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A Dirichlet series for Hermitian modular forms of degree 2

by

S. RAGHAVAN and J. SENGUPTA (Bombay)

Introduction. In a recent paper [5], Kohnen and Skoruppa, using the Rankin–Selberg method, investigated a novel Dirichlet series $D_{F,G}(s)$ associated with a pair F, G of Siegel cusp forms of integral weight for the modular group $\mathrm{Sp}_2(\boldsymbol{Z})$ of degree 2; prompted by the form of a functional equation it satisfies, they established its "proportionality" with the "spinor" zeta function $Z_F(s)$ attached to F by Andrianov, whenever F is a Hecke eigenform and G is in the Maass space.

An analogous zeta function $Z_F(s)$ (but with "Euler factors of degree 6") for eigen cusp forms F of weight k with respect to the Hermitian modular group Γ_2 of degree 2 over $\mathcal{O} := \boldsymbol{Z}[\sqrt{-1}]$ has been studied by Gritsenko [2], [3]. Using the results of Kojima [6] on the "Saito–Kurokawa descent" $F \mapsto f$ for F in the relevant Maass space, Gritsenko [2] proved (for k divisible by 4) the identity

$$Z_F(s) = \zeta(s-k+1)\, L\left(s-k+2, \left(\frac{-4}{\cdot}\right)\right) \zeta(s-k+3)\, R_f(s)$$

where R_f is the (Rankin) symmetric square zeta function associated to f and $L(\cdot, (\frac{-4}{\cdot}))$ is a Dirichlet L-series for the character $(\frac{-4}{\cdot})$.

This article is concerned with an analogue of $D_{F,G}$ for Hermitian cusp forms of weight k (divisible by 4) for Γ_2 and its possible relationship with Z_F. Even when F and G are both in the Maass space, we find that $D_{F,G}$ and Z_F are no longer "proportional", unlike in [5]. Although both $D_{F,G}(s)$ and $Z_F(s)$ admit functional equations under $s \mapsto 2k-3-s$, one faces at the same time a disturbing difference in the respective Γ-factors involved therein. Nevertheless, we can prove that

$$D_{F,G}(s) = \text{constant} \times \frac{\zeta(s-k+2)}{L(s-k+2, (\frac{-4}{\cdot}))} Z_F(s).$$

For a fundamental operator identity needed in the arguments, §4 provides a proof (quite algebraic and) different from the rather sketchy proof in [5] for the corresponding identity there; of course, both these identities have the same structure. We also need to work out anew, for the "Saito–Kurokawa descent", detailed arguments in regard to the aspect of its Hecke-equivariance which is somewhat slurred over in [6] and barely mentioned in [2].

1. Notation and terminology. For any complex matrix $P = (p_{ij})$, let $\bar{P} := (\bar{p}_{ij})$ where the bar denotes complex conjugation in $\boldsymbol{C}$; let ${}^tP$ denote the transpose of P. For a square matrix P over $\boldsymbol{C}$, let $\sigma(P)$, $\det P$ and $\|P\|$ stand respectively for the trace of P, determinant of P and absolute value of $\det P$. For any $(2, 2)$ matrix $P = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$, we define $P^* = \left(\begin{smallmatrix} s & -q \\ -r & p \end{smallmatrix}\right)$. By $\mathrm{Diag}(a_1, \ldots, a_n)$ we mean the (n, n) diagonal matrix with $a_1, \ldots, a_n$ as its entries in that order. For any (commutative) ring R, let $\mathcal{M}_n(R)$ denote the ring of all (n, n) matrices with entries from R. We denote the (n, n) identity matrix in $\mathcal{M}_n(\boldsymbol{C})$ by E_n; by 0, we mean a matrix of the appropriate size with all entries equal to 0. Whenever the product ${}^t\bar{G}HG$ of complex matrices G, H is defined, we write $H[G]$ for the same. For $t \in \boldsymbol{N}$, let $\mathcal{M}_2(\boldsymbol{Z})_t := \{M \in \mathcal{M}_2(\boldsymbol{Z}) \mid \det M = t\}$ and $\Gamma_1 := \mathcal{M}_2(\boldsymbol{Z})_1 = \mathrm{SL}(2, \boldsymbol{Z})$, the elliptic modular group. For any matrix M with entries from $\boldsymbol{Z}$, let $\gcd(M)$ denote the greatest common divisor of the entries of M. If $n \in \boldsymbol{N}$ and $n = m^2$ for m in $\boldsymbol{Z}$, we write $n = \square$.

The ring $\boldsymbol{Z}[\sqrt{-1}]$ of algebraic integers in $\boldsymbol{K} := \boldsymbol{Q}(\sqrt{-1})$ is denoted by $\mathcal{O}$; by $\sqrt{-1}$, we always mean that square root with argument $\pi/2$. For α in $\boldsymbol{K}$ its "trace" $\alpha + \bar{\alpha}$ is denoted by $\mathrm{tr}(\alpha)$. Moreover, for $z = (z_1, z_2) \in \boldsymbol{C} \times \boldsymbol{C}$ and $\alpha \in \boldsymbol{K}$, we shall simply write $\mathrm{tr}(\alpha z)$ for $\alpha z_1 + \bar{\alpha} z_2$, without risk of confusion. By a positive-definite matrix H, we mean a complex matrix $H = {}^t\bar{H}$ with all its eigenvalues positive and we then write $H > 0$. For any $\gamma \in \boldsymbol{C}$, $e(\gamma)$ stands for $\exp(2\pi\sqrt{-1}\gamma)$.

Let $\mathcal{H} = \mathcal{H}_1$ denote the upper half-plane $\{\tau = u + \sqrt{-1}v \in \boldsymbol{C} \mid u, v \in \boldsymbol{R}, v > 0\}$. For $k \in \boldsymbol{N}$ and any Dirichlet character ε (on $\boldsymbol{Z}$) modulo N, the space of holomorphic cusp forms of weight k and character ε for the congruence subgroup $\Gamma_0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 \mid c \in N\boldsymbol{Z}\}$ is denoted by $S_k(\Gamma_0(N); \varepsilon)$.

The generalized upper half-plane $\{Z \in \mathcal{M}_2(\boldsymbol{C}) \mid -\sqrt{-1}(Z - {}^t\bar{Z}) > 0\}$ is denoted by $\mathcal{H}_2$. The Hermitian modular group Γ_2 of degree 2 over $\mathcal{O}$ consists of all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $\mathcal{M}_4(\mathcal{O})$ with $(2, 2)$ matrices A, B, C, D for which $A\,{}^t\bar{B} = B\,{}^t\bar{A}$, $C\,{}^t\bar{D} = D\,{}^t\bar{C}$ and $A\,{}^t\bar{D} - B\,{}^t\bar{C} = E_2$. For any such M in Γ_2 and Z in $\mathcal{H}_2$, let $M\langle Z\rangle := (AZ + B)(CZ + D)^{-1}$. The analytic homeomorphisms $Z \mapsto M\langle Z\rangle$ for M in Γ_2 give a representation of Γ_2 as a discontinuous group of automorphisms of $\mathcal{H}_2$ and we denote by $\mathcal{F}_2$ a standard fundamental domain for Γ_2 in $\mathcal{H}_2$. Writing $X := \frac{1}{2}(Z + {}^t\bar{Z})$ and $Y = \mathrm{Im}(Z) := (1/2\sqrt{-1})(Z - {}^t\bar{Z}) > 0$ for Z in $\mathcal{H}_2$, we have on $\mathcal{H}_2$ an invariant volume element $(\det Y)^{-4} dX\, dY$. A complex-valued function F holomorphic on $\mathcal{H}_2$ is called a *modular form of integral weight k for Γ_2* if $F(M\langle Z\rangle)\det(CZ + D)^{-k} = F(Z)$ for every $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in Γ_2. Such an F has a Fourier expansion

$$(1) \qquad F(Z) = \sum_T a(T)\, e(\sigma(TZ))$$

indexed by $(2, 2)$ Hermitian matrices $T = (t_{ij})$ with t_{ii} and $2t_{ij}$ in $\mathcal{O}$ and having (only) non-negative eigenvalues. We call F a *cusp form of weight k for Γ_2* if in (1), $a(T)$ vanishes for all T which are not positive-definite. The space of (holomorphic) cusp forms of weight k for Γ_2 is denoted by $S_k(\Gamma_2)$; in the following, we assume, as in [6], that k is divisible by 4.

We denote the subgroup $\{M \in \Gamma_2 \mid M$ has $(0\ 0\ 0\ 1)$ as its last row$\}$ by $\mathscr{C}$.

Writing any Z in $\mathcal{H}_2$ as $Z = \left(\begin{smallmatrix} \tau & z_1 \\ z_2 & \tau' \end{smallmatrix}\right)$, the Jacobi modular group $\Gamma_1^J = \mathrm{SL}(2, \boldsymbol{Z}) \propto \mathcal{O}^2$ acts [1] in accordance with its imbedding in Γ_2, taking

$$(\tau, z_1, z_2) \in \mathcal{H} \times \boldsymbol{C} \times \boldsymbol{C} \quad \text{to} \quad \left(\frac{a\tau + b}{c\tau + d}, \frac{z_1 + \lambda\tau + \mu}{c\tau + d}, \frac{z_2 + \bar{\lambda}\tau + \bar{\mu}}{c\tau + d}\right)$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}(2, \boldsymbol{Z})$ and (λ, μ) in $\mathcal{O}^2$. For $(k, m) \in \boldsymbol{N}^2$, let $J_{k,m}^0$ denote the space of Jacobi cusp forms φ of weight k and index m for Γ_1^J, viz. holomorphic $\varphi\colon \mathcal{H} \times \boldsymbol{C} \times \boldsymbol{C} \mapsto \boldsymbol{C}$ such that for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}(2, \boldsymbol{Z})$ and $(\lambda, \mu) \in \mathcal{O}^2$,

$$(2) \qquad (\varphi|_{k,m}M)(\tau, z_1, z_2) := \varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z_1}{c\tau + d}, \frac{z_2}{c\tau + d}\right) e\left(-\frac{mcz_1z_2}{c\tau + d}\right) = \varphi(\tau, z_1, z_2),$$

$$(3) \qquad (\varphi|[\lambda, \mu])(\tau, z_1, z_2) := \varphi(\tau, z_1 + \lambda\tau + \mu, z_2 + \bar{\lambda}\tau + \bar{\mu})\, e(\lambda\bar{\lambda}\tau + \bar{\lambda}z_1 + \lambda z_2) = \varphi(\tau, z_1, z_2)$$

and the function $\varphi(\tau, z_1, z_2)e(m\tau')$ has on $\mathcal{H}_2$ a Fourier expansion of the form (1) indexed by $T > 0$. We note that any F in $S_k(\Gamma_2)$ has a Fourier–Jacobi expansion

$$F(Z) = \sum_{m \geq 1} \varphi_m(\tau, z_1, z_2)\, e(m\tau')$$

with $\varphi_m(\tau, z_1, z_2)$ in $J_{k,m}^0$ which we shall refer to as the mth *Fourier–Jacobi coefficient* of F.

