Hermitian modular group \( \Gamma_2 \) of degree 2 over \( \mathcal{E} \) consists of all \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{H}(\mathbb{E}) \) with \( (2, 2) \) matrices \( A, B, C, D \) for which \( A^* = B^*, C^* = D^* \) and \( A^*D - B^*C = E_2 \). For any such \( M \) in \( \Gamma_2 \) and \( Z \in \mathcal{H} \), let \( M(Z) = (AZ + B)(CZ + D)^{-1} \). The analytic homeomorphisms \( \mathbb{E} \to \mathbb{E} \) for \( M \) in \( \Gamma_2 \) give a representation of \( \Gamma_2 \) as a discontinuous group of automorphisms of \( \mathcal{H} \) and we denote \( \mathcal{H} \) a standard fundamental domain for \( \Gamma_2 \) in \( \mathcal{H} \).

Writing \( X := \frac{1}{2}(Z + \overline{Z}) \) and \( Y := \text{Im}(Z) := \frac{1}{2} \text{Im}(Z) \in \mathbb{R} \) for \( Z \in \mathcal{H} \), we have on \( \mathcal{H} \) an invariant volume element \( dV = X^* dX dY \). A complex-valued function \( F \) holomorphic on \( \mathcal{H} \) is called a modular form of integral weight \( k \) for \( \Gamma_2 \), if \( F(M(Z)) \det(CZ + D)^{-k} = F(Z) \) for every \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2 \). Such an \( F \) has a Fourier expansion

\[
F(Z) = \sum_{T} a(T) e(\sigma(TZ))
\]
with the accent on the first infinite product indicating that \( p \) (in \( N \)) runs over all primes for which \( p^\ell \) is prime in \( \mathcal{O} \), while the second product is extended over all the remaining primes from \( N \). Moreover, in the foregoing, the polynomial \( Q_p^{(2)}(t) \) with \( t \) indeterminate is defined by \( Q_p^{(2)}(t) F = Q_p^{(2)}(t) F \) for the following explicitly given operator-valued polynomials \( Q_p^{(2)}(t) \) involving the standard operators

\[
T_2 := \Gamma_2 \text{Diag}(1, 1, p, p) \Gamma_2, \quad T_{1,:} := \Gamma_2 \text{Diag}(1, p, p^2, p) \Gamma_2,
\]

\[
T_1 := \Gamma_2 \text{Diag}(1, p, p, \pi) \Gamma_2 \quad \text{for} \quad p = \pi \pi \text{ in } \mathcal{O},
\]

\[
A_{\lambda} := \Gamma_2 (\lambda E_2) \Gamma_2 \quad \text{for} \quad \lambda \in \mathcal{O}
\]
in the Hecke ring for \( \Gamma_2 \):

(i) \( Q_p^{(2)}(t) := -t + (pT_1 + p^2T_2 + p^3T_3 + p^4T_4) t^2 + p^5T_5 t^3 + p^6T_6 t^4 \)

for \( p^\ell \) prime in \( \mathcal{O} \), i.e., \( (\tau_p) = -1 \),

(ii) \( Q_p^{(2)}(t) := -t + (pT_1 + p^2T_2 + p^3T_3 + p^4T_4) t^2 + p^5T_5 t^3 + p^6T_6 t^4 \)

for \( p = \pi \pi \) in \( \mathcal{O} \), i.e., \( (\tau_p) = -1 \), and

(iii) \( Q_p^{(2)}(t) := 0 \) for \( p = \pi \pi \) in \( \mathcal{O} \).

We also know from [3] that if \( \xi(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s) \), then

\[
Z_p(s) := \pi^{-3s} \Gamma(s)(s-k+3) \Gamma^2 \left( \frac{s-k+3}{2} \right) \xi(s-k+2) Z_p(s) = Z_p(2-k-3-s).
\]

2. An Eisenstein series. Towards obtaining the meromorphic continuation and functional equation of \( D_{\mathcal{O}}(s) \), we first address ourselves precisely to these two questions but in the context of an Eisenstein series \( E_s(Z) \) on \( \mathcal{O} \), of the Klingen–Siegel type, relative to the subgroup \( \mathcal{O} \) of \( \Gamma_2 \). The latter is defined (cf. [4]) for \( Z \) in \( \mathcal{O} \) and \( s \) in \( \mathcal{C} \) with \( \Re(s) > 3 \), by

\[
E_s(Z) := \sum_{\pi \in \mathcal{O}} \det(M(Z)) \frac{1}{\Im(M(Z))_1}
\]

where \( (\Im(M(Z)))_1 \) is the first diagonal element of \( \Im(M(Z)) \). Let us rewrite the general term of the series in (5) in a more convenient form, for \( M = (\zeta \delta) \) in \( \Gamma_2 \) with \( (b \ a \ c \ d) \) as its last row. For any \((2, 2)\) matrix \( P \), we have \( P \delta \star P = \delta \star P \), and in particular, for \( P = Y = \Im(Z) \) and \( P = \mathcal{C} + D \), also we note that \( \mathcal{C} \star \mathcal{C} = \mathcal{C} + \sqrt{-1} \mathcal{Y} \) and further \( \overline{b} \mathcal{A} + \mathcal{A} \overline{b} = 0 \). It is now easy to check that

\[
\Im(M(Z)) = Y[(\mathcal{C} + D)^{-1}] = [(\mathcal{C} + D)^{-1}]^2 Y[(\mathcal{C} + D)^{-1}]
\]

\[
(\Im(M(Z)))_1 = [(\mathcal{C} + D)^{-1}] Y[\mathcal{C} \star \mathcal{C} + \mathcal{Y}][\mathcal{C} \star \mathcal{C} + \mathcal{Y}]
\]

