Now take an arbitrary positive ε. By definition of u, v and Lemma 2 we can find a K such that we have

$$S(x) \leqslant (v+\varepsilon)x+K,$$
$$S(a_k) \leqslant (u+\varepsilon)a_k+K,$$
$$S(x) \geqslant (u-\varepsilon)x-K$$

for all x and k. Applying these inequalities in this order to the terms of (14) we obtain

$$S(y) \leqslant vy-(v-u)a_n+\varepsilon(y+2a_{n+1}-2a_n)+3K.$$

Taking into account that $a_n \geqslant y/2$ and $a_{n+1} \leqslant 2y$, this gives

$$S(y) \leqslant \frac{u+v}{2}y+5\varepsilon y+3K.$$

Dividing by y and taking the lim sup we get

$$v \leqslant (u+v)/2+5\varepsilon.$$

Since this holds for every $\varepsilon > 0$, we have $v \leqslant u$. ■

Remark. (1) was used several times in the course of the proof. The crucial one seems to be that in the proof of Lemma 2 to infer $d_{i+1}-d_i \leqslant a_m$: in the rest weaker assumptions would also work.

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An additive problem of prime numbers

by

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1. Introduction. Let $\Lambda(x) = \log p$ if $x = p^m$ with a prime number p and an integer $m \geqslant 1$, and $= 0$ otherwise. We put

$$r_2(n) = \sum_{m+k=n} \Lambda(m)\Lambda(k), \quad S_2(n) = \prod_{p|n}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid n}\left(1-\frac{1}{(p-1)^2}\right).$$

It is a long standing conjecture of Goldbach that

$$r_2(n) > 0 \quad \text{for even } n \geqslant 6.$$

Quantitatively, it is a conjecture of Hardy and Littlewood that

$$r_2(n) \sim nS_2(n) \quad \text{as even } n \to \infty.$$

In this article, we are concerned with the asymptotic behavior of the sum

$$\sum_{n \leqslant X} (r_2(n)-nS_2(n)) \quad \text{as } X \to \infty.$$

We recall a well-known result related with this problem. It is shown by van der Corput [2], Chudakov [3] and Estermann [4] that

$$\sum_{n \leqslant X} (r_2(n)-nS_2(n))^2 \ll X^3(\log X)^{-A},$$

where A is any positive constant. This implies, in particular, that

$$\sum_{n \leqslant X} r_2(n) = \tfrac{1}{2}X^2+O\big(X^2(\log X)^{-A}\big),$$

since by Lemma 1 of Montgomery and Vaughan [8]

$$\sum_{n \leqslant X} nS_2(n) = \tfrac{1}{2}X^2+O(X\log X).$$

The purpose of the present article is to refine this under the Riemann Hypothesis (RH) as follows.

THEOREM (on RH).

$$\sum_{n\leqslant X} r_2(n)=\tfrac{1}{2}X^2+O(X^{3/2}).$$

The key idea of the proof of this theorem is to apply Gallagher's Lemma 1 of [5].

We assume RH throughout the rest of this article.

2. Proof of Theorem. We put

$$R(y)=\sum_{n\leqslant y}\Lambda(n)-y \qquad \text{for } y>0.$$

We suppose first that X is an integer N. By 8.59 of p. 258 of Prachar [9]

$$\sum_{n\leqslant N} r_2(n)=\sum_{m\leqslant N}\Lambda(m)\sum_{n\leqslant N-m}\Lambda(n)$$

$$=\sum_{m\leqslant N}\Lambda(m)(N-m)+\sum_{2\leqslant m\leqslant N-2}\Lambda(m)R(N-m)+O(\log N)$$

$$=\tfrac{1}{2}N^2+\sum_{2\leqslant m\leqslant N-2}\Lambda(m)R(N-m)+O(N^{3/2})$$

$$=\tfrac{1}{2}N^2+S+O(N^{3/2}),\quad \text{say.}$$

By Satz 4.5 of p. 231 of Prachar [9], we get for $T\geqslant 2$,

$$S=\sum_{2\leqslant m\leqslant N-2}\Lambda(N-m)R(m)$$

$$=\sum_{2\leqslant m\leqslant N-2}\Lambda(N-m)\left\{-\sum_{|\gamma|\leqslant T}\frac{m^{\varrho}}{\varrho}+O\left(\frac{m}{T}\log^2(mT)\right)+O(\log m)\right\}$$