For $(\tau, z_1, z_2) \in \mathcal{H} \times \boldsymbol{C} \times \boldsymbol{C}$, writing $\tau = u + \sqrt{-1}v$, $z_j = x_j + \sqrt{-1}y_j$ $(j = 1, 2)$ with $u, v\ (> 0), x_1, y_1, x_2, y_2$ in $\boldsymbol{R}$, we have on $\mathcal{H} \times \boldsymbol{C} \times \boldsymbol{C}$ an invariant volume element $d\mu := v^{-4} du\, dv\, dx_1\, dy_1\, dx_2\, dy_2$. For φ, ψ in $J_{k,m}^0$, we have the Petersson inner product

$$(4) \qquad \langle\varphi, \psi\rangle := \int_{\mathcal{F}^J} \varphi(\tau, z_1, z_2)\overline{\psi(\tau, z_1, z_2)}\, v^k e^{-\pi m|z_1 - \bar{z}_2|^2/v}\, d\mu$$

where $\mathcal{F}^J$ is a fundamental domain for Γ_1^J.

Besides the zeta functions $\zeta(s) := \sum_{n \in \boldsymbol{N}} n^{-s}$ and $\zeta_{\boldsymbol{K}}(s) := \frac{1}{4}\sum_{0 \neq \lambda \in \mathcal{O}} (\lambda\bar{\lambda})^{-s}$ and the Dirichlet L-series $\sum_{n \in \boldsymbol{N}} \chi(n) n^{-s}$ for a Dirichlet character χ, we recall also (from [3]), the "spinor" zeta function $Z_F(s)$ defined, for any Hecke eigenform F in $S_k(\Gamma_2)$, by

$$Z_F(s) := \prod_p{}' (1 + p^{k-2-s})^{-2} Q_{F,p}^{(2)}(p^{-s})^{-1} \prod_p Q_{F,p}^{(2)}(p^{-s})^{-1}$$

with the accent on the first infinite product indicating that p (in N) runs over all primes for which $p\mathcal{O}$ is prime in $\mathcal{O}$, while the second product is extended over all the remaining primes from N. Moreover, in the foregoing, the polynomial $Q^{(2)}_{F,p}(t)$ with t indeterminate is defined by $Q^{(2)}_p(t)F = Q^{(2)}_{F,p}(t)F$ for the following explicitly given operator-valued polynomials $Q^{(2)}_p(t)$ involving the standard operators

$$T_p := \Gamma_2 \operatorname{Diag}(1, 1, p, p)\Gamma_2, \quad T_{1,p} := \Gamma_2 \operatorname{Diag}(1, p, p^2, p)\Gamma_2,$$
$$T_\pi := \Gamma_2 \operatorname{Diag}(1, \pi, p, \pi)\Gamma_2 \quad \text{for } p = \pi\bar{\pi} \text{ in } \mathcal{O},$$
$$\Delta_\lambda := \Gamma_2(\lambda E_4)\Gamma_2 \quad \text{for } \lambda \in \mathcal{O}$$

in the Hecke ring for Γ_2:

(i) $Q^{(2)}_p(t) := 1 - T_p t + (pT_{1,p} + p(p^3+p^2-p+1)\Delta_p)t^2 - p^4\Delta_p T_p t^3 + p^8\Delta_p^2 t^4$
for $p\mathcal{O}$ prime in $\mathcal{O}$, i.e. $(\frac{-4}{p}) = -1$,

(ii) $Q^{(2)}_p(t) := 1 - T_p t + p(T_\pi T_{\bar{\pi}} - p^4\Delta_p)t^2 - p^3(T_\pi^2\Delta_{\bar{\pi}} + T_{\bar{\pi}}^2\Delta_\pi - 2p\Delta_p T_p)t^3$
$+ p^4\Delta_p(pT_\pi T_{\bar{\pi}} - p^4\Delta_p)t^4 - p^8\Delta_p^2 T_p t^5 + p^{12}\Delta_p^3 t^6,$
for $p = \pi\bar{\pi}$ in $\mathcal{O}$, i.e. $(\frac{-4}{p}) = 1$, and

(iii) $Q^{(2)}_2(t) := 1 - (T_2 - 3\Delta_{1+i})t + 2(T_{1+i}^2 - 8\Delta_{1+i}(T_2 + \Delta_{1+i}))t^2$
$-(4\Delta_{1+i})^2(T_2 - 3\Delta_{1+i})t^3 + (4\Delta_{1+i})^4 t^4$ with $i := \sqrt{-1}$.

We also know from [3] that if $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$, then

$$Z^*_F(s) := \pi^{-3s}\Gamma(s)\Gamma(s-k+3)\Gamma^2\left(\frac{s-k+3}{2}\right)\xi(s-k+2)Z_F(s) = Z^*_F(2k-3-s).$$

2. An Eisenstein series. Towards obtaining the meromorphic continuation and functional equation of $D_{F,G}(s)$, we first address ourselves precisely to these two questions but in the context of an Eisenstein series $E_s(Z)$ on $\mathcal{H}_2$, of the Klingen–Siegel type, relative to the subgroup $\mathscr{C}$ of Γ_2. The latter is defined (cf. [4]) for Z in $\mathcal{H}_2$ and s in C with $\operatorname{Re}(s) > 3$, by

$$(5) \qquad E_s(Z) := \sum_{M \in \mathscr{C}\backslash\Gamma_2} (\det \operatorname{Im}(M\langle Z\rangle)/(\operatorname{Im}(M\langle Z\rangle))_1)^s$$

where $(\operatorname{Im}(M\langle Z\rangle))_1$ is the first diagonal element of $\operatorname{Im}(M\langle Z\rangle)$. Let us rewrite the general term of the series in (5) in a more convenient form, for $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ in Γ_2 with $(b\ a\ d\ c)$ as its last row. For any (2, 2) matrix P, we have $PP^* = P^*P = (\det P)E_2$ and in particular, for $P = Y = \operatorname{Im}(Z)$ and $P = CZ + D$; also we note that $Z^* = X^* + \sqrt{-1}\,Y^*$ and further $b\bar{d} + a\bar{c} - d\bar{b} - c\bar{a} = 0$. It is now easy to check that

$$\operatorname{Im}(M\langle Z\rangle) = Y[(CZ+D)^{-1}] = \|CZ+D\|^{-2}\,Y[(CZ+D)^*],$$

$$(\operatorname{Im}(M\langle Z\rangle))_1 = \|CZ+D\|^{-2}\,Y\left[Z^*\begin{pmatrix} a \\ -b\end{pmatrix} + \begin{pmatrix} c \\ -d\end{pmatrix}\right],$$

$$(6) \qquad \det Y(\operatorname{Im}(M\langle Z\rangle))_1/\det \operatorname{Im}(M\langle Z\rangle)$$
$$= Y\left[X^*\begin{pmatrix} a \\ -b\end{pmatrix} + \begin{pmatrix} c \\ -d\end{pmatrix}\right] + Y\left[Y^*\begin{pmatrix} a \\ -b\end{pmatrix}\right]$$
$$= (\det Y)\left(Y^{*-1}\left[X^*\begin{pmatrix} a \\ -b\end{pmatrix} + \begin{pmatrix} c \\ -d\end{pmatrix}\right] + Y^*\begin{pmatrix} a \\ -b\end{pmatrix}\right).$$

Under $M \to (0\ 0\ 0\ 1)M$, we can identify $\mathscr{C}\backslash\Gamma_2$ with

$$(7) \qquad \{(a, b, c, d) \in \mathcal{O}^4 \mid a\bar{c} + b\bar{d} - c\bar{a} - d\bar{b} = 0,\ a\mathcal{O} + b\mathcal{O} + c\mathcal{O} + d\mathcal{O} = \mathcal{O}\}$$

which is invariant with respect to $(b, d) \mapsto (-b, -d)$. If we take $g_1 := {}^t(a\ b)$, $g_2 := {}^t(c\ d)$ and $g := {}^t({}^tg_1\ {}^tg_2)$ the defining conditions in (7) go over into

$$(8) \qquad {}^t\bar{g}_2 g_1 - {}^t\bar{g}_1 g_2 = 0, \quad g \text{ is "primitive" over } \mathcal{O}$$

where by a "primitive" column over $\mathcal{O}$, we mean a column whose entries together generate $\mathcal{O}$. Let S, H be (4, 4) Hermitian matrices defined by

$$(9) \qquad S = \begin{pmatrix} 0 & \sqrt{-1}\,E_2 \\ -\sqrt{-1}\,E_2 & 0\end{pmatrix}, \quad H = \begin{pmatrix} Y^* & 0 \\ 0 & Y^{*-1}\end{pmatrix}\begin{pmatrix} E_2 & 0 \\ X^* & E_2\end{pmatrix}.$$

Then $H > 0$ and it is a "majorant" of the indefinite (Hermitian) matrix S of "signature" (2, 2), i.e. $HS^{-1}H = S$. After replacing (b, d) by $(-b, -d)$ in (6), we see that

$$(10) \qquad (\operatorname{Im}(M\langle Z\rangle))_1/\det \operatorname{Im}(M\langle Z\rangle) = H[g]$$

with, of course g "primitive" and $S[g] = 0$, by (8). Any non-zero column over $\mathcal{O}$ becomes "primitive" under multiplication by λ^{-1} for a suitable $\lambda \neq 0$ from $\mathcal{O}$; thus, from (5), (10) and multiplication by $\zeta_K(s)$ to eliminate the inconvenient "primitivity" condition in (8), we obtain

$$(11) \qquad \zeta_K(s)E_s(Z) = \sum_g{}' (H[g])^{-s}$$

where the accent indicates that g runs over all non-zero (4, 1) columns over $\mathcal{O}$ with $S[g] = 0$. This formula provides a link with a theta series associated with S and H and the possibility of using towards our own objectives, its (transformation) properties, by exhibiting (11) as a "transform" of the theta series; tackling the condition $S[g] = 0$ has to be done at the same time, by adopting an idea from [3].