Under \( M \to (0 \ 0 \ 1 \ 1) \), we can identify \( \mathcal{O} \) with

\[
\{(a, b, c, d) \in \mathcal{O}^4 \mid ac + \overline{b} \overline{c} = 0, ac + b \overline{c} = 0 \}\]

which is invariant with respect to \((b, d) \to (-b, -d)\). If we take \( y_1 := (a, b) \) and \( y_2 := (c, d) \) the defining conditions in (7) go over into

\[
i_{\delta_2} y_1 - i_{\delta_1} y_2 = 0, \quad g \text{ is "primitive" over } \mathcal{O}
\]

where, by a "primitive" column over \( \mathcal{O} \), we mean a column whose entries together generate \( \mathcal{O} \). Let \( S, H \) be \((4, 4)\) Hermitian matrices defined by

\[
S = \begin{pmatrix} 0 & \sqrt{-1} E_2 \\ -\sqrt{-1} E_2 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} Y^* & 0 \\ 0 & Y^* \end{pmatrix} \begin{pmatrix} E_2 \\ 0 \end{pmatrix}
\]

Then \( H > 0 \) and it is a "majorant" of the indefinite (Hermitian) matrix \( S \) of signature \((2, 2)\), i.e., \( HS^{-1} H = S \). After replacing \((b, d) \) by \((-b, -d)\) in (6), we see that

\[
(\Im(M(Z)))_1 \frac{1}{\Im(M(Z))_1} = H[g]
\]

with, of course \( g \) "primitive" and \( S[g] = 0 \), by (8). Any non-zero column over \( \mathcal{O} \) becomes "primitive" under multiplication by \( \lambda^{-1} \) for a suitable \( \lambda \neq 0 \) from \( \mathcal{O} \); thus, from (5), (10) and multiplication by \( \zeta_k(s) \) to eliminate the inconvenient "Primitivity" condition in (8), we obtain

\[
(\Im(M(Z)))_1 \frac{1}{\Im(M(Z))_1} = \sum_g (H[g])^{-s}
\]

where the accent indicates that \( g \) runs over all non-zero \((4, 1)\) columns over \( \mathcal{O} \) with \( S[g] = 0 \). This formula provides a link with a theta series associated with \( S \) and \( H \) and the possibility of using towards our own objectives, its (transformation) properties, by exhibiting (11) as a "transform" of the theta series; tackling the condition \( S[g] = 0 \) has to be done at the same time, by adopting an idea from [3].

For \( S, H \) in (9) and \( \tau = u + \sqrt{-1} v \in \mathcal{O} \), with \( u, v \in \mathcal{R} \), let us consider the theta series

\[
\theta(\tau) = \theta(\tau, Z) := \sum_{n \in \mathcal{O}} e(\frac{i}{4} u S[n] + \sqrt{-1} v H[n])
\]
where \( g \) runs over all \((4, 1)\) columns over \( \mathcal{O} \); the absolute convergence of the series is ensured by \( "H > 0" \). There exists an invertible \((4, 4)\) complex matrix \( V \) such that \( H = E_4[V] \) and \( S = D[V] \) with \( D \) diagonal; also \( D^2 = E_4 \), in view of the relation \( (HS^{-1})^2 = E_4 \). Since \( S \) has "signature" \((2, 2)\), we may, after multiplying \( V \) on the left by an appropriate permutation matrix, assume already that \( D = \text{Diag}(1, 1, -1, -1) \). If now \( K = K(\tau) = vH - \sqrt{-1}uS \), then it is easily verified that

\[
K^{-1} = \frac{1}{v^3 + u^3} H^{-1} + \sqrt{-1} \frac{1}{v^3 + u^3} S^{-1} = (K(-1/\tau))[S^{-1}], \quad \det K = (u^3 + v^3)^2.
\]

We are thus led to the theta transformation formula

\[
\theta(-1/\tau, Z) = \sum_g \exp(-\pi K(-1/\tau)[g]) = \sum_{\tau \in \text{GL}(4, \mathcal{O})} \exp(-\pi K^{-1}[g])
\]

\[
= (\text{det} K) \sum_{\tau \in \text{GL}(4, \mathcal{O})} \exp(-\pi K[g]) = |\tau|^4 \theta(\tau, Z).
\]

This transformation formula can also be proved by going over to the theta series associated with the \((8, 8)\) matrix \( vP - \sqrt{-1}uQ \) and the lattice \( Z^8 \) (in lieu of \( \mathcal{O}^4 \) as above) where

\[
2P = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad 2Q = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad A := \begin{pmatrix} E_4 & E_4 \\ -E_4 & -E_4 \end{pmatrix}
\]

and applying the well-known theta transformation formula from [(16), §2].

Now, for any \((4, 1)\) column \( g \) over \( \mathcal{O} \), \( S[g] \) is always in \( 2Z \) and \( S(\tau + 1) = S(\tau) \). This together with the transformation formula (12) implies the invariance of \( \theta^2 \theta(\tau, Z) \) for every substitution \( \tau \rightarrow (\tau + b)(\tau + d)^{-1} \) from \( \Gamma_1 \). Consequently, the same invariance holds good also for

\[
\theta(\tau, Z) = L(\theta^2 \theta(\tau, Z)) + 2\theta^2 \theta(\tau, Z)
\]

where \( L \) is the invariant differential operator \(-v^2(\partial^2/\partial u^2 + \partial^2/\partial v^2)\). Writing \( s, \beta \) for \( H[g], S[g] \) respectively and taking \( f(u, v) = \exp(-\pi u^2 + \pi v^2) \), it is immediate that

\[
L(v^2f(u, v)) + 2v^2f(u, v) = (4\pi v^2 - \pi^2(\alpha^2 - \beta^2)^2 + \beta^2) f(u, v).
\]