$$=-\sum_{2\leqslant m\leqslant N-2}\Lambda(N-m)\sum_{|\gamma|\leqslant T}\frac{m^{\varrho}}{\varrho}+O\left(\frac{N^2}{T}\log^2(NT)\right)+O(N\log N)$$

$$=S_1+O\left(\frac{N^2}{T}\log^2(NT)\right)+O(N\log N),\quad \text{say,}$$

where $\varrho=1/2+i\gamma$ runs over the zeros of $\zeta(s)$. Hereafter we suppose that $1\ll T\ll N$. We have

$$S_1=-\sum_{2\leqslant m\leqslant N-2}\sqrt{m}\,\Lambda(N-m)\sum_{|\gamma|\leqslant T}\frac{m^{i\gamma}}{1/2+i\gamma}$$

$$=-2\operatorname{Im}\left\{\sum_{2\leqslant m\leqslant N-2}\sqrt{m}\,\Lambda(N-m)\sum_{0<\gamma\leqslant T}\frac{m^{i\gamma}}{\gamma}\right\}$$

$$+O\left(\sum_{m\leqslant N}\sqrt{m}\,\Lambda(N-m)\sum_{0<\gamma\leqslant T}\frac{1}{\gamma^2}\right)=-2\operatorname{Im}(S_2)+S_3,\quad \text{say,}$$

with $S_3\ll N^{3/2}$.

To evaluate S_2, we notice first that by the Riemann–von Mangoldt formula, we get for $Y>Y_0$,

$$\sum_{0<\gamma\leqslant Y}\frac{1}{\gamma}=\frac{1}{4\pi}\log^2 Y-\frac{\log(2\pi)}{2\pi}\log Y+\int_1^{\infty}\frac{S(t)}{t^2}dt-\frac{1+\log(2\pi)}{2\pi}+\frac{7}{8}+\int_1^{\infty}\frac{\eta(t)}{t^2}dt+B(Y)$$

$$=A(Y)+B(Y),\quad \text{say,}$$

where $S(t)=(1/\pi)\arg\zeta(1/2+it)$ as usual, $\eta(t)$ satisfies $\eta(t)=O(1/t)$ for $t>t_0$ and we put

$$B(Y)=\frac{S(Y)}{Y}-\int_Y^{\infty}\frac{S(t)}{t^2}dt+\frac{\eta(Y)}{Y}-\int_Y^{\infty}\frac{\eta(t)}{t^2}dt.$$

We notice next that if $N>Z\geqslant 1$,

$$\sum_{m\leqslant Z}\sqrt{m}\,\Lambda(N-m)=\sum_{N-Z<n\leqslant N-1}\sqrt{N-n}\,\Lambda(n)+\Lambda(N-Z)\sqrt{Z}$$

$$=\int_{N-Z}^{N-1}\sqrt{N-y}\,d\big(y+R(y)\big)+\Lambda(N-Z)\sqrt{Z}$$

$$=\int_{N-Z}^{N-1}\sqrt{N-y}\,dy+R(N-1)-\sqrt{Z}R(N-Z)$$

$$+\frac{1}{2}\int_{N-Z}^{N-1}\frac{R(y)}{\sqrt{N-y}}dy+\Lambda(N-Z)\sqrt{Z}$$

$$=C(Z)+D(Z),\quad \text{say,}$$

where we put

$$C(Z)=\tfrac{2}{3}Z^{3/2}-\tfrac{2}{3},$$

$$D(Z)=R(N-1)-\sqrt{Z}R(N-Z)+\frac{1}{2}\int_{N-Z}^{N-1}\frac{R(y)}{\sqrt{N-y}}dy+\Lambda(N-Z)\sqrt{Z}.$$

If $N\leqslant Z$,

$$\sum_{m\leqslant Z}\sqrt{m}\,\Lambda(N-m)=\sum_{1<n\leqslant N-1}\sqrt{N-n}\,\Lambda(n)$$

$$=\int_1^{N-1}\sqrt{N-y}\,dy+\sqrt{N-1}+R(N-1)+\frac{1}{2}\int_1^{N-1}\frac{R(y)}{\sqrt{N-y}}dy.$$