For S, H in (9) and $\tau = u + \sqrt{-1}\,v \in \mathcal{H}$ with $u, v \in \mathbf{R}$, let us consider the theta series

$$\vartheta(\tau) = \vartheta(\tau, Z) := \sum_g e(\tfrac{1}{2}(uS[g] + \sqrt{-1}\,vH[g]))$$

where g runs over all (4, 1) columns over $\mathcal{O}$; the absolute convergence of the series is ensured by "$H > 0$". There exists an invertible (4, 4) complex matrix V such that $H = E_4[V]$ and $S = D[V]$ with D diagonal; also $D^2 = E_4$, in view of the relation $(HS^{-1})^2 = E_4$. Since S has "signature" (2, 2), we may, after multiplying V on the left by an appropriate permutation matrix, assume already that $D = \mathrm{Diag}(1, 1, -1, -1)$. If now $K = K(\tau) := vH - \sqrt{-1}\,uS$, then it is easily verified that

$$K^{-1} = \frac{v}{v^2+u^2}H^{-1} + \frac{\sqrt{-1}\,u}{v^2+u^2}S^{-1} = (K(-1/\tau))[S^{-1}], \quad \det K = (u^2+v^2)^2.$$

We are thus led to the theta transformation formula

(12) $$\vartheta(-1/\tau, Z) = \sum_g \exp(-\pi K(-1/\tau)[g]) = \sum_g \exp(-\pi K^{-1}[S^{-1}g])$$

$$= \sum_g \exp(-\pi K^{-1}[g]) \quad \text{since } S \in \mathrm{GL}(4, \mathcal{O})$$

$$= (\det K)\sum_g \exp(-\pi K[g]) = |\tau|^4\,\vartheta(\tau, Z).$$

This transformation formula can also be proved by going over to the theta series associated with the (8, 8) matrix $vP - \sqrt{-1}\,uQ$ and the lattice $\mathbf{Z}^8$ (in lieu of $\mathcal{O}^4$ as above) where

$$2P := {}^t A\begin{pmatrix} 0 & \bar{H} \\ H & 0\end{pmatrix}A, \quad 2Q = {}^t A\begin{pmatrix} 0 & \bar{S} \\ S & 0\end{pmatrix}A, \quad A := \begin{pmatrix} E_4 & E_4 \\ \sqrt{-1}E_4 & -\sqrt{-1}E_4\end{pmatrix}$$

and applying the well-known theta transformation formula from ([6], §2). Now, for any (4, 1) column g over $\mathcal{O}$, $S[g]$ is always in $2\mathbf{Z}$ and so $\vartheta(\tau+1) = \vartheta(\tau)$. This together with the transformation formula (12) implies the invariance of $v^2\vartheta(\tau, Z)$ for every substitution $\tau \mapsto (a\tau+b)(c\tau+d)^{-1}$ from Γ_1. Consequently, the same invariance holds good also for

$$\Theta(\tau, Z) := L(v^2\vartheta(\tau, Z)) + 2v^2\vartheta(\tau, Z)$$

where L is the invariant differential operator $-v^2(\partial^2/\partial u^2 + \partial^2/\partial v^2)$. Writing α, β for $H[g]$, $S[g]$ respectively and taking $f(u, v) := \exp(-\pi\alpha v + \pi\sqrt{-1}\,\beta u)$, it is immediate that

(13) $$L(v^2 f(u, v)) + 2v^2 f(u, v) = (4\pi\alpha v^3 - \pi^2(\alpha^2-\beta^2)v^4)\,f(u, v).$$

For s in $\mathbf{C}$ with large enough $\mathrm{Re}(s)$, integration over a standard fundamental domain for the subgroup $\Gamma_{1,\infty} := \{(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}) \in \Gamma_1\}$ in $\mathscr{H}$ yields

(14)
$$\int_0^\infty\int_0^1 v^{s-1}\Theta(\tau, Z)v^{-2}\,du\,dv = \int_0^\infty v^{s-3}\,dv\int_0^1\sum_g(4\pi H[g]v^3 - \pi^2(H[g]^2 - S[g]^2v^4)$$
$$\times\exp(-\pi v H[g] + 2\pi\sqrt{-1}\,uS[g])\,du, \quad \text{by (13)}$$
$$= \sum_{\substack{0\neq g\\ S[g]=0}}\int_0^\infty \exp(-\pi v H[g])$$
$$\times(4\pi H[g]v^{s+1} - \pi^2 H[g]^2 v^{s+2})\frac{dv}{v}$$
$$= \pi^{-s}s(3-s)\Gamma(s)\zeta_{\mathbf{K}}(s)E_s(Z), \quad \text{in view of (11).}$$

On "folding back" the domain of integration for the left-hand side to recover the (usual) fundamental domain $\mathscr{F}_1$ for Γ_1 in $\mathscr{H}$, we see that the left-hand side is nothing but

$$\int_{\mathscr{F}_1}\Theta(\tau, Z)E_{\Gamma_1}(\tau, 2s-2)v^{-2}\,du\,dv$$

where

$$E_{\Gamma_1}(\tau, \varrho) := v^{\varrho/2}\sum_{(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix})\in\Gamma_{1,\infty}\backslash\Gamma_1}|c\tau+d|^{-\varrho}$$

is the usual Eisenstein series for Γ_1. It is well known that $\pi^{-\varrho/2}\Gamma(\varrho/2)\zeta(\varrho)E_{\Gamma_1}(\tau, \varrho)$ is holomorphic in the entire ϱ-plane except for simple poles at $\varrho = 0$, 2 and is further invariant under $\varrho \mapsto 2-\varrho$. For $\tau \in \mathscr{F}_1$ and s in any compact set, we have

$$|E_{\Gamma_1}(\tau, 2s-2)| \leqslant c_1 v^{\nu}, \quad |\Theta(\tau, Z)| \leqslant c_2\exp(-c_3 v)$$

for suitable constants ν, c_1, c_2, c_3 independent of τ. These facts give rise (as usual) to the required meromorphic continuation as well as a functional equation under $s \mapsto 3-s$ (or correspondingly $2s-2 \mapsto 4-2s$). Multiplying the expression in (14) by $\pi^{-(s-1)}\Gamma(s-1)\zeta(2s-2)$, we see that

$$\pi^{-(2s-1)}\Gamma(s)\Gamma(s-1)s(3-s)\zeta(2s-2)\zeta_{\mathbf{K}}(s)E_s(Z)$$

is holomorphic in the entire s-plane except for possible simple poles at $s = 1$, 2 and is further invariant under $s \mapsto 3-s$. We have hence the following

LEMMA 1. *The function*

$$E_s^*(Z) := \pi^{-2s}\Gamma(s)\Gamma(s-1)\zeta(2s-2)\zeta_{\mathbf{K}}(s)E_s(Z)$$

admits analytic continuation in s to all of $\mathbf{C}$ and is holomorphic except for possible simple poles at $s = 0, 1, 2, 3$. Moreover, $E_s^*(Z) = E_{3-s}^*(Z)$.

Remark. For Z in $\mathscr{F}_2$ (and hence, due to Γ_2-invariance, also for all Z in $\mathscr{H}_2$), $E_s(Z)$ behaves at most like a power of $\det(\mathrm{Im}(Z))$ as the latter goes to infinity.

3. The Dirichlet series $D_{F,G}(s)$. For any two holomorphic cusp forms F, G of integral weight k for the Hermitian modular group Γ_2 over $\mathcal{O}$, we associate, following [5], a Dirichlet series $D_{F,G}(s)$ of the Rankin–Selberg type. Namely, if

$$F(Z) = \sum_{m\in N} \varphi_m(\tau, z_1, z_2)e(m\tau') \quad \text{and} \quad G(Z) = \sum_{n\in N} \psi_n(\tau, z_1, z_2)e(n\tau'),$$

we first consider, for s in C with $\mathrm{Re}(s) > k+1$, the Dirichlet series $\sum_{m\in N} \langle\varphi_m, \psi_m\rangle m^{-s}$ whose absolute convergence is immediate from "$\langle\varphi_m, \psi_m\rangle = O(m^k)$". To get this estimate, we start from the formula $\varphi_m(\tau, z_1, z_2) = \int_\gamma^{\gamma+1} F(Z)e(-m\tau')\,d\tau'$ with $\gamma = \sqrt{-1}c$ and any $c > 0$ and note that for all Z in $\mathscr{H}_2$, $(\det Y)^{k/2}F(Z) = (vv' - |z_1 - \bar{z}_2|^2/4)^{k/2}F(Z)$ is bounded where $v := \mathrm{Im}(\tau)$, $v' := \mathrm{Im}(\tau')$. We then choose $c = (|z_1 - \bar{z}_2|^2/4v) + 1/m$ and are led, from the foregoing, to the estimate

$$\varphi_m(\tau, z_1, z_2) = O\left((v/m)^{-k/2}\exp(\pi m|z_1 - \bar{z}_2|^2/2v)\right)$$

and eventually to conclude that

$$\varphi_m(\tau, z_1, z_2)\psi_m(\tau, z_1, z_2)v^k\exp(-\pi m|z_1 - \bar{z}_2|^2/v) = O(m^k) \quad \text{on } \mathscr{F}^J,$$

yielding the estimate asserted for $\langle\varphi_m, \psi_m\rangle$ at once.

For s in C with $\mathrm{Re}(s) > k+1$, we can now define

$$D_{F,G}(s) = \zeta_K(s-k+3)\zeta(2s-2k+4)\sum_{m\in N}\langle\varphi_m, \psi_m\rangle m^{-s}$$

and proceed to state

THEOREM 1. *The Dirichlet series $D_{F,G}(s)$ associated to F, G in $S_k(\Gamma_2)$ can be continued meromorphically to the entire s-plane. The function*

$$D^*_{F,G}(s) := (4\pi^3)^{-s}\Gamma(s)\Gamma(s-k+2)\Gamma(s-k+3)D_{F,G}(s)$$

is holomorphic in s except for possible simple poles at $s = k-3, k-2, k-1, k$ and satisfies the functional equation $D^*_{F,G}(s) = D^*_{F,G}(2k-3-s)$.