For \( s \) in \( C \) with large enough \( \text{Re}(s) \), integration over a standard fundamental domain for the subgroup \( \Gamma_{1, \infty} := \{ \gamma \in \Gamma_1 \} \) in \( \mathcal{X} \) yields

\[
\left[ \int_0^1 \int_0^1 \Theta(\tau, Z) v^{-2} du dv \int_0^1 \int_0^1 \sum (4\pi H[g]v^3 - \pi^2(H[g]^2 - S[g]^2)v^4) \exp(-\pi u^2/2) + 2\pi \sqrt{-1}uv \} \right] dv
\]

\[
= \sum_{\tau \in \text{GL}(4, \mathcal{O})} \exp(-\pi u^2/2) \times (4\pi H[g]v^2 + \pi^2 \sqrt{-1}uv) \right] dv, \quad \text{by (13)}
\]

\[
= \sum_{\tau \in \text{GL}(4, \mathcal{O})} \exp(-\pi u^2/2) \times (4\pi H[g]v^2 + \pi^2 \sqrt{-1}uv) \right] dv
\]

\[
= \pi^{-8} \Gamma(s-3) \Gamma(s) \zeta(s) \theta(s, Z), \quad \text{in view of (11)}
\]

On "folding back" the domain of integration for the left-hand side to recover the (usual) fundamental domain \( \mathcal{F}_1 \) for \( \Gamma_1 \) in \( \mathcal{X} \), we see that the left-hand side is nothing but

\[
\int \frac{\theta(\tau, Z) E_{r_1}(\tau, 2s-2) v^{-2} du dv}{\zeta(s)}
\]

where

\[
E_{r_1}(\tau, \varphi) := v^{\alpha_2} \sum (\tau^r)_{\varphi, \mathcal{X}, r} \cdot \varphi
\]

is the usual Eisenstein series for \( \Gamma_1 \). It is well known that \( \pi^{-s} \Gamma(s/2) \zeta(s) \theta(s, Z) \) is holomorphic in the entire \( \varphi \)-plane except for simple poles at \( \varphi = 0, 2 \) and is further invariant under \( \varphi \rightarrow \varphi + 2 \). For \( \tau \in \mathcal{F}_1 \) and \( s \) in any compact set, we have

\[
|E_{r_1}(\tau, 2s-2)| \leq c_1 v^s, \quad |\theta(\tau, Z)| \leq c_2 \exp(-c_3 v)
\]

for suitable constants \( v, c_1, c_2, c_3 \) independent of \( \tau \). These facts give rise (as usual) to the required meromorphic continuation as well as a functional equation under \( s \rightarrow 3-s \) (or correspondingly \( 2s-2 \rightarrow 4-2s \)). Multiplying the expression in (14) by \( \pi^{-s} \Gamma(s-1) \zeta(2s-2) \), we see that

\[
\pi^{-s} \Gamma(s) \Gamma(s-1) \zeta(s) \zeta(2s-2) \theta(\tau, Z)
\]

is holomorphic in the entire \( s \)-plane except for possible simple poles at \( s = 1, 2 \) and is further invariant under \( s \rightarrow 3-s \). We have hence the following

**Lemma 1.** The function

\[
E_{r_1}(Z) := \pi^{-s} \Gamma(s) \Gamma(s-1) \zeta(s) \zeta(2s-2) \theta(\tau, Z)
\]

admits analytic continuation in \( s \) to all of \( \mathbb{C} \) and is holomorphic except for possible simple poles at \( s = 0, 1, 2, 3 \). Moreover, \( E_{r_1}(Z) = E_{r_1}(Z) \zeta(s) \theta(s, Z) \)

**Remark.** For \( Z \) in \( \mathcal{F}_2 \) (and hence, due to \( \Gamma_2 \)-invariance, also for all \( Z \) in \( \mathcal{X} \)), \( E_{r_1}(Z) \) behaves at most like a power of \( \text{det} \{\text{Im}(Z)\} \) as the latter goes to infinity.
3. The Dirichlet series $D_{F,G}(s)$. For any two holomorphic cusp forms $F$, $G$ of integral weight $k$ for the Hermitian modular group $\Gamma_2$ over $C$, we associate, following [5], a Dirichlet series $D_{F,G}(s)$ of the Rankin–Selberg type. Namely, if

$$F(Z) = \sum_{m \in \mathbb{N}} \varphi_m(t, z_1, z_2)e(mt) \quad \text{and} \quad G(Z) = \sum_{m \in \mathbb{N}} \psi_m(t, z_1, z_2)e(mt),$$

we first consider, for $s$ in $C$ with $\Re(s) > k + 1$, the Dirichlet series

$$\sum_{m \in \mathbb{N}} \langle \varphi_m, \psi_m \rangle m^{-s}$$

whose absolute convergence is immediate from $\langle \varphi_m, \psi_m \rangle = O(m^s)$. To get this estimate, we start from the formula $\varphi_m(t, z_1, z_2) = \int_{T_2} F(Z)e(-mt)\,dz$ with $\gamma = -1$. We then choose $c = (z_1 - z_2)^2/4v$ and $m/n = 1$: and note that for all $Z$ in $H_2$, $(\det Y)\psi_m(Z) = ((v - |z_1 - z_2|^2/4v - |z_2|^2/4v)^{1/2} F(Z)$ is bounded where $v := \Im(t), \, t' := \Im(t')$. Then we choose $c = (z_1 - z_2)^2/4v) + 1/m$ and are led from the foregoing, to the estimate

$$\varphi_m(t, z_1, z_2) = O((m/n)^{-k/2} \exp(vm |z_1 - z_2|^2/2v))$$

and eventually to conclude that

$$\varphi_m(t, z_1, z_2) \psi_m(t, z_1, z_2) = O((m/n)^{-k/2} \exp(-vm |z_1 - z_2|^2/2v)) = O(m^s) \quad \text{on } \mathcal{S}_2,$$

yielding the estimate asserted for $\langle \varphi_m, \psi_m \rangle$ at once.