Consequently, we put for $N \leqslant Z$,

$$D(Z)=\sum_{m\leqslant Z}\sqrt{m}\,\Lambda(N-m)-\left(\frac{2}{3}Z^{3/2}-\frac{2}{3}\right)$$
$$=\frac{2}{3}(N-1)^{3/2}-\frac{2}{3}Z^{3/2}+\sqrt{N-1}+R(N-1)+\frac{1}{2}\int_1^{N-1}\frac{R(y)}{\sqrt{N-y}}\,dy.$$

Now,

$$S_2=\int_1^T\int_1^N v^{it}\,d\big(C(v)+D(v)\big)\,d\big(A(t)+B(t)\big)$$
$$=\int_1^T\int_0^{\log N} e^{itx}\,d\big(C(e^x)+D(e^x)\big)\,d\big(A(t)+B(t)\big)$$
$$=\int_1^T\int_0^{\log N} e^{itx}\big\{-dC(e^x)\,dA(t)+dC(e^x)\,d\big(A(t)+B(t)\big)$$
$$+d\big(C(e^x)+D(e^x)\big)\,dA(t)+dD(e^x)\,dB(t)\big\}$$
$$=S_4+S_5+S_6+S_7,\quad \text{say,}$$

$$S_4=-\int_1^T\int_0^{\log N} e^{itx}\left(\frac{1}{2\pi}\log t\cdot\frac{1}{t}-\frac{\log 2\pi}{2\pi}\cdot\frac{1}{t}\right)e^{3x/2}\,dx\,dt \ll N^{3/2}\int_1^T\frac{\log t}{t^2}\,dt \ll N^{3/2},$$

$$S_5=\int_1^T\int_0^{\log N} e^{itx}e^{3x/2}\,dx\,d\big(A(t)+B(t)\big)$$
$$\ll N^{3/2}\left(\left[\frac{1}{t}\sum_{0<\gamma<t}\frac{1}{\gamma}\right]_1^T+\int_1^T\frac{1}{t^2}\sum_{0<\gamma<t}\frac{1}{\gamma}\,dt\right)\ll N^{3/2}.$$

Since

$$\int_1^T e^{itx}\left(\frac{1}{2\pi}\log t\cdot\frac{1}{t}-\frac{\log 2\pi}{2\pi}\cdot\frac{1}{t}\right)dt \ll \min(1/x,\ \log^2 T),$$

we get

$$S_6=\int_0^1 O(\log^2 T)\,d\Big(\sum_{m<e^x}\sqrt{m}\,\Lambda(N-m)\Big)+\int_1^{\log N} O(1/x)\,d\Big(\sum_{m<e^x}\sqrt{m}\,\Lambda(N-m)\Big)$$
$$\ll N^{3/2}.$$

By Gallagher's lemma (cf. Lemma 1 of [5]), we get

$$S_7 \ll T\log N\cdot \max_{0<\delta\ll 1/T}\Big(\int_0^{\log N}\big(D(e^{y+\delta})-D(e^y)\big)^2\,dy\Big)^{1/2}$$
$$\times\left(\int_1^T\left(B\left(t+\frac{1}{2\log N}\right)-B(t)\right)^2 dt\right)^{1/2}.$$

We denote the last two integrals by S_8 and S_9, respectively. We have

$$S_8 \ll \int_0^{\log N-\delta}\Big\{-\sqrt{e^{y+\delta}}R(N-e^{y+\delta})+\frac{1}{2}\int_{N-e^{y+\delta}}^{N-1}\frac{R(u)}{\sqrt{N-u}}\,du$$
$$+\sqrt{e^y}R(N-e^y)-\frac{1}{2}\int_{N-e^y}^{N-1}\frac{R(u)}{\sqrt{N-u}}\,du\Big\}^2\,dy$$
$$+\int_{\log N-\delta}^{\log N}\Big\{\frac{2}{3}\big((N-1)^{3/2}-e^{3(y+\delta)/2}\big)+\frac{1}{2}\int_1^{N-1}\frac{R(u)}{\sqrt{N-u}}\,du+\sqrt{N-1}$$
$$+\sqrt{e^y}R(N-e^y)-\frac{1}{2}\int_{N-e^y}^{N-1}\frac{R(u)}{\sqrt{N-u}}\,du\Big\}^2\,dy$$
$$=S_{10}+S_{11},\quad\text{say,}$$