Proof. Clearly $D_{F,G}(s)$ is holomorphic for $\mathrm{Re}(s) > k+1$. To continue it meromorphically to the left, we first note that for F, G in $S_k(\Gamma_2)$ and $\mathrm{Re}(s) > k+1$,

$$\langle FE_s, G\rangle := \int_{\mathscr{F}_2} (\det Y)^k F(Z)E_s(Z)\overline{G(Z)}(\det Y)^{-4}\,dX\,dY$$

is well-defined and by the (definition of $E_s(Z)$ and the) usual "unfolding argument", it is equal to

$$\int_{\mathscr{C}\backslash\mathscr{H}_2} F(Z)\overline{G(Z)}v^{-s}(\det Y)^{k-4+s}\,dX\,dY.$$

Now, a fundamental domain for $\mathscr{C}$ in $\mathscr{H}_2$ is given by the set

$$\left\{Z = \begin{pmatrix}\tau & z_1\\ z_2 & \tau'\end{pmatrix}\in\mathscr{H}_2 \;\middle|\; (\tau, z_1, z_2)\in\mathscr{F}^J,\right.$$
$$\left.\tau' = u' + \sqrt{-1}v',\ v' > |z_1 - \bar{z}_2|^2/4v,\ 0 \leqslant u' \leqslant 1\right\}.$$

Hence $\langle FE_s, G\rangle$ equals

$$(15)\quad \int_{\mathscr{F}^J}\int_{v'>|z_1-\bar{z}_2|^2/4v} du\,dv\,dv'\,dx_1\,dy_1\,dx_2\,dy_2\int_0^1\sum_{m,n\in N}\varphi_m(\tau, z_1, z_2)\overline{\psi_n(\tau, z_1, z_2)}$$
$$\times\exp(-2\pi(m+n)v')e((m-n)u')v^{k-4}\left(v' - \frac{|z_1-\bar{z}_2|^2}{4v}\right)^{k-4+s}du'$$
$$= \int_{\mathscr{F}^J}\sum_{m\in N}\varphi_m(\tau, z_1, z_2)\bar{\psi}_m(\tau, z_1, z_2)\exp(-\pi m|z_1-\bar{z}_2|^2/v)v^k\,d\mu$$
$$\times\int_0^\infty\exp(-4\pi nt)t^{k-4+s}\,dt \quad (t := v' - |z_1-\bar{z}_2|^2/4v)$$
$$= (4\pi)^{-(s+k-3)}\Gamma(s+k-3)\sum_{m\in N}\langle\varphi_m, \psi_m\rangle m^{-(s+k-3)},$$

as the interchange of the summation over m and the integration over $\mathscr{F}^J$ is easily justified. From the "rapid decay at infinity" of F and G and the "polynomial growth" of E_s in $\mathscr{F}_2$ (see the Remark following Lemma 1), the left-hand side of (15) represents a meromorphic function of s all over C and provides the meromorphic continuation required for $D_{F,G}$, as well. The asserted properties of $D^*_{F,G}(s) = \pi^{6-2k}\langle FE^*_{s-k+3}, G\rangle$ by (15) including its functional equation under $s\mapsto 2k-3-s$ are now immediate, proving Theorem 1.

4. A proposition. This section is devoted to the proof of a basic identity (needed in §6) involving certain operators $\mathscr{V}_l$ and $\mathscr{T}_l$ in the spaces $J^0_{k,1}$. For $l\in N$, the operators $\mathscr{V}_l$: $J^0_{k,1}\to J^0_{k,l}$ and $\mathscr{T}_l$: $J^0_{k,1}\to J^0_{k,1}$ are defined as follows: namely, for any φ in $J^0_{k,1}$ (cf. [1], [6]),

$$(16)\quad (\varphi|_{k,1}\mathscr{V}_l)(\tau, z_1, z_2)$$
$$:= l^{k-1}\sum_{S=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_1\backslash\mathscr{M}_2(Z)_l} e\left(-\frac{lcz_1z_2}{c\tau+d}\right)\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{lz_1}{c\tau+d}, \frac{lz_2}{c\tau+d}\right)(c\tau+d)^{-k},$$

$$(\varphi|_{k,1}\mathscr{T}_l)(\tau, z_1, z_2) := l^{-4}\sum_{\lambda,\mu\in\mathcal{O}/l\mathcal{O}} e(\lambda\bar{\lambda}\tau + \bar{\lambda}z_1 + \lambda z_2)\cdot l^{k-2}$$
$$\times\sum_{\substack{R\in\Gamma_1\backslash\mathscr{M}_2(Z)_{l^2}\\ \gcd(R)=\square}}(\varphi|_{k,1}R)(\tau, z_1 + \lambda\tau + \mu, z_2 + \bar{\lambda}\tau + \bar{\mu}).$$

For the representatives S, R here we can choose ones in the triangular form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. The definition of the operator $\mathcal{T}_l^0\colon J_{k,1}^0 \to J_{k,1}^0$ is the same as that of $\mathcal{T}_l$, except that the condition $\gcd(R) = \square$ is replaced by $\gcd(R) = 1$. We have between $\mathcal{T}_l$ and $\mathcal{T}_r^0$ the identity $\mathcal{T}_l = \sum_{d^2 | l} d^{2k-4} \mathcal{T}_{l/d^2}^0$ just as in [1].

For the adjoint operator $\mathscr{V}_l^*\colon J_{k,l}^0 \to J_{k,1}^0$ corresponding to $\mathscr{V}_l$, we have the formula

$$(17)\quad (\psi | \mathscr{V}_l^*)(\tau, z_1, z_2) = l^{k-5} \sum_{\lambda, \mu \in \mathcal{O}/l\mathcal{O}} e\big(l(\lambda\bar{\lambda}\tau + \bar{\lambda} z_1 + \lambda z_2)\big) \times \sum_{S = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Gamma_1 \backslash \mathscr{M}_2(\mathbf{Z})_l} \psi\left(\frac{a\tau + b}{d}, \frac{z_1 + \lambda\tau + \mu}{d}, \frac{z_2 + \bar{\lambda}\tau + \bar{\mu}}{d}\right) d^{-k}.$$

We omit its proof, since it is on the same lines as in [5].

The following proposition deals with an identity connecting $\mathscr{V}_l$, $\mathscr{V}_l^*$ with the Hecke operators $\mathcal{T}_r$ on $J_{k,1}^0$ and it corresponds to assertion (ii) of a proposition due to Kohnen and Skoruppa ([5], p. 549), where, however, a good part of the details in the proof has been left to the reader. Our proof is different and algebraic in nature.

PROPOSITION. *For $l \in \mathbf{N}$,*

$$\mathscr{V}_l^* \mathscr{V}_l = \sum_{1 \leqslant t | l} \psi(l/t)(l/t)^{k-2} \mathcal{T}_t \quad \text{on } J_{k,1}^0,$$

where, for any $r \in \mathbf{N}$,

$$\psi(r) := r \prod_{p | r} (1 + 1/p),$$

the product being extended over all primes p dividing r.

Proof. It is clear from the definition (16) of $\mathscr{V}_l$ (taking the representatives S in upper triangular form as is more convenient) that for $l = l_1 l_2$ and $(l_1, l_2) = 1$, $\mathscr{V}_l = \mathscr{V}_{l_1}\mathscr{V}_{l_2} = \mathscr{V}_{l_2}\mathscr{V}_{l_1}$ and $\mathscr{V}_l^* = \mathscr{V}_{l_2}^*\mathscr{V}_{l_1}^* = \mathscr{V}_{l_1}^*\mathscr{V}_{l_2}^*$. Moreover, the mapping $t \mapsto t^{k-2}\psi(t)$ from $\mathbf{N}$ is multiplicative. It is not hard to verify directly from the definitions that, for $(l_1, l_2) = 1$, $\mathcal{T}_{l_1}$ commutes both with $\mathscr{V}_{l_2}$ and with $\mathcal{T}_{l_2}$ (cf. [1], p. 51). Thus it suffices to prove the proposition for $l = p^n$ for any given prime number p; we assume $l = p^n$ in the sequel. From (16) and (17), we have

$$(18)\quad \big((\mathscr{V}_l^* \mathscr{V}_l)(\varphi)\big)(\tau, z_1, z_2) = l^{2k-6} \sum_{\substack{\lambda, \mu \in \mathcal{O}/l\mathcal{O} \\ S_i = \left(\begin{smallmatrix} a_i & b_i \\ 0 & d_i \end{smallmatrix}\right) \in \Gamma_1 \backslash \mathscr{M}_2(\mathbf{Z})_l,\ i = 1, 2}} e\big(l(\lambda\bar{\lambda}\tau + \bar{\lambda} z_1 + \lambda z_2)\big) \times \varphi\left(\frac{a_1 a_2 \tau + a_1 b_2 + a_2 b_1}{d_1 d_2}, \frac{l(z_1 + \lambda\tau + \mu)}{d_1 d_2}, \frac{l(z_2 + \bar{\lambda}\tau + \bar{\mu})}{d_1 d_2}\right)(d_1 d_2)^{-k}$$

$$= l^{k-6} \sum_{\lambda, \mu} e\big(l(\lambda\bar{\lambda}\tau + \bar{\lambda} z_1 + \lambda z_2)\big) \sum_{S_1, S_2} (\varphi|_{k,1} S_1 S_2)(\tau, z_1 + \lambda\tau + \mu, z_2 + \bar{\lambda}\tau + \bar{\mu})$$

$$= \frac{1}{l^4} \sum_{\lambda, \mu \in \mathcal{O}/l\mathcal{O}} \Big(\big(l^{k-2} \sum_{S_1, S_2} \varphi|_{k,1} S_1 S_2\big) | [\lambda, \mu]\Big)(\tau, z_1, z_2)$$

where λ, μ run independently over a complete system of residues of $\mathcal{O}$ modulo $l\mathcal{O}$ and S_1, S_2 run again independently over

$$\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, 1 \leqslant d | l, 0 \leqslant b < d, ad = l\right\} \simeq \Gamma_1 \backslash \mathscr{M}_2(\mathbf{Z})_l.$$

For given $r \in \mathbf{Z}$ with $0 \leqslant r \leqslant n$, let

$$\mathscr{R}(r, n-r) = \mathscr{R}_p(r, n-r) := \left\{\begin{pmatrix} p^r & a \\ 0 & p^{n-r} \end{pmatrix} \,\middle|\, 0 \leqslant a < p^{n-r}, a \in \mathbf{Z}\right\}$$

and $\mathscr{R}^*(r, n-r) = \mathscr{R}_p^*(r, n-r)$ the subset for which $\gcd(a, p^r, p^{n-r}) = 1$. For $r = 0$ or n, we clearly have $\mathscr{R}(r, n-r) = \mathscr{R}^*(r, n-r)$. To deal further with (18), we need to know the structure of

$$\mathscr{S}_{r,s} := \mathscr{R}(r, n-r) \cdot \mathscr{R}(s, n-s) := \{AB \mid A \in \mathscr{R}(r, n-r), B \in \mathscr{R}(s, n-s)\}.$$