For $s$ in $C$ with $\Re(s) > k + 1$, we can now define

$$D_{F,G}(s) = \zeta(s-k-3)\xi(2s-2k+4) \sum_{m \in \mathbb{N}} \langle \varphi_m, \psi_m \rangle m^{-s}$$

and proceed to state

**Theorem 1.** The Dirichlet series $D_{F,G}(s)$ associated to $F, G$ in $S_k(\Gamma_2)$ can be continued meromorphically to the entire s-plane. The function

$$D_{F,G}(s) := \frac{1}{\Gamma(s-k-3)}(2s-2k+4) \sum_{m \in \mathbb{N}} \langle \varphi_m, \psi_m \rangle m^{-s}$$

is holomorphic in $s$ except for possible simple poles at $s = k - 3, k - 2, k - 1, k$ and satisfies the functional equation $D_{F,G}(s) = D_{G,F}(2k-3-s)$.

**Proof.** Clearly $D_{F,G}(s)$ is holomorphic for $\Re(s) > k + 1$. To continue it meromorphically to the left, we first note that for $F, G$ in $S_k(\Gamma_2)$ and $\Re(s) > k + 1$,

$$\langle FE, G \rangle := \int_{\mathcal{S}_2} (\det Y)^s F(Z) E_2(Z) G(Z) (\det Y)^{-s} \, dX \, dY$$

is well-defined and by the (definition of $E_2(Z)$ and the) usual “unfolding argument”, it is equal to

$$\sum_{m \in \mathbb{N}} (\varphi_m(Z) e(mt)^{-s} \, dX \, dY.$$

Now, a fundamental domain for $\mathcal{C}$ in $H_2$ is given by the set

$$\left\{ \begin{array}{l}
Z = (t, z_1, z_2) \in H_2 \mid (t, z_1, z_2) \in \mathcal{F}, \\
t' = t' + \sqrt{-1} t', \, t' > |z_1 - z_2|^2/4v, \, 0 \leq u' \leq 1
\end{array} \right\}.$$}

Hence $\langle FE, G \rangle$ equals

$$\sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{N}} \langle \varphi_m, \psi_n \rangle (2s-2k+4) \sum_{m \in \mathbb{N}} \langle \psi_n, \chi_m \rangle m^{-s}$$

yielding the estimate asserted for $\langle \psi_n, \chi_m \rangle$ at once.

For $s$ in $C$ with $\Re(s) > k + 1$, we can now define

$$D_{F,G}(s) = \zeta(s-k-3)\xi(2s-2k+4) \sum_{m \in \mathbb{N}} \langle \varphi_m, \psi_m \rangle m^{-s},$$

as the interchange of the summation over $m$ and the integration over $\mathcal{S}_2$ is easily justified. From the “rapid decay at infinity” of $F$ and $G$ and the “polynomial growth” of $E_2$ in $H_2$ (see the Remark following Lemma 1), the left-hand side of (15) represents a meromorphic function of $s$ all over $C$ and provides the meromorphic continuation required for $D_{F,G}$, as well. The asserted properties of $D_{F,G}(s) = \pi^{-k} \langle FE_2, G \rangle$ by (15) including its functional equation under $s \rightarrow 2k-3-s$ are now immediate, proving Theorem 1.

4. A proposition. This section is devoted to the proof of a basic identity (needed in §6) involving certain operators $\mathcal{Y}$ and $\mathcal{S}$ in the spaces $J_{2,1}^0$. For $s \in \mathbb{N}$, the operators $\mathcal{Y}; \mathcal{J}_{2,1}^0 \rightarrow \mathcal{J}_{2,1}^0$ and $\mathcal{S}; \mathcal{J}_{2,1}^0 \rightarrow \mathcal{J}_{2,1}^0$ are defined as follows: namely, for any $\varphi$ in $\mathcal{J}_{2,1}^0$ (cf. [1], [6]),

$$\langle \varphi, \mathcal{Y} \rangle(t, z_1, z_2) := \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{N}} \varphi(mz_1 + \lambda z_2) \psi(mz_1 + \lambda z_2) e \left( -\frac{mz_1 + \lambda z_2}{\lambda z_1 + \lambda z_2} \right) \left( \frac{mz_1 + \lambda z_2}{\lambda z_1 + \lambda z_2} \right)^{-s},$$

$$\langle \varphi, \mathcal{S} \rangle(t, z_1, z_2) := \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{N}} \varphi(mz_1 + \lambda z_2) \psi(mz_1 + \lambda z_2) e \left( -\frac{mz_1 + \lambda z_2}{\lambda z_1 + \lambda z_2} \right) \left( \frac{mz_1 + \lambda z_2}{\lambda z_1 + \lambda z_2} \right)^{-s} \times \sum_{\mathcal{R}_1(\mathcal{G}_0) = 0} (\varphi(\mathcal{R}_1 \mathcal{G}_0)(t, z_1 + \lambda z + \mu, z_2 + \lambda z + \mu)).$$
For the representatives $S, R$ here we can choose ones in the triangular form \((0, \cdot)\). The definition of the operator \(\mathcal{F}_l^*\): \(J_{k,1} \rightarrow J_{k,1}^*\) is the same as that of \(\mathcal{F}_l\) except that the condition \(\gcd(R) = \square\) is replaced by \(\gcd(R) = 1\). We have the formula for \(\mathcal{F}_l^*\) and \(\mathcal{F}_l^*\) the identity \(\mathcal{F}_l^* = \sum_{l} a_l d^{k-2} \mathcal{F}_{l,1}^*\) just as in [1].

For the adjoint operator \(\mathcal{F}_l^{**}\): \(J_{k,1}^* \rightarrow J_{k,1}^{**}\) corresponding to \(\mathcal{F}_l\), we have the formula

\[
(\psi \mathcal{F}_l^*)(x, y) = \psi \frac{A}{B} \sum_{1 \leq l \leq L} \sum_{\lambda \mod \ell} \sum_{\sigma \mod \ell} \sum_{\nu \in \mathcal{F}_l} \sum_{\mu \mod \ell} \sum_{\nu \mod \ell} \sum_{\omega \mod \ell} \sum_{\theta \mod \ell} \sum_{\pi \mod \ell} \sum_{\sigma \mod \ell} \sum_{\nu \mod \ell}
\]

\[
\sum_{\lambda \mod \ell} \psi \left(\frac{A}{B} \sum_{1 \leq l \leq L} \sum_{\lambda \mod \ell} \sum_{\sigma \mod \ell} \sum_{\nu \in \mathcal{F}_l} \sum_{\mu \mod \ell} \sum_{\nu \mod \ell} \sum_{\omega \mod \ell} \sum_{\theta \mod \ell} \sum_{\pi \mod \ell} \sum_{\sigma \mod \ell} \sum_{\nu \mod \ell}
\]

We omit its proof, since it is on the same lines as in [5].

