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Fractional and integral parts of p^{λ}

by

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1. Introduction. In this paper { } denotes fractional part, [ ] means integral part, and p is always a prime. The distribution of $\{\sqrt{p}\}$ is, at first sight, merely an interesting problem on the borderline of diophantine approximation and multiplicative number theory. However, a moment's thought reveals that the question is linked with two of the hardest problems in analytic number theory. One of these concerns the difference between consecutive primes. The other one is the conjecture that if $f(x)$ is an irreducible polynomial over $\mathbf{Z}$, with positive leading coefficient, and having no fixed prime divisor, then $f(n)$ is a prime infinitely often. If we could prove that

$$\{p^{1/2}\} < Kp^{-1/2} \tag{1.1}$$

for some K, then it would be known that there is some quadratic polynomial taking infinitely many prime values. This would be a major advance, and so it is not surprising that we are far from proving (1.1)! The first approximation to (1.1) was obtained by Vinogradov [16]. His method was improved by Kaufman [10]. The best known result to date has been obtained by A. Balog [1] and the author [4]. We proved that, for any $\varepsilon > 0$, there are infinitely many solutions to

$$\{p^{1/2}\} < p^{-1/4+\varepsilon}. \tag{1.2}$$

The first problem we consider in this paper concerns the still harder conjecture that if $f(n)$ is an irreducible quadratic polynomial over $\mathbf{Z}$, with positive leading coefficient, and $nf(n)$ has no fixed prime divisor, then $f(p)$ will be a prime for infinitely many p. Examples of such polynomials include p^2-2, p^2+4. The following result gives an approximation to this question.

THEOREM 1. *Let θ be a real number, and let $\varepsilon > 0$ be given. Then there are infinitely many solutions to*

$$\{p^{1/2}-\theta\} < p^{-1/8+\varepsilon}, \quad \textit{with } [p^{1/2}] \textit{ prime}. \tag{1.3}$$

Theorem 1 is an immediate corollary of the following more general theorem, which is no harder to prove than (1.3) directly. We write $\Lambda(n)$ for the von Mangoldt function.

THEOREM 2. *Let $\mathscr{A}$ be a set of integers, and suppose that $\varepsilon > 0$, θ a real, $N \geqslant 2$, and $\delta \in [N^{\varepsilon - 1/8}, 1)$ are given. Then*

$$\sum_{\substack{n \leqslant N \\ \{n^{1/2} - \theta\} < \delta \\ [n^{1/2}] \in \mathscr{A}}} \Lambda(n) = 2 \sum_{\substack{n \in \mathscr{A} \\ n \leqslant N^{1/2}}} n\delta + O\big(N\delta \exp(-(\log N)^{1/4})\big). \tag{1.4}$$

It is not surprising that (1.3) is even further from what is believed to be the truth than is (1.2). On the Riemann hypothesis it is not difficult to show that the 1/8 in (1.3) can be increased to 1/4. The assumption of the Riemann hypothesis does not appear to improve (1.2) though. A weaker result than (1.3) can be deduced from work on the differences between consecutive primes. For example, one can obtain (1.3) with the exponent 3/32 using [3]. This can be improved slightly using the methods of [7] and [8], although Heath-Brown's method does not yield the exponent 1/8 (his method is, of course, more efficient for the problem he considers).

The second problem in this paper concerns the number of solutions to

$$\{p^\lambda - \theta\} < N^{-a(\lambda) + \varepsilon} \quad \text{for } p \leqslant N. \tag{1.5}$$

Asymptotic formulae in this situation were derived in [1], [4] and [6]. For $\lambda < 1/2$ one might hope to establish such a formula with $a(\lambda) = (1-\lambda)/2$. Balog [2] managed to prove a lower bound of the correct order of magnitude when $\lambda > 2/5$. An asymptotic formula is given by Theorem 2 of [6] which holds when $a(\lambda) \leqslant \min(\lambda, 1/4)$. It follows from Huxley's prime number theorem [9] that the formula holds when $a(\lambda) \leqslant 5/12 - \lambda$, which is better than [6] if $\lambda < 5/24$. We write

$$\psi(N) = \sum_{n \leqslant N} \Lambda(n),$$

and prove the following stronger result.

THEOREM 3. *For $1/5 \leqslant \lambda \leqslant 3/8$, $\theta \in \mathbf{R}$, and $\delta \in (0, 1)$, we have, for $N \geqslant 2$,*

$$\sum_{\substack{n \leqslant N \\ \{n^\lambda - \theta\} < \delta}} \Lambda(n) = \delta\psi(N) + O\big(\delta N^{1-\varepsilon} + \delta^{2/3} N^{1 - E(\lambda)/3} (\log N)^{40}\big), \tag{1.6}$$

where $E(\lambda) = 5/14 - 2\lambda/7$, and $\varepsilon = 1/160$.