We now show that indeed modulo factors from Γ_1 on the left

$$(19)\quad \mathscr{R}(r, n-r) \cdot \mathscr{R}(s, n-s) = \begin{cases} \overset{p^r}{\dot{\perp}} \coprod\limits_{0 \leqslant u \leqslant s} p^{r+u} \mathscr{R}^*(s-u, 2n-2r-s-u) & \text{for } r+s \leqslant n, \\ \overset{p^{n-s}}{\dot{\perp}} \coprod\limits_{0 \leqslant v \leqslant n-r} p^{n-s+v} \mathscr{R}^*(r+2s-n-v, n-r-v) & \text{for } r+s \geqslant n \end{cases}$$

where $\dot{\perp}$ signifies that the sets involved occur with the indicated multiplicity $*$ and $t\mathscr{R}^*(a, b) := \{tA \mid A \in \mathscr{R}(a, b)\}$. The left-hand side is just $\{M = \left(\begin{smallmatrix} p^{r+s} & ap^{n-s} + bp^r \\ 0 & p^{2n-r-s} \end{smallmatrix}\right) \mid 0 \leqslant a < p^{n-r}, 0 \leqslant b < p^{n-s}\}$. First, when $r+s = n$, we see that the general element M becomes $p^r\left(\begin{smallmatrix} p^s & a+b \\ 0 & p^s \end{smallmatrix}\right)$; (modulo factors from Γ_1 on the left) this covers $p^r \cdot \mathscr{R}(s, s)$ exactly once, for each fixed b, as a runs over $\mathbf{Z}/p^{n-r}\mathbf{Z}$. Hence, for $r+s = n$,

$$\mathscr{S}_{r,s} = \overset{p^r}{\perp} p^r \cdot \mathscr{R}(s, s) = \overset{p^r}{\perp} \coprod_{0 \leqslant u \leqslant s} p^{r+u} \mathscr{R}^*(p^{s-u}, p^{s-u})$$

proving (19) in this case. When $r+s < n$ (respectively $r+s > n$), we have

$$M = p^r \begin{pmatrix} p^s & ap^{n-s-r} + b \\ 0 & p^{2n-2r-s} \end{pmatrix}, \quad \text{resp. } M = p^{n-s} \begin{pmatrix} p^{2s+r-n} & a + bp^{r+s-n} \\ 0 & p^{n-r} \end{pmatrix}.$$

Writing $b = b_1 + p^{n-s-r} b_2$ with $b_1 \in \mathbf{Z}/p^{n-s-r}\mathbf{Z}$ and $b_2 \in \mathbf{Z}/p^r\mathbf{Z}$, $ap^{n-s-r} + b$

$=(b_1+ap^{n-s-r})+b_2p^{n-s-r}=c+b_2p^{n-s-r}$ covers, for each fixed b_2, the elements of $\mathbf{Z}/p^{2n-2r-s}\mathbf{Z}$ once, as a and b_1 run respectively over $\mathbf{Z}/p^{n-r}\mathbf{Z}$ and $\mathbf{Z}/p^{n-s-r}\mathbf{Z}$. Thus, for $r+s<n$,

$$\mathcal{S}_{r,s}=\overset{p^r}{\perp}\, p^r\mathcal{R}(s,\, 2n-r-s).$$

If $r+s>n$ on the other hand, $a+bp^{r+s-n}$ covers $\mathbf{Z}/p^{n-r}\mathbf{Z}$ once along with a, for each fixed b from $\mathbf{Z}/p^{n-s}\mathbf{Z}$ and hence

$$\mathcal{S}_{r,s}=\overset{p^r}{\perp}\, p^{n-s}\mathcal{R}(2s+r-n,\, n-r).$$

The proof of (19) is complete, since $\mathcal{R}(e,f)=\coprod_{0\leqslant g\leqslant e}p^g\mathcal{R}^*(e-g,f-g)$ for $e\leqslant f$.

Remark. The second relation in (19) can formally be obtained from the first by using $(r,s,u)\mapsto(n-s,n-r,v)$ followed by $\mathcal{R}^*(a,b)\mapsto\mathcal{R}^*(b,a)$. Thus $\mathcal{S}_{r',s'}$ with $r'+s'>n$ are determined (already) by $\mathcal{S}_{r,s}$ with $r+s<n$.

For $l^{k-2}\sum_{S_1,S_2}\varphi|_{k,1}S_1S_2$ in (18), we get by (19) the expression

$$(20)\quad p^{n(k-2)}\Big\{\sum_{\substack{0\leqslant r,s\\ r+s\leqslant n}}p^r\sum_{\substack{0\leqslant u\leqslant s\\ M_1}}\varphi|p^{r+u}M_1+\sum_{\substack{0\leqslant r',s'\leqslant n\\ r'+s'>n}}p^{n-s'}\sum_{\substack{0\leqslant u'\leqslant s'\\ M_2'}}\varphi|p^{n-s'+u'}M_2'\Big\}$$

where M_1 runs over $\mathcal{R}^*(s-u,\,2n-2r-s-u)$ and M_2' over $\mathcal{R}(2r'+2s-n-r'-u',\,n-r'-u')$. Under $(r',s')\mapsto(n-s,n-r)$, the condition "$r'+s'>n$" goes over precisely into "$r+s<n$". The part of the sum over r, s, u and M_1 in (20) corresponding to the condition $r+s=n$ clearly reduces under $u\mapsto v\ (:=s-u)$ to

$$(21)\quad \sum_{0\leqslant r\leqslant n}p^r\sum_{0\leqslant v\leqslant n-r}\sum_{M\in\mathcal{R}^*(v,v)}\varphi|p^{n-v}M$$

$$=\sum_{0\leqslant v\leqslant n}(1+p+\ldots+p^{n-v})\,\varphi|_{k,1}\mathcal{R}^*(v,v),$$

using the abbreviation $\varphi|p^\lambda\mathcal{R}^*(\cdot,\cdot)$ for $\sum_{M\in\mathcal{R}^*(\cdot,\cdot)}\varphi|_{k,1}p^\lambda M$ and noting that $\varphi|_{k,1}p^\lambda M=\varphi|_{k,1}M$. The rest of the sum over r, s, u, M_1 as well as over r', s', u', M_2 in (20) yields in all

(22)

$$\sum_{\substack{0\leqslant r,s\\ r+s<n}}p^r\sum_{0\leqslant u\leqslant s}\big(\varphi|_{k,1}\mathcal{R}^*(s-u,\,2n-2r-s-u)+\varphi|_{k,1}\mathcal{R}^*(2n-2r-s-u,\,s-u)\big)$$

$$=\sum_{t=0}^{n-1}\sum_{\substack{0\leqslant r,v\\ r+v\leqslant t}}p^r\big(\varphi|\mathcal{R}^*(v,\,2(n-t)+v)+\varphi|\mathcal{R}^*(2(n-t)+v,\,v)\big)$$

(with $t:=r+s$, $v:=s-u$)

$$=\sum_{w=1}^{n}\sum_{\substack{0\leqslant r,v\\ w+r+v\leqslant n}}p^r\big(\varphi|\mathcal{R}^*(v,\,v+2w)+\varphi|\mathcal{R}^*(v+2w,\,v)\big)\quad(\text{with } w:=n-t)$$

$$=\sum_{\substack{1\leqslant w,0\leqslant v\\ v+w\leqslant n}}\Big(\sum_{0\leqslant r\leqslant n-v-w}p^r\Big)\big(\varphi|\mathcal{R}^*(v,\,v+2w)+\varphi|\mathcal{R}^*(v+2w,\,v)\big).$$

Hence, for the innermost sum over S_1, S_2 in (18), we have, in view of (20)–(22), derived the expression

(23)

$$p^{n(k-2)}\sum_{\substack{0\leqslant v,w\\ v+w\leqslant n}}(1+p+\ldots+p^{n-v-w})\big(\varphi|_{k,1}\mathcal{R}^*(v,\,v+2w)+\varphi|_{k,1}\mathcal{R}^*(v+2w,\,v)\big)\frac{1}{1+\delta}$$

with $\delta=\delta_{w,0}:=1$ for $w=0$ and 0, otherwise. Taking $a=n-v-w$, $b=v$ and $c=v+2w$, (23) becomes

$$(24)\quad p^{n(k-2)}\sum_{\substack{0\leqslant a,b,c\\ 2a+b+c=2n}}(1+p+\ldots+p^a)\,\varphi|_{k,1}\mathcal{R}^*(b,c)$$

$$=p^{n(k-2)}\sum_{\substack{0\leqslant a,b,c\\ 2a+b+c=2n}}\{\psi(p^a)+\psi(p^{a-2})+\psi(p^{a-4})+\ldots$$

$$\ldots+\psi(p^{(1-(-1)^a)/2})\}\,\varphi|_{k,1}\mathcal{R}^*(b,c)$$

$$=p^{n(k-2)}\sum_{0\leqslant d\leqslant n}\psi(p^d)\sum_{\substack{0\leqslant b,c,r(\in\mathbf{Z})\\ b+c+4r=2(n-d)}}\varphi|_{k,1}\mathcal{R}^*(b,c),$$

collecting together $\varphi|\mathcal{R}^*(\cdot,\cdot)$ corresponding to the same d

$$=\sum_{0\leqslant d\leqslant n}\psi(p^{n-d})\,p^{(n-d)(k-2)}p^{d(k-2)}\sum_{\substack{0\leqslant b,c,r(\in\mathbf{Z})\\ b+c+4r=2d}}\varphi|\mathcal{R}^*(b,c).$$

We know, from the definition of $\mathcal{T}_{p^d}$, that, for $\varphi\in J^0_{k,1}$,

$$\mathcal{T}_{p^d}(\varphi)=p^{d(k-6)}\sum_{\lambda,\mu\in\mathcal{O}/p^d\mathcal{O}}\;\sum_{\substack{0\leqslant b,c,r\\ b+c+4r=2d}}\varphi|\mathcal{R}^*(b,c)|[\lambda,\mu].$$

Consequently, from (18), (23) and (24), we obtain finally, for $l=p^n$,

$$(\mathcal{V}_l^*\mathcal{V}_l)(\varphi)=(1/l^4)\sum_{\lambda,\mu\in\mathcal{O}/l\mathcal{O}}\Big(\sum_d\psi(p^{n-d})\,p^{(n-d)(k-2)}p^{d(k-2)}\sum_{b,c,r}\varphi|\mathcal{R}^*(b,c)\Big)|[\lambda,\mu]$$

$$=\sum_{0\leqslant d\leqslant n}\psi(p^{n-d})\,p^{(n-d)(k-2)}p^{d(k-2)}\cdot(1/p^{4n})\sum_{\lambda,\mu\in\mathcal{O}/p^n\mathcal{O}}\Big(\sum_{b,c,r}\varphi|\mathcal{R}^*(b,c)\Big)|[\lambda,\mu]$$

$$= \sum_d \psi(p^{n-d})\, p^{(n-d)(k-2)} p^{d(k-2)} \cdot (1/p^{4d}) \cdot \sum_{\lambda',\mu' \in \mathcal{O}/p^d\mathcal{O}} \Big(\sum_{b,c,r} \varphi \mid \mathscr{R}^*(b,c)\Big) \mid [\lambda',\mu']$$

$$= \sum_{0\leqslant d\leqslant n} \psi(p^{n-d})(p^{n-d})^{k-2}\, \mathscr{T}_{p^d}(\varphi)$$

and the proposition is proved.