The following proposition deals with an identity connecting \(\psi \mathcal{F}_l^*\) and \(\psi \mathcal{F}_l^*\) with the Hecke operators \(\mathcal{F}_l\) on \(J_{k,1}\) and it corresponds to assertion (ii) of a proposition due to Kohnen and Skoruppa ([5], p. 549), where, however, a good part of the details in the proof has been left to the reader. Our proof is different and algebraic in nature.

**Proposition.** For \(l \in \mathbb{N}\),

\[
\psi \mathcal{F}_l^* \psi \mathcal{F}_l^* = \sum_{1 \leq l \leq L} \psi \frac{A}{B} \sum_{1 \leq l \leq L} \sum_{\lambda \mod \ell} \sum_{\sigma \mod \ell} \sum_{\nu \in \mathcal{F}_l} \sum_{\mu \mod \ell} \sum_{\nu \mod \ell} \sum_{\omega \mod \ell} \sum_{\theta \mod \ell} \sum_{\pi \mod \ell} \sum_{\sigma \mod \ell} \sum_{\nu \mod \ell}
\]

where, for any \(r \in \mathbb{N}\),

\[
\psi(r) = \frac{r}{\prod_{p \mid r} (1 + 1/p)}
\]

the product being extended over all primes \(p\) dividing \(r\).

**Proof.** It is clear from the definition \(\psi \mathcal{F}_l^*\) (taking the representatives \(S\) in upper triangular form as is more convenient) that for \(l = l_1, l_2\) and \((l_1, l_2) = 1\), \(\psi \mathcal{F}_l^* \psi \mathcal{F}_l^* = \psi \mathcal{F}_l^* \psi \mathcal{F}_l^*\). Moreover, the mapping \(l \mapsto \psi \mathcal{F}_l \psi \mathcal{F}_l\) from \(N\) to \(N\) is multiplicative. It is not hard to verify directly from the definitions that, for \((l_1, l_2) = 1\), \(\psi \mathcal{F}_l \psi \mathcal{F}_l\) commutes both with \(\psi \mathcal{F}_l\) and with \(\mathcal{F}_l\) (cf. [1], p. 51). Thus it suffices to prove the proposition for \(l = p^k\) for any given prime number \(p\); we assume \(l = p^k\) in the sequel. From \(16\) and \(17\), we have

\[
(\psi \mathcal{F}_l^* \psi \mathcal{F}_l^*) (x, y) = \sum_{k=0}^{k} \sum_{\lambda \mod \ell} \sum_{\sigma \mod \ell} \sum_{\nu \in \mathcal{F}_l} \sum_{\mu \mod \ell} \sum_{\nu \mod \ell} \sum_{\omega \mod \ell} \sum_{\theta \mod \ell} \sum_{\pi \mod \ell} \sum_{\sigma \mod \ell} \sum_{\nu \mod \ell}
\]

where \(\mathcal{F}_l^* \psi \mathcal{F}_l^*\) signifies that the sets involved occur with the indicated multiplicity * and \(\mathcal{F}_l^* \psi \mathcal{F}_l^*\) (a, b) = \(\{A | A \in \mathcal{F}_l^* (a, b)\}\). The left-hand side is just \(\{M = (p_1^{a_1} p_2^{b_1} \cdots p_n^{c_n}) | 0 \leq a < p_1^{-1}, 0 < b < p_2^{-1}\}\). First, when \(r = s = n\), we see that the general element \(M\) becomes \(\psi(p_1^{a_1} p_2^{b_1} \cdots p_n^{c_n})\) (modulo factors from \(\mathcal{F}_l\) on the left) this covers \(p^r \mathcal{F}_l \mathcal{F}_l\) exactly once, for each fixed \(b\), as \(a\) runs over \(Z/p^r Z\). Hence, for \(r + s = n\),

\[
\mathcal{F}_l^* \psi \mathcal{F}_l^* = \sum_{0 \leq a < p_1^{-1}} \sum_{0 \leq b < p_2^{-1}} \sum_{0 \leq \cdots} \sum_{0 \leq \cdots}
\]

Proving \(19\) in this case. When \(r + s < n\) (respectively \(r + s > n\), we have

\[
M = p \left(p^{p_1^{a_1} p_2^{b_1} \cdots p_n^{c_n}} + b\right), \quad \text{resp.} \quad M = p^{p_1^{a_1} p_2^{b_1} \cdots p_n^{c_n}} + b\]

Writing \(b = b_1 + p^{r+s} b_2\) with \(b_1 \in Z/p^r Z\) and \(b_2 \in Z/p^r Z\), \(p^{r+s} + b\)
\[ (b_1 + a p^{x-s-r}) + b_2 p^{x-s-r} = c + b_2 p^{x-s-r} \]
for each fixed \(b_2\), the elements of \(Z/p^{2n-2r-s}Z\) once, as \(a\) and \(b_1\) run respectively over \(Z/p^{2n-2r-s}Z\) and \(Z/p^{x-s-r}Z\). Thus, for \(r+s < n\),
\[ \mathcal{S}_{r,s} = \prod_{r+s < n} p' \mathcal{R}(s, 2n-r-s). \]

If \(r+s > n\) on the other hand, \(a + b p^{x-s-r}\) covers \(Z/p^{x-s-r}Z\) once along with \(a\),
for each fixed \(b\) from \(Z/p^{x-s-r}Z\) and hence
\[ \mathcal{S}_{r,s} = \prod_{r+s > n} p' \mathcal{R}(2s+n-r-s, n-r). \]

The proof of (19) is complete, since \(\mathcal{R}(e, f) = \prod_{0 \leq g \leq e} p' \mathcal{R}(e-g, f-g)\) for \(e < f\).