This provides an asymptotic formula for

$$a(\lambda) \leqslant 5/14 - 2\lambda/7.$$

For smaller λ technical problems make it difficult to establish a result of the above form. However, it is possible to prove the following result, whose proof we shall sketch briefly at the end of the paper.

THEOREM 4. *For $0 < \lambda < 1/5$, $\varepsilon > 0$, $B > 0$, $N \geqslant 2$, $\delta \leqslant N^{-E(\lambda) + \varepsilon}$, we have*

$$\sum_{\substack{n \leqslant N \\ \{n^\lambda\} < \delta}} \Lambda(n) = \delta\psi(N)\big(1 + O((\log N)^{-B})\big), \tag{1.7}$$

where

$$E(\lambda) = \max_{h \geqslant 1} \min\left(\frac{5h}{12h-6} - \lambda,\ \frac{5h - (2h+4)\lambda}{12h+4}\right). \tag{1.8}$$

Clearly $E(\lambda) = 5/12 - \lambda/2 + O(\lambda^2)$ as $\lambda \to 0$. Theorem 4 can be extended to include the condtion $[p^\lambda] \in \mathscr{A}$ with the same value of $E(\lambda)$ given by (1.8) for any $\lambda \leqslant 1/2$ (note that $E(1/2) = 1/8$).

2. Auxiliary results. To prove our results we shall be using the familiar formula of Landau [11]:

$$\psi(N) = N - \sum_{|\gamma| \leqslant T} \frac{N^\varrho}{\varrho} + O\left(\frac{N(\log N)^2 + N \log T}{T} + \log N\right). \tag{2.1}$$

Here $\varrho = \beta + i\gamma$ is a zero of the Riemann zeta-function with $0 < \beta < 1$. The following lemma contains the information we need on the distribution of these zeros.

LEMMA 1. *Write*

$$N(\sigma, T) = \sum_{\substack{\varrho \\ \beta \geqslant \sigma \\ |\gamma| \leqslant T}} 1.$$

Then, if T is sufficiently large, we have

$$N(\sigma, T) = 0 \quad \text{for } \sigma > 1 - (\log T)^{-7/10}, \tag{2.2}$$

$$N(\sigma, T) \ll T^{2-2\sigma} \quad \text{for } \sigma \geqslant 4/5, \tag{2.3}$$

$$N(\sigma, T) \ll T^{A(\sigma)} (\log T)^{44} \quad \text{for } 1/2 \leqslant \sigma \leqslant 4/5, \tag{2.4}$$

with

$$A(\sigma) = \begin{cases} 3(1-\sigma)/(3\sigma - 1) & \text{for } 3/4 \leqslant \sigma \leqslant 4/5, \\ 3(1-\sigma)/(2-\sigma) & \text{for } 1/2 \leqslant \sigma \leqslant 3/4, \end{cases} \tag{2.5}$$

and

$$N(\sigma, T+1) - N(\sigma, T) \ll \log T. \tag{2.6}$$

Proof. See Chapters 6 and 9 of [15], in particular the end of chapter notes where references to the original papers may be found.

LEMMA 2. *Let $\theta \in [0, 1)$, $N, T \geqslant 1$, a_n a sequence of complex numbers, and suppose that $\mathscr{B}$ is a set of distinct points in $\mathbf{R}^2$ with*

$$|t| \leqslant T, \quad \sigma_0 \leqslant \sigma \leqslant \sigma_0 + (\log 2N)^{-1},$$

$$|t - t'| \geqslant \delta \quad \text{if} \quad (\sigma, t) \neq (\sigma', t') \quad \text{for } (\sigma, t), (\sigma', t') \in \mathscr{B}.$$

Then

(2.7) $$\sum_{(\sigma,t)\in\mathscr{B}} \Big|\sum_{n=N}^{2N} a_n(n+\theta)^{it-\sigma}\Big|^2 \ll (\log 2N+\delta^{-1})(N+T)\sum_{N\leqslant n\leqslant 2N}|a_n|^2 n^{-2\sigma_0}.$$

Proof. From Corollary 2 to Theorem 2 of [13] we obtain

(2.8) $$\int_{-T}^{T}\Big|\sum_{n=N}^{2N} a_n(n+\theta)^{it}\Big|^2 dt \ll (T+N)\sum_{n=N}^{2N}|a_n|^2.$$

The bound (2.7) follows from (2.8) using the argument of Theorems 7.3 and 7.5 in [12].