5. The Maass space and Saito–Kurokawa descent. The subspace $M_k(\Gamma_2) := \{F \in S_k(\Gamma_2)$ for whose Fourier coefficients $a(T)$ in (1), there exists $\tilde a$ defined on $\mathbf{Z}$ with $a\left(\begin{pmatrix} n & \alpha/2 \\ \bar\alpha/2 & m\end{pmatrix}\right) = \sum_{1\leqslant d|(m,n,\alpha)} d^{k-1}\tilde a((4mn-|\alpha|^2)/d^2)\}$ is called the *Maass space* for Γ_2 and weight k. It is stable under the action of all the Hecke operators and if φ is the first Fourier–Jacobi coefficient of F, the map $F \mapsto \varphi$ from $M_k(\Gamma_2)$ to $J^0_{k,1}$ is a Hecke-equivariant isomorphism. If F is a Hecke eigenform in $M_k(\Gamma_2)$, so is φ and let us denote by $\lambda_\varphi(l)$ the eigenvalue of φ corresponding to the Hecke operator $\mathscr{T}_l$. Defining the theta series

$$\theta_h(\tau, z_1, z_2) := \sum_{m\in\mathcal{O}} e\big(\tau|m+h|^2 + \mathrm{tr}((m+h)z)\big)$$

for $(\tau, z_1, z_2) \in \mathscr{H}\times \mathbf{C}\times\mathbf{C}$ and any h in $\frac{1}{2}\mathcal{O}/\mathcal{O}$, we know that any φ in $J^0_{k,1}$ can be written as a finite linear combination

$$\varphi(\tau, z_1, z_2) = \sum_{h\in\frac{1}{2}\mathcal{O}/\mathcal{O}} \varphi_h(\tau)\,\theta_h(\tau, z_1, z_2)$$

and the "θ_0-component" φ_0 of φ actually belongs to $S_{k-1}(\Gamma_0(4);\chi)$ with $\chi(n) := (\frac{-4}{n})$.

The map $\mathscr{D}$: $\varphi(\tau, z_1, z_2) \mapsto \varphi_0(\tau)$ from $J^0_{k,1}$ to $S_{k-1}(\Gamma_0(4);\chi)$ is injective. Defining $\tilde\varphi_0$ by $(\tilde\varphi_0|_{k-1}W_4)(\tau) = \varphi_0(\tau)$ under the Fricke involution $z\mapsto -1/4z$ corresponding to $W_4 := (\begin{smallmatrix}0 & -1\\ 4 & 0\end{smallmatrix})$, one obtains as isomorphism $\tilde{\mathscr{D}}$: $J^0_{k,1} \xrightarrow{\sim} S^+_{k-1}(\Gamma_0(4);\chi)$ where $S^+_{k-1}(\Gamma_0(4);\chi) := \{g\in S_{k-1}(\Gamma_0(4);\chi) \mid g(\tau) = \sum_{n\geqslant 1} b(n)e(n\tau),\ b(n) = 0$ for $\chi(n) = 1\}$. This subspace has a basis given by $\{f_i - f_i^\varrho \mid i = 1, 2, \ldots, a\}$ where each f_i is a normalized Hecke eigenform in $S_{k-1}(\Gamma_0(4);\chi)$ in the sense of Atkin–Lehner, $f_i^\varrho(\tau) := \bar f_i(-\bar\tau)$ and $a :=$ dimension over $\mathbf{C}$ of $S^+_{k-1}(\Gamma_0(4);\chi)$. The isomorphism I: $F\mapsto \tilde\varphi_0$ is precisely the "Saito–Kurokawa descent" from $M_k(\Gamma_2)$ to $S^+_{k-1}(\Gamma_0(4);\chi)$ [6].

We will show that the map $\tilde{\mathscr{D}}$ is Hecke-equivariant with respect to a homomorphism, say δ, between the associated Hecke algebras and thereby compute the eigenvalues $\lambda_\varphi(l)$ of φ in $J^0_{k,1}$ under $\mathscr{T}_l$, where, in view of our proposition, it clearly suffices to take $l = p^n$ for prime $p\in \mathbf{N}$ and $n\geqslant 1$. (This *direct* computation of $\lambda_\varphi(p^n)$ spares us from having to find explicit generators and relations in the local Hecke algebra for $J^0_{k,1}$.) Again, in view of the relation between $\mathscr{T}_l$ and $\mathscr{T}^0_r$, it is enough to find the eigenvalues $\lambda^0_\varphi(l)$ of φ under $\mathscr{T}^0_l$, for $l = p^n$; along with these simplifying steps (and symbols) we shall also take, in the sequel, φ to be of the form $\tilde{\mathscr{D}}^{-1}(f - f^\varrho)$ for $f = f_i$ with some i $(1\leqslant i\leqslant a)$.

With obvious notation, we have the following

LEMMA 2. *With $l = p^n$, φ and f as above and $f(\tau) = \sum_{t\geqslant 1} a(t)e(t\tau)$,*

$$\lambda^0_\varphi(l) = \begin{cases} a(p^{2n}) - p^{k-3}\chi(p)\,a(p^{2n-2}) & \textit{for any odd prime } p, \\ a(2)^{2n} + \overline{a(2)}^{2n} & \textit{for } p = 2.\end{cases}$$

Proof. By definition, $(\mathscr{T}^0_{p^n}\varphi)(\tau, z_1, z_2)$ equals

$$p^{n(k-6)} \sum_{0\leqslant s\leqslant 2n} p^{k(s-n)} \sum_{u_{2n-s};\lambda;\mu} \varphi\left(\frac{p^s\tau + u_{2n-s}}{p^{2n-s}}, \frac{z_1+\lambda\tau+\mu}{p^{n-s}}, \frac{z_2+\bar\lambda\tau+\bar\mu}{p^{n-s}}\right) \times e\big(\lambda\bar\lambda\tau + \mathrm{tr}(\bar\lambda z)\big)$$

$$= p^{n(k-6)} \sum_{0\leqslant s\leqslant 2n} p^{k(s-n)} \sum_{u_{2n-s};\lambda;\mu}\ \sum_h \sum_{m\in\mathcal{O}} \varphi_h\left(\frac{p^s\tau + u_{2n-s}}{p^{2n-s}}\right) \times e\big((p^s\tau + u_{2n-s})p^{s-2n}|m+h|^2 + (z_1+\lambda\tau+\mu)(m+h)p^{s-n} + (z_2+\bar\lambda\tau+\bar\mu)(\bar m+\bar h)p^{s-n} + \lambda\bar\lambda\tau + \mathrm{tr}(\bar\lambda z)\big)$$

where, as in the rest of this proof, in the summations carried out, u_r runs over $\mathbf{Z}/p^r\mathbf{Z}$ subject to the additional proviso that $p\nmid u_r$ for $0<r<2n$, h runs over $\frac{1}{2}\mathcal{O}/\mathcal{O}$ and λ, μ run independently over $\mathcal{O}/p^n\mathcal{O}$.

We need to compute the θ_0-component of $\mathscr{T}^0_{p^n}\varphi$. First, we restrict ourselves to that part of the sum above defining it where $n\leqslant s\leqslant 2n$ and take up the remaining part later. For $0\leqslant s\leqslant 2n$, the coefficient of $\varphi_h\left(\frac{p^s\tau + u_{2n-s}}{p^{2n-s}}\right)$ therein (for h, s, u_{2n-s} fixed) is, upto the factor $p^{n(k-6)+k(s-n)}$, precisely

$$(25)\quad \sum_{\lambda,\mu}\sum_{m\in\mathcal{O}} e\big(\tau(|p^{s-n}(m+h)|^2 + \lambda\bar\lambda + \mathrm{tr}(p^{s-n}(m+h)\lambda)) + \mathrm{tr}((\bar\lambda + p^{s-n}(m+h)z)\big)\, e\big(u_{2n-s}|m+h|^2 p^{s-2n} + \mathrm{tr}((m+h)p^{s-n}\mu)\big).$$

If we write $m = m_1 + p^{2n-s}m_2$ with m_2 running over $\mathcal{O}$ and m_1 over $\mathcal{O}/p^{2n-s}\mathcal{O}$ then

$$e(u_{2n-s}|m+h|^2 p^{s-2n}) = e\big(u_{2n-s}|p^{s-n}(m_1+h)|^2 p^{-s}\big)$$

is independent of m_2 and therefore the sum over m, λ, μ in (25) reduces, for $n\leqslant s\leqslant 2n$, to

$$p^{2n} \sum_{\substack{m_2\in\mathcal{O};\lambda\\ m_1\in\mathcal{O}/p^{2n-s}\mathcal{O}}} e\big(\tau|p^n m_2 + \bar\lambda + p^{s-n}(m_1+h)|^2 + \mathrm{tr}((p^n m_2 + \bar\lambda + p^{s-n}(m_1+h))z)\big) \times e(u_{2n-s}|m_1+h|^2 p^{s-2n}).$$