Remark. The second relation in (19) can formally be obtained from the first by using \((r, s, u) \rightarrow (n-s, n-r, v)\) followed by \(\mathcal{R}^*(a, b) \rightarrow \mathcal{R}^*(b, a)\). Thus \(\mathcal{S}_{r', s'}\) with \(r'+s' > n\) are determined already by \(\mathcal{S}_{r,s}\) with \(r+s < n\).

For \(r^2-2 \sum_{s, t} \varphi_{k,1} s, t, s, t, t\) in (18), we get by (19) the expression
\[ \sum_{r^2-2} p' \left[ \sum_{r+s < n} \varphi_{r+s} \mathcal{R}^*(s, 2n-r-s) + \sum_{r+s > n} \varphi_{r+s} \mathcal{R}^*(n-s-n-r, n-r) \right], \]
where \(M_1\) runs over \(\mathcal{R}^*(s-u, 2n-2r-s-u)\) and \(M_2\) over \(\mathcal{R}^*(2r+2s-n-r'-u', n-r'-u')\). Under \((r', s') \rightarrow (n-s, n-r)\), the condition \(r'+s' > n\) goes over precisely into \(r+s < n\). The part of the sum over \(r, s, u\) and \(M_1\) in (20) corresponding to the condition \(r+s = n\) clearly reduces under \(u \rightarrow 0\) to
\[ \sum_{0 \leq r \leq n} p' \left[ \sum_{0 \leq u \leq n-r} \varphi_{r+u} \mathcal{R}^*(u, v, r) \right], \]
using the abbreviation \(\varphi_{r+s} \mathcal{R}^*(s-u, 2n-2r-s-u)\) for \(\varphi_{k,1} p^r M\) and noting that \(\varphi_{k,1} p^r M = \varphi_{k,1} M\). The rest of the sum over \(r, s, u, M_1\) as well as over \(r', s', u', M_2\) in (20) yields in all
\[ \sum_{0 \leq r \leq n} p' \left[ \sum_{0 \leq s \leq n} \varphi_{k,1} \mathcal{R}^*(s-u, 2n-2r-s-u, s-u) \right]. \]

We know, from the definition of \(\mathcal{S}_{r,s}\), that, for \(\varphi \in \mathcal{S}_{k,1}\),
\[ \mathcal{F}_{p'}(\varphi) = p^{p-6} \sum_{0 \leq k \leq m} \varphi_{k,1} \mathcal{S}^*(b, c)[\lambda, \mu]. \]

Consequently, from (18), (23) and (24), we obtain finally, for \(l = p^r\),
\[ \mathcal{F}_{p^r}(\varphi) = (1/p^r) \sum_{l \leq k \leq n} p^{p-6} \sum_{0 \leq k \leq m} \varphi_{k,1} \mathcal{S}^*(b, c)[\lambda, \mu]. \]
for \( i = p^n \); along with these simplifying steps (and symbols) we shall also take, in the sequel, \( \varphi \) to be of the form \( \mathcal{G}^{-1}(f - f) \) for \( f = f_i \), with some \( i (1 \leq i \leq a) \).

With obvious notation, we have the following

**Lemma 2.** With \( i = p^n \), \( \varphi \) and \( f \) as above and \( f(t) = \sum_{a \leq n} a(t)e(\xi t) \),

\[
\lambda_p^a(t) = \left\{ \begin{array}{ll}
(a(2^n) - p^{n-3} \chi(p)a(2^{n-1})) & \text{for any odd prime } p, \\
(a(2^n) + a(2^{n-1}) & \text{for } p = 2.
\end{array} \right.
\]

**Proof.** By definition, \((\mathcal{F}_p^0, \varphi)(\tau, z_1, z_2)\) equals

\[
p(z, \tau, \mu) \sum_{a \leq n} p(z, \tau, \mu) \sum_{\sigma \leq m} \frac{\varphi(t)}{p(z, \tau, \mu) + \chi(z; z_1, z_2)}
\]

where, as in the rest of this proof, in the summations carried out, \( u \) runs over \( \mathcal{Z}/p^n \mathcal{Z} \) subject to the additional proviso that \( p \not| u \) for \( 0 < r < 2n, h \) runs over \( \mathcal{Z}/p \mathcal{Z} \) and \( \lambda, \mu \) run independently over \( \mathcal{Z}/p \mathcal{Z} \).

We need to compute the \( \lambda_0 \)-component of \( \mathcal{F}_p^0 \). First, we restrict ourselves to that part of the sum above defining it where \( n \leq s \leq 2s \) and take the remaining part later. For \( 0 \leq s \leq 2n \), the coefficient of \( \varphi_{s+1}(p(z, \tau, \mu) + \chi(z; z_1, z_2)) \) therein (for \( h, s, u_2n-s \) fixed) is, up to the factor \( p^{s-n-1} \), precisely

\[
\sum_{\lambda, \mu} e(t((p(z, \tau, \mu) + \chi(z; z_1, z_2)) \lambda + \tau + \mu))
\]

is independent of \( m_2 \) and therefore the sum over \( m, \lambda, \mu \) in (25) reduces, for \( n \leq s \leq 2n \), to

\[
p^{2n} \sum_{\lambda, \mu} e(t((p(z, \tau, \mu) + \chi(z; z_1, z_2)) \lambda + \tau + \mu))
\]

where, as in the rest of this proof, in the summations carried out, \( u \) runs over \( \mathcal{Z}/p^2 \mathcal{Z} \) subject to the additional proviso that \( p \not| u \) for \( 0 < r < 2n, h \) runs over \( \mathcal{Z}/p \mathcal{Z} \) and \( \lambda, \mu \) run independently over \( \mathcal{Z}/p \mathcal{Z} \).