LEMMA 3. *Let a_n be a sequence of complex numbers with $|a_n| \leqslant 1$. Then, for $N, T \geqslant 1$, we have*

(2.9) $$\int_{-T}^{T}\Big|\sum_{N\leqslant n<2N} a_n(n+\theta)^{it}\Big|^4 dt \ll (T+N^2)N^2(\log 2N)^3.$$

Proof. Write $H(A, N)$ for the number of solutions to

(2.10) $$|(n_1+\theta)(n_2+\theta)-(n_3+\theta)(n_4+\theta)| \leqslant A, \quad \text{with } n_j \sim N.$$

Here we have written $n \sim N$ to mean $N \leqslant n < 2N$. By expanding the left hand side of (2.9) and integrating the resulting fourfold series termwise we obtain a bound for the integral which is

(2.11) $$\ll \sum_A H(A, N)A^{-1} + TH(1, N),$$

where A takes the values 1, 2, 4, 8, ... less than $8N^2$. To produce (2.11) we have made use of the inequalities:

$$\int_0^T x^{it}dt \ll \min(T, |\log x|^{-1}) \quad \text{for } x>0,$$

and

$$|\log(1+y)| \geqslant |y|/4 \quad \text{for } -3/4 \leqslant y \leqslant 3.$$

To bound the number of solutions to (2.10) write

$$u = n_1 - n_4, \quad v = n_3 - n_2.$$

The inequality then becomes

(2.12) $$|u(n_3+\theta)-v(n_1+\theta)| \leqslant A.$$

The number of solutions to (2.12) can be estimated by the method used to prove Lemma 7 in [5]. If we suppose that $u \sim U$ and $v \sim V$, then the number of solutions to (2.12) is

$$\ll A(\log 2A)\big(\min(U, V)N(1+|\log|U/V||)+(U+V)^{5/3}\big)(\log 2N).$$

From this it quickly follows that

$$H(A, N) \ll A(\log 2A)(\log 2N)N^2 \ll AN^2(\log 2N)^2.$$

Thus (2.9) follows from (2.11).

LEMMA 4. *Suppose the hypotheses of Lemma 2 are given, but with $|a_n| \leqslant 1$. Then*

(2.13) $$\sum_{(\sigma,t)\in\mathscr{B}}\Big|\sum_{N\leqslant n<2N} a_n(n+\theta)^{it-\sigma}\Big|^4 \ll (\delta^{-1}+\log N)(\log 2N)^3(T+N^2)N^{2-4\sigma_0}.$$

Proof. This follows from Lemma 3, together with Lemma 1.2 of [12], an application of Hölder's inequality, and the argument used to prove Theorem 7.5 in [12].

3. Proof of Theorem 2. Clearly we may assume that $\theta\in[0, 1)$. Write

$$S(X, \delta) = \sum_{\substack{X\leqslant m<2X\\ m\in\mathscr{A}}}\ \sum_{n\in\mathscr{I}_m} \Lambda(n),$$

where $\mathscr{I}_m = \big((m+\theta)^2, (m+\theta+\delta)^2\big]$. We then note that it suffices to establish that

(3.1) $$S(X, \delta) = \sum_{\substack{X\leqslant m<2X\\ m\in\mathscr{A}}} 2\delta m + O\big(\delta X^2\exp(-(\log X)^{1/4})\big)$$

for $X^{-1/4+\varepsilon} < \delta \leqslant 1$. By (2.1) with $T = X^2$, we have

(3.2) $$S(X, \delta) = \sum_{\substack{X\leqslant m<2X\\ m\in\mathscr{A}}} 2\delta m + O(X(\log X)^2)$$

$$-\sum_{\substack{\varrho\\ |\gamma|\leqslant X^2}}\ \sum_{\substack{X\leqslant m<2X\\ m\in\mathscr{A}}} \frac{(m+\theta+\delta)^{2\varrho}-(m+\theta)^{2\varrho}}{\varrho}.$$

Let $T_1 = X\delta^{-1}$. We write S_1 for that portion of the final term in (3.2) with $|\gamma| \leqslant T_1$ and write S_2 for the remaining part of the sum. If we prove that

(3.3) $$S_j \ll X^2\delta\exp(-(\log X)^{1/4})$$

for $j = 1, 2$, then (3.1) will follow from (3.2). We write

$$X_y(s) = \sum_{\substack{X\leqslant m<2X\\ m\in\mathscr{A}}} (m+y)^{s-1}.$$

Then

(3.4) $$S_1 = 2\int_\theta^{\theta+\delta}\sum_{\substack{\varrho\\ |\gamma|\leqslant T_1}} X_y(2\varrho)dy.$$

Put $\eta = (\log X)^{-1}$ and write

$$S_y(\sigma) = \sum_{\substack{\varrho\\ |\gamma|\leqslant T_1\\ \sigma\leqslant\beta\leqslant\sigma+\eta}} |X_y(2\varrho)|.$$

Then it follows from (3.4) that

(3.5) $$S_1 \ll (\log X)\delta \sup_{\substack{0\le\sigma\le 1\\ 0\le y\le 1}} |S_y(\sigma)|.$$

For $\sigma \geq 4/5$ we use Hölder's inequality to obtain

$$S_y(\sigma) \leq N(\sigma, T)^{3/4}\Big(\sum_{\substack{\