$$S_{11}\ll\int_{\log N-\delta}^{\log N}\left\{\frac{N^{3/2}}{T}+\sqrt{N}\left(\frac{N}{T}\right)^{1/2}\log^2 N\right\}^2 dy \ll \frac{1}{T}\left(\frac{N^3}{T^2}+\frac{N^2}{T}\log^4 N\right),$$

since for $\log N-\delta\leqslant y\leqslant \log N$,

$$\int_1^{N-e^y}\frac{R(u)}{\sqrt{N-u}}\,du \ll \int_1^{N/T}\frac{R(u)}{\sqrt{N-u}}\,du \ll \left(\frac{N}{T}\right)^{3/2}\frac{1}{\sqrt{N}}\log^2 N.$$

Next,

$$S_{10}\ll\int_0^{\log N-\delta} e^y\big(R(N-e^{y+\delta})-R(N-e^y)\big)^2\,dy$$
$$+\int_0^{\log N-\delta}\left(\int_{N-e^{y+\delta}}^{N-e^y}\frac{R(u)}{\sqrt{N-u}}\,du\right)^2 dy+\int_0^{\log N-\delta}\left(\sqrt{e^y}R(N-e^{y+\delta})\frac{1}{T}\right)^2 dy$$
$$=S_{12}+S_{13}+S_{14},\quad\text{say,}$$

$$S_{12}=\int_{N(1-e^{-\delta})}^{N-1}\big\{R\big(x+x(e^\delta-1)-N(e^\delta-1)\big)-R(x)\big\}^2\,dx$$
$$\ll\int_{N(1-e^{-\delta})}^{N-1}\big\{R\big(x+x(e^\delta-1)-N(e^\delta-1)\big)-R\big(x+x(e^\delta-1)\big)\big\}^2\,dx$$
$$+\int_{N(1-e^{-\delta})}^{N-1}\big\{R\big(x+x(e^\delta-1)\big)-R(x)\big\}^2\,dx$$
$$\ll\int_{N(e^\delta-1)}^{(N-1)e^\delta}\big(R\big(y-N(e^\delta-1)\big)-R(y)\big)^2\,dy$$
$$+\int_{N(1-e^{-\delta})}^{N-1}\big(R\big(x+x(e^\delta-1)\big)-R(x)\big)^2\,dx.$$

The last two types of the mean values are treated in Saffari and Vaughan [10] (cf. also Goldston and Montgomery [6]). Using their results, we get

$$S_{12} \ll N^2 \max_{0<\delta \ll 1/T} \delta \log^2(1/\delta) \ll \frac{N^2}{T} \log^2 N.$$

Since

$$S_{13} \ll N \log^4 N \int_0^{\log N - \delta} \left(\int_{N-e^{y+\delta}}^{N-e^y} \frac{du}{\sqrt{N-u}} \right)^2 dy$$

$$\ll \frac{N}{T^2} \log^4 N \int_0^{\log N - \delta} e^y \, dy \ll \frac{N^2}{T^2} \log^4 N$$

and

$$S_{14} \ll \frac{N^2}{T^2} \log^4 N,$$

we get

$$S_8 \ll \frac{N^2}{T} \log^2 N + \frac{N^3}{T^3} + \frac{N^2}{T^2} \log^4 N.$$

To conclude our estimate on S_7, we notice that

$$S_9 \ll \int_1^T \left(\frac{\log t}{t} \right)^2 dt \ll 1.$$

Consequently,

$$S_7 \ll T \log N \left(\frac{N}{\sqrt{T}} \log N + \frac{N^{3/2}}{T^{3/2}} + \frac{N}{T} \log^2 N \right).$$

Choosing $T = \sqrt{N} \log^2 N$, we get $S = O(N^{3/2})$. Thus

$$\sum_{n \leq N} r_2(n) = \tfrac{1}{2} N^2 + O(N^{3/2}).$$

It is clear that the restriction on X imposed at the beginning may be removed within the remainder term.

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