We will show that the map \( \mathcal{F}_p^0 \) is Hecke-equivalent with respect to a homomorphism, say \( \delta \), between the associated Hecke algebras and thereby compute the eigenvalues \( \lambda_p^a(l) \) of \( \varphi \) in \( J_{p^n}^1 \) under \( \mathcal{F}_p^0 \), where, in view of our proposition, it clearly suffices to take \( t = p^n \) for \( p \in \mathcal{N} \) and \( n \geq 1 \). (This direct computation of \( \lambda_p^a(p^n) \) spares us from having to find explicit generators and relations in the local Hecke algebra for \( J_{p^n}^1 \).) Again, in view of the relation between \( \mathcal{F}_p^0 \) and \( \mathcal{F}_p^0 \), it is enough to find the eigenvalues \( \lambda_p^a(l) \) of \( \varphi \) under \( \mathcal{F}_p^0 \),
When $m_2$ runs over $\mathcal{O}$ and $\lambda$ over $\mathcal{O}/p\mathcal{O}$, $x := p^m m_2 + \lambda$ (and likewise, for every fixed $m_1 \in \mathcal{O}/p^{n-1}\mathcal{O}$, also $x + p^{n-1} m_1$) runs through $\mathcal{O}$ exactly once. Thus the last-mentioned sum (for $n \leq s \leq 2n$) becomes

$$p^{2n} \sum_{\mu \in \mathcal{O}/p^n \mathcal{O}} e\left( \frac{x + p^{n-1} \mu}{2^{n-1}} \right) \sum_{x \in \mathcal{O}/p^n \mathcal{O}} e\left( u_{2n-2s} \mu + \frac{h}{2} \right) p^{-2n}.$$

Here, let us note that the series over $x$ is nothing but $\theta_0(x, z, z)$ for odd $p$ and $n \leq s \leq 2n$ or $p = 2$ with $s = n$ and $\theta_0(x, z, z)$ for $p = 2$ and $s > n$. The sum over $m_1$ is a generalized Gauss sum which can be evaluated. In fact, it can be shown that, for $p, h \in \mathbb{Z}$, $r \geq 0$ and $h \in \mathcal{O}/\mathcal{O},$

$$G(u, h; p^n) = \sum_{\mu \in \mathcal{O}/p^n \mathcal{O}} e\left( u \mu + h \right) p^{-r},$$

where

$$
= \begin{cases} 
(p(x)p^n) & \text{for odd primes } p \text{ and } r > 0, \\
2 + 1 \chi(u) \sqrt{-1} & \text{if } h = 0, r \geq 2, p = 2 \text{ or } \\
0 & \text{otherwise (for } p = 2 \text{ and } r > 0). 
\end{cases}
$$

We see therefore that the total contribution to the $\theta_0$-component of $\mathcal{F}_0^p \varphi$ from terms for which $n \leq s \leq 2n$

$$\sum_{n \leq s \leq 2n} \sum_{p \text{ odd}} p^{2n-k-1} \sum_{x \in \mathcal{O}/p^n \mathcal{O}} \varphi_0\left((p x + u_{2n-2s}) p^{-2n}\right)$$

for all odd primes $p$; the relevant contribution for $p = 2$ from terms with $n \leq s \leq 2n$ can be seen to be

$$2^{n-k-1} \sum_{n \leq s \leq 2n} \varphi_0\left((2^{n-k} + u_{2n-2s}) 2^{2n-2s}\right) G(u, 0; 2^n)$$

$$+ \sum_{n < s \leq 2n} 2^{n-k} \sum_{h \in \mathcal{O}/\mathcal{O}} \varphi_0\left((2^{n-k} + u_{2n-2s}) 2^{n-2s}\right) G(u_{2n-2s}, h; 2^{n-2s}).$$

The contribution arising from terms with $0 \leq s < n$ can be treated similarly. We give the detailed arguments for the case $p = 2$, since the case of odd $p$ is simpler. Carrying out in (25) the summation over $\mu$ (while keeping the other summation-indices fixed) and noting that

$$\sum_{\mu} e\left( \frac{\mu (m + h)}{2^{n-s}} \right) \begin{cases} 
2^{2n} & \text{if } (m + h) 2^{n-s} \in \mathcal{O}, \\
0 & \text{otherwise},
\end{cases}$$

we have non-zero contribution only from terms with $h = 0$ and $m \in 2^{n-s-1} \mathcal{O}$ (as the first alternative above on the right-hand side would entail, under "$n > s$"). Consequently, writing $m = 2^{n-s-1} \mu$ with $\mu \in \mathcal{O},$ (25) now takes the form

$$2^{2n} \sum_{i \in \mathcal{O}/2^{n-i}} e\left( \frac{\mu (m + h)}{2^{n-s}} \right) \sum_{x \in \mathcal{O}/p^n \mathcal{O}} e\left( u_{2n-2s} \mu + \frac{h}{2} \right) p^{-2n}.$$
Thus, for $2 \leq s \leq 2n-2$, we obtain

\begin{equation}
2\sqrt{-1} \chi(u_{2n-s}) \phi_0 | W_4 \left( \begin{array}{c} 2s \ 0 \\ 2^{2n-s} \ 0 \end{array} \right) W_4^{-1} = - \chi(u_{2n-s}) \sum_{c \in \mathbb{Z}/2n} \phi_0 | P_1
\end{equation}

(with $P_1$ as above)

\begin{equation}
= \sum_{c \in \mathbb{Z}/2n} \phi_0 \left( \begin{array}{c} 1 \ 0 \\ 2^{2n} \ 0 \end{array} \right)
\end{equation}

Together with the residue classes $2^{2n-2}c, 2^{2n-2}c - 2^{2n-3}$, the residues above cover all the residue classes modulo $2^{2n}$. Hence we get by (27) and (28),

\begin{equation}
\mathcal{H}_n \phi_0 = \phi_0 | G_0(4) \left( \begin{array}{c} 2s \ 0 \\ 0 \ 1 \end{array} \right) G_0(4) + \phi_0 | G_0(4) \left( \begin{array}{c} 0 \ 0 \\ 0 \ 2^{2n} \end{array} \right) G_0(4)
\end{equation}

\begin{equation}
= (a(2)2^n + a(2)^2) \phi_0,
\end{equation}

i.e.,

\begin{equation}
\delta(\mathcal{H}_n) = (a(2)2^n + a(2)2^n) \text{Id}
\end{equation}

where $\text{Id}$ denotes the identity operator. As a result, we have

\begin{equation}
\lambda_0(2^n) = a(2)2^n + a(2)2^n
\end{equation}

and this concludes the proof of Lemma 2.