When m_2 runs over $\mathcal{O}$ and λ over $\mathcal{O}/p^n\mathcal{O}$, $x := p^n m_2 + \bar{\lambda}$ (and likewise, for every fixed $m_1 \in \mathcal{O}/p^{2n-s}\mathcal{O}$, also $x + p^{s-n} m_1$) runs through $\mathcal{O}$ exactly once. Thus the last-mentioned sum (for $n \leqslant s \leqslant 2n$) becomes

$$p^{2n} \sum_{x \in \mathcal{O}} e\left(\tau |x + p^{s-n} h|^2 + \operatorname{tr}((x + p^{s-n} h) z)\right) \sum_{m_1 \in \mathcal{O}/p^{2n-s}\mathcal{O}} e(u_{2n-s} |m_1 + h|^2 p^{s-2n}).$$

Here, let us note that the series over x is nothing but $\theta_h(\tau, z_1, z_2)$ for odd p and $n \leqslant s \leqslant 2n$ or $p = 2$ with $s = n$ and $\theta_0(\tau, z_1, z_2)$ for $p = 2$ and $s > n$. The sum over m_1 is a generalized Gauss sum which can be evaluated. In fact, it can be shown that, for $p \nmid u \in \mathbf{Z}$, $r \geqslant 0$ and $h \in \frac{1}{2}\mathcal{O}/\mathcal{O}$,

$$G(u, h; p^r) := \sum_{m \in \mathcal{O}/p^r\mathcal{O}} e(u |m + h|^2 p^{-r})$$
$$= \begin{cases} (p\chi(p))^r & \text{for odd primes } p \text{ and } r > 0, \\ 2^{r+1}\chi(u)\sqrt{-1} & \text{if } h = 0,\ r \geqslant 2,\ p = 2 \text{ or} \\ & \text{if } h = \frac{1}{2}(1 + \sqrt{-1}),\ r = 1,\ p = 2, \\ 0 & \text{otherwise (for } p = 2 \text{ and } r > 0). \end{cases}$$

We see therefore that the total contribution to the θ_0-component of $\mathcal{T}^0_{p^n}\varphi$ from terms for which $n \leqslant s \leqslant 2n$ is

$$\sum_{n \leqslant s \leqslant 2n} p^{n(k-6) + k(s-n) + 2n} (p\chi(p))^{2n-s} \sum_{u_{2n-s}} \varphi_0((p^s\tau + u_{2n-s}) p^{s-2n})$$

for all odd primes p; the relevant contribution for $p = 2$ from terms with $n \leqslant s \leqslant 2n$ can be seen to be

$$2^{n(k-6)+2n} \{\sum_{u_n} \varphi_0(\tau + u_n 2^{-n}) G(u_n, 0; 2^n)$$
$$+ \sum_{n < s \leqslant 2n} 2^{k(s-n)} \sum_{h; u_{2n-s}} \varphi_h((2^s\tau + u_{2n-s}) 2^{s-2n}) G(u_{2n-s}, h; 2^{2n-s})\}.$$

The contribution arising from terms with $0 \leqslant s < n$ can be treated similarly. We give the detailed arguments for the case $p = 2$, since the case of odd p is simpler. Carrying out in (25) the summation over μ (while keeping the other summation-indices fixed) and noting that

$$\sum_{\mu} e\left(\operatorname{tr}(\mu(m+h) 2^{s-n})\right) = \begin{cases} 2^{2n} & \text{if } (m+h) 2^{s-n} \in \frac{1}{2}\mathcal{O}, \\ 0 & \text{otherwise,} \end{cases}$$

we can have non-zero contribution only from terms with $h = 0$ and $m \in 2^{n-s-1}\mathcal{O}$ (as the first alternative above on the right-hand side would entail, under "$n > s$"). Consequently, writing $m = 2^{n-s-1} l$ with $l \in \mathcal{O}$, (25) now takes the form

$$2^{2n} \sum_{l \in \mathcal{O}; \lambda} e\left(\tau |(l/2) + \bar{\lambda}|^2 + \operatorname{tr}(((l/2) + \bar{\lambda}) z) + u_{2n-s} |l/2|^2 2^{-s}\right).$$

Again, we put $l/2 = x + h_1$ with $x \in \mathcal{O}$ and $h_1 \in \frac{1}{2}\mathcal{O}/\mathcal{O}$ and further $x = x_1 + 2^s x_2$ with $x_1 \in \mathcal{O}/2^s\mathcal{O}$ and $x_2 \in \mathcal{O}$. We also note that

$$e(u_{2n-s} |x_1 + 2^s x_2 + h_1|^2 2^{-s}) = e(u_{2n-s} |x_1 + h_1|^2 2^{-s})$$

is independent of $x_2 \in \mathcal{O}$ and moreover, as λ and x_2 run respectively over $\mathcal{O}/2^n\mathcal{O}$ and $\mathcal{O}$, $\bar{\lambda} + 2^s x_2$ runs through $\mathcal{O}$ precisely $2^{2(n-s)}$ times. Thus (25) finally reduces to

$$2^{2n+2n-2s} \sum_{h_1 \in \frac{1}{2}\mathcal{O}/\mathcal{O}} G(u_{2n-s}, h_1; 2^s) \theta_{h_1}(\tau, z_1, z_2)$$

and using the above-mentioned values of the Gauss sums, we see that the contribution to the θ_0-component of $\mathcal{T}^0_{2^n}\varphi$ from the terms with $0 \leqslant s < n$ is 0 for $s = 1$,

$$2^{-2n} \sum_{u_{2n}} \varphi_0((\tau + u_{2n})/2^{2n}) \quad \text{for } s = 0$$

and

$$2^{n(2k-4)+1} \sqrt{-1} \sum_{u_{2n-s}} \chi(u_{2n-s}) \varphi_0((2^s\tau + u_{2n-s}) 2^{s-2n}) 2^{(s-2n)(k-1)}$$

for $1 < s < n$. Now, using the relation $\sum_h \varphi_h(\tau) = -2\sqrt{-1}\, \varphi_0(-1/\tau) \tau^{1-k}$ (see [6], p. 221), we conclude that the θ_0-component of $\mathcal{T}^0_{2^n}\varphi$ is

(26) $$2^{n(2k-4)}$$
$$\times \left\{ \sum_{u_{2n}} \varphi_0\left(\frac{\tau + u_{2n}}{2^{2n}}\right) 2^{-2n(k-1)} + 2\sqrt{-1} \left(\sum_{1 < s < 2n-1; u_{2n-s}} \chi(u_{2n-s}) \varphi_0\left(\frac{2^s\tau + u_{2n-s}}{2^{2n-s}}\right) \right.\right.$$
$$\left.\left. \times 2^{-(2n-s)(k-1)} + \varphi_{(1+\sqrt{-1})/2}\left(\frac{2^{2n-1}\tau + 1}{2}\right) 2^{-(k-1)} - \varphi_0\left(\frac{-1}{2^{2n}\tau}\right) (2^{2n}\tau)^{-(k-1)} \right) \right\}.$$

The θ_0-component of $\mathcal{T}^0_{p^n}\varphi$ for odd primes p is given by

$$\sum_{0 \leqslant s \leqslant 2n} p^{n(k-4) + k(s-n) + 2n - s} \chi(p^{2n-s}) \varphi_0((p^s\tau + u_{2n-s}) p^{s-2n})$$
$$= p^{n(2k-4)} \sum_{0 \leqslant s \leqslant 2n} \sum_{u_{2n-s}} \varphi_0\left(\frac{p^s\tau + u_{2n-s}}{p^{2n-s}}\right) \chi(p^{2n-s}) p^{-(2n-s)(k-1)}$$

which is nothing but $\varphi_0 | T(1, p^{2n})$ where, for $a, b \in \mathbf{N}$, we denote the Hecke operator corresponding to the double coset $\Gamma_0(4)\,\mathrm{Diag}(a, b)\,\Gamma_0(4)$ by $T(a, b)$. Now $T(1, p^{2n})$ commutes with W_4 and hence $\delta(\mathcal{T}^0_{p^n}) = T(1, p^{2n})$. This gives us $\lambda^0_\varphi(p^n) = a(p^{2n}) - p^{k-3}\chi(p) a(p^{2n-2})$, at once for any odd prime p.

Using also the relation $\varphi_{(1+\sqrt{-1})/2}(\tau) = \varphi_0(\tau/(2\tau+1))/(2\tau+1)^{k-1}$ (see [6], p. 222), we obtain from (26),

(27) $$(\widetilde{\mathcal{T}^0_{2^n}\varphi})_0 = \tilde{\varphi}_0|T(2^{2n},1)+2\sqrt{-1}$$
$$\times\Big\{\sum_{2\le s\le 2n-2;\,u_{2n-s}} \chi(u_{2n-s})\,\tilde{\varphi}_0|W_4\begin{pmatrix}2^s & u_{2n-s}\\ 0 & 2^{2n-s}\end{pmatrix}W_4^{-1}$$
$$+\tilde{\varphi}_0|W_4\begin{pmatrix}2^{2n-1} & 1\\ 2^{2n} & 4\end{pmatrix}W_4^{-1}-\tilde{\varphi}_0|W_4\begin{pmatrix}0 & -1\\ 2^{2n} & 0\end{pmatrix}W_4^{-1}\Big\}.$$

From

$$2\sqrt{-1}\,\tilde{\varphi}_0|\,W_4 = -\sum_{c\in \mathbf{Z}/4\mathbf{Z}}\tilde{\varphi}_0|\begin{pmatrix}1 & c\\ 0 & 4\end{pmatrix}$$

(see [6], p. 226), it is seen that

$$-2\sqrt{-1}\,\tilde{\varphi}_0|\,W_4\begin{pmatrix}0 & -1\\ 2^{2n} & 0\end{pmatrix}W_4^{-1} = \sum_{c\in \mathbf{Z}/4\mathbf{Z}}\tilde{\varphi}_0|\begin{pmatrix}1 & 2^{2n-2}c\\ 0 & 2^{2n}\end{pmatrix};$$

likewise,

$$-2\sqrt{-1}\,\tilde{\varphi}_0|W_4\begin{pmatrix}2^{2n-1} & 1\\ 2^{2n} & 4\end{pmatrix}W_4^{-1} = \sum_{c\in \mathbf{Z}/4\mathbf{Z}}\tilde{\varphi}_0|\begin{pmatrix}1 & 2^{2n-2}c-2^{2n-3}\\ 0 & 2^{2n}\end{pmatrix},$$

if we note that

$$\begin{pmatrix}-4c-1 & c^2\\ -16 & 4c-1\end{pmatrix}\in\Gamma_0(4)\quad\text{and}\quad \chi(4c-1) = -1.$$