Remarks. For $n = 1$, the results in Lemma 2 appear as assertions (without detailed proof) in Propositions 4.2 and 4.3 of [2], wherein one should read $2^{2n-3}$ instead of $2^{2n-2}$ in the expression for $\psi_0(t)$ in Proposition 4.3 on page 19 and $3 \cdot 2^{k-2}$ instead of $3 \cdot 2^{k-4}$ in the last line of page 20; the second term in the formula for $\text{Desc}(T_{1,1})$ on page 20 should read $p^{2k-8} (p^3 + p^2 + p + 1)$.

We are thankful to Professor A. Krieg for having rushed to us, at our request, a copy of [2] from Münster.

6. A relation between $D_{F,0}$ and $Z_F$. For the Fourier–Jacobi coefficients $\phi_m$, $\psi_m$ respectively of a Hecke eigenform $F$ and any cusp form $G$ in $M_0(\Gamma)$, we have $\langle \phi_m, \psi_m \rangle = \langle \gamma_m \psi_1, \gamma_m \psi_1 \rangle = \langle \gamma_m \gamma \psi_1, \psi_1 \rangle$ and so, by the proposition,

\begin{equation}
D_{F,0}(s) = \zeta(s-k+3) \zeta(2s-2k+4) \langle \phi_1, \psi_1 \rangle \sum_{m \in \mathbb{Z}/d} \psi(d) d^{k-2} \lambda_m(t/d) t^{-s}
\end{equation}

\begin{equation}
= \langle \phi_1, \psi_1 \rangle \zeta(s-k+3) \zeta(2s-2k+4) \sum_{m \in \mathbb{Z}/d} \psi(d) d^{k-2} \lambda_m(t/d) t^{-s}
\end{equation}

\begin{equation}
= \langle \phi_1, \psi_1 \rangle \zeta(s-k+3) \zeta(s-k+1) \zeta(s-k+2) \sum_{m \in \mathbb{Z}/d} \lambda_m(t) t^{-s},
\end{equation}

\[ \text{Acta Arithmetica LVIII.2} \]
on noting that
\[ \sum_{n \in \mathbb{N}} \psi(d) d^{-s} = \zeta(q-1) \zeta(q) / \zeta(2q). \]

Now
\[ \sum_{n \in \mathbb{N}} \chi_n(t) t^{-s} = \prod_p \sum_{n \geq 0} \chi_n(p^n) p^{-ns} \]
where the product is taken over all primes \( p \in \mathbb{N} \). We know from Lemma 2 that for odd primes \( p \),
\[ \lambda_n(p^r) = \sum_{\substack{0 \leq j \leq k \leq n \leq [n/2]}} p^{2jk - r} \cdot a(\frac{n}{2k} - 1) \cdot a(\frac{n}{2k} - 2) \cdot \chi(p) \sum_{\substack{0 \leq j \leq k \leq n \leq [n/2]}} p^{2jk - r} \cdot a(\frac{n}{2k} - 1) \cdot a(\frac{n}{2k} - 2) \]
where \([x] := \text{largest integer not exceeding } x \in \mathbb{R}\). Therefore
\[ \sum_{n \geq 0} \chi_n(p^n) p^{-ns} = \left( \sum_{j > 0} p^{2j(n-1)} - p^{-1} \chi(p) \sum_{j > 0} p^{2j+1(n-1)} \right) \left( \sum_{n \geq 0} a(p^n) p^{-ns} \right) \]
which is precisely the \( p \)-th Euler factor of \( R_f(s)/L(s-k+3, \chi) \), where
\[ R_f(s) := \left(1-a(2)^{2s} \right) \left(1-\bar{a}(2)^{2s} \right) \prod_{p \neq 2} \left(1-a^2 p^{-s}\right) \left(1-\bar{a}^2 p^{-s}\right) \left(1-\chi(p) a \bar{a} p^{-s} \right) \left(1-\bar{a}^2 p^{-s} \right)^{-1} \]
is the symmetric square zeta function associated to \( f \) and \( a_f + \chi(p) \bar{a}_f = \chi(p) \bar{a}_f \). By Lemma 2 again
\[ \sum_{n \geq 0} \lambda_n(2^n) 2^{-ns} = \sum_{n \geq 0} \left( \sum_{n \geq 0} a(2^{2n}) a(2^{2n}) 2^{-ns} \right) \]
\[ = \left(1-a(2)^{2s} \right) \left(1-\bar{a}(2)^{2s} \right) \left(1-\chi(p) a \bar{a} p^{-s} \right) \left(1-\bar{a}^2 p^{-s} \right)^{-1} \]
Hence
\[ \sum_{n \in \mathbb{N}} \chi_n(t) t^{-s} = R_f(s)/L(s-k+3, \chi) \]
which implies that
\[ D_{F,G}(s) = \langle \varphi_1, \psi_1 \rangle \zeta(s-k+3) \zeta(s-k+2) \zeta(s-k+1) R_f(s). \]
Since Gritsenko [2] has shown that
\[ Z(s-3) R_f(s), \] and we finally have the following

**Theorem 2.** Let \( F, G \in M_k(\Gamma_2), F \) being a Hecke eigenform. Then
\[ D_{F,G}(s) = \langle \varphi_1, \psi_1 \rangle \zeta(s-k+2) L(s-k+2, \left(\frac{-4}{.}\right))^{-1} Z(s). \]