Now for $2\le s\le 2n-2$, let

$$t := s-2 \quad\text{and}\quad P_1 := \begin{pmatrix}-u_{2n-s}-2^{2n-2-t}c & 2^t\\ -2^{2n-t} & 0\end{pmatrix}$$

with given odd u_{2n-s} modulo 2^{2n-s} and c modulo 4. There exists (an odd) u'_{2n-s} unique modulo 2^{2n-2-t} such that $w := u'_{2n-s}(u_{2n-s}+2^{2n-2-t}c) \equiv -1 \pmod{2^{2n-2-t}}$ and again $w\equiv -(1+2^{2n-2-t}u_{2n-s}c_1)\pmod{2^{2n-t}}$ for a uniquely determined c_1 modulo 4. If we set $r := 2^t(2^{2n-2-t}c_1+u'_{2n-s})$, then $r/2^t\in\mathbf{Z}$ and $2^t+(u_{2n-s}+2^{2n-2-t}c)r\in 2^{2n}\mathbf{Z}$. Moreover,

$$P_2 := P_1\begin{pmatrix}1 & -r/2^{2n}\\ 0 & 2^{-2n}\end{pmatrix} = \begin{pmatrix}* & *\\ -2^{2n-t} & r/2^t\end{pmatrix}\in\Gamma_0(4).$$

The pair (u_{2n-s}, c) determines (u'_{2n-s}, c_1) uniquely and this gives rise to a bijection of $(\mathbf{Z}/2^{2n-s}\mathbf{Z})^\times\times\mathbf{Z}/4\mathbf{Z}$, where the first factor is the group of odd residue classes modulo 2^{2n-s}. Since $\chi(u'_{2n-s}) = -\chi(u_{2n-s})$ and $P_2\in\Gamma_0(4)$, we have

$$\tilde{\varphi}_0|P_2 = \chi(r/2^t)\,\tilde{\varphi}_0 = -\chi(u_{2n-s})\,\tilde{\varphi}_0.$$

Thus, for $2\le s\le 2n-2$, we obtain

(28)
$$2\sqrt{-1}\,\chi(u_{2n-s})\,\tilde{\varphi}_0|\,W_4\begin{pmatrix}2^s & u_{2n-s}\\ 0 & 2^{2n-s}\end{pmatrix}W_4^{-1} = -\chi(u_{2n-s})\sum_{c\in\mathbf{Z}/4\mathbf{Z}}\tilde{\varphi}_0|\,P_1$$
(with P_1 as above)
$$= \sum_{c_1\in\mathbf{Z}/4\mathbf{Z}}\tilde{\varphi}_0\Big|\begin{pmatrix}1 & 2^t(2^{2n-2-t}c_1+u'_{2n-s})\\ 0 & 2^{2n}\end{pmatrix}.$$

Together with the residue classes $2^{2n-2}c$, $2^{2n-2}c-2^{2n-3}$, the residues r above cover all the residue classes modulo 2^{2n}. Hence we get by (27) and (28),

$$(\widetilde{\mathcal{T}^0_{2^n}\varphi})_0 = \tilde{\varphi}_0|\,\Gamma_0(4)\begin{pmatrix}2^{2n} & 0\\ 0 & 1\end{pmatrix}\Gamma_0(4)+\tilde{\varphi}_0|\Gamma_0(4)\begin{pmatrix}1 & 0\\ 0 & 2^{2n}\end{pmatrix}\Gamma_0(4)$$
$$= (a(2)^{2n}+\overline{a(2)}^{2n})\,\tilde{\varphi}_0,$$

i.e.

$$\delta(\mathcal{T}^0_{2^n}) = (a(2)^{2n}+\overline{a(2)}^{2n})\,\mathrm{Id}$$

where Id denotes the identity operator. As a result, we have

$$\lambda^0_\varphi(2^n) = a(2)^{2n}+\overline{a(2)}^{2n}$$

and this concludes the proof of Lemma 2.

Remarks. For $n = 1$, the results in Lemma 2 appear as assertions (without detailed proof) in Propositions 4.2 and 4.3 of [2], wherein one should read 2^{k-3} instead of 2^{2k-6} in the expression for $\tilde{\psi}_0(\tau)$ in Proposition 4.3 on page 19 and $3\cdot 2^{k-2}$ instead of $3\cdot 2^{k-4}$ in the last line of page 20; the second term in the formula for $\mathrm{Desc}(T_{1,p})$ on page 20 should read $p^{2k-8}(p^3+p^4+p-1)$.

We are thankful to Professor A. Krieg for having rushed to us, at our request, a copy of [2] from Münster.

6. A relation between $D_{F,G}$ and Z_F. For the Fourier–Jacobi coefficients φ_m, ψ_m respectively of a Hecke eigenform F and any cusp form G in $M_k(\Gamma_2)$, we have $\langle\varphi_m,\psi_m\rangle = \langle\mathcal{V}_m\varphi_1,\mathcal{V}_m\psi_1\rangle = \langle\mathcal{V}^*_m\mathcal{V}_m\varphi_1,\psi_1\rangle$ and so, by the proposition,

$$D_{F,G}(s) = \zeta_{\mathbf{K}}(s-k+3)\,\zeta(2s-2k+4)\langle\varphi_1,\psi_1\rangle\sum_{t\in\mathbf{N}}\sum_{d|t}\psi(d)\,d^{k-2}\lambda_{\varphi_1}(t/d)\,t^{-s}$$
$$= \langle\varphi_1,\psi_1\rangle\,\zeta_{\mathbf{K}}(s-k+3)\,\zeta(2s-2k+4)\sum_{d\in\mathbf{N}}\psi(d)\,d^{k-2-s}\sum_{t\in\mathbf{N}}\lambda_{\varphi_1}(t)\,t^{-s}$$
$$= \langle\varphi_1,\psi_1\rangle\,\zeta_{\mathbf{K}}(s-k+3)\,\zeta(s-k+1)\,\zeta(s-k+2)\sum_{t\in\mathbf{N}}\lambda_{\varphi_1}(t)\,t^{-s},$$

on noting that

$$\sum_{d\in N} \psi(d)\,d^{-\varrho} = \zeta(\varrho-1)\zeta(\varrho)/\zeta(2\varrho).$$

Now

$$\sum_{t\in N} \lambda_{\varphi_1}(t)\,t^{-s} = \prod_p \Big(\sum_{n\geq 0} \lambda_{\varphi_1}(p^n)\,p^{-ns}\Big)$$

where the product is taken over all primes $p\in N$. We know from Lemma 2 that, for odd primes p,

$$\lambda_{\varphi_1}(p^n)$$

$$= \sum_{0\leq r\leq [n/2]} p^{2r(k-2)} a(p^{2(n-2r)}) - p^{-1}\chi(p) \sum_{0\leq r\leq [(n-1)/2]} p^{(2r+1)(k-2)} a(p^{2(n-2r-1)})$$

where $[x] :=$ largest integer not exceeding $x\in \boldsymbol{R}$. Therefore

$$\sum_{n\geq 0} \lambda_{\varphi_1}(p^n)\,p^{-ns} = \Big(\sum_{j\geq 0} p^{2j(k-2-s)} - p^{-1}\chi(p) \sum_{j\geq 0} p^{(2j+1)(k-2-s)}\Big)\Big(\sum_{u\geq 0} a(p^{2u})\,p^{-us}\Big)$$

which is precisely the pth Euler factor of $R_f(s)/L(s-k+3, \chi)$, where

$$R_f(s)$$

$$:= \{(1-a(2)^2 2^{-s})(1-\overline{a(2)}^2 2^{-s}) \prod_{p\neq 2} (1-\alpha_p^2 p^{-s})(1-\chi(p)\alpha_p\bar{\alpha}_p p^{-s})(1-\bar{\alpha}_p^2 p^{-s}\}^{-1}$$

is the symmetric square zeta function associated to f and $\alpha_p + \chi(p)\bar{\alpha}_p = a(p)$, $\alpha_p\bar{\alpha}_p = p^{k-2}$. By Lemma 2 again

$$\sum_{n\geq 0} \lambda_{\varphi_1}(2^n)\,2^{-ns} = \sum_{n\geq 0} \Big(\sum_{\substack{n_1, n_2\geq 0\\ n_1+n_2=n}} a(2)^{2n_1}\overline{a(2)}^{2n_2}\Big) 2^{-ns}$$

$$= \{(1-a(2)^2 2^{-s})(1-\overline{a(2)}^2 2^{-s})\}^{-1}.$$

Hence

$$\sum_{t\in N} \lambda_{\varphi_1}(t)\,t^{-s} = R_f(s)/L(s-k+3, \chi)$$

which implies that

$$D_{F,G}(s) = \langle \varphi_1, \psi_1\rangle \zeta(s-k+3)\zeta(s-k+2)\zeta(s-k+1)\,R_f(s).$$

Since Gritsenko [2] has shown that $Z_F(s) = \zeta(s-k+1)\,L(s-k+2, \chi) \times \zeta(s-k+3)\,R_f(s)$, we have finally the following

THEOREM 2. *Let F, $G\in M_k(\Gamma_2)$, F being a Hecke eigenform. Then*

$$D_{F,G}(s) = \langle \varphi_1, \psi_1\rangle \zeta(s-k+2)\,L\Big(s-k+2, \Big(\frac{-4}{\cdot}\Big)\Big)^{-1} Z_F(s).$$

References

[1] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Math., Vol. 55, Birkhaüser, Boston 1985.

[2] V. A. Gritsenko, *The Maass space for SU(2, 2), Hecke operators and zeta functions*, LOMI R-7-85 (Preprint).

[3] — *Zeta function of degree six for Hermite modular forms of genus two*, Zap. Nauchn. Semin. LOMI 154 (1986), 46–66.

[4] H. Klingen, *Zum Darstellungssatz für Siegelsche Modulformen*, Math. Z. 102 (1967), 30–43.

[5] W. Kohnen and N.-P. Skoruppa, *A certain Dirichlet series attached to Siegel modular forms of degree two*, Invent. Math. 95 (1989), 541–558.

[6] H. Kojima, *An arithmetic of Hermitian modular forms of degree two*, ibid. 69 (1982), 217–227.

[7] C. L. Siegel, *Indefinite quadratische Formen und Funktionentheorie I*, Math. Ann. 124 (1951), 17–54.

SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
Homi Bhabha Road
Bombay 400 005
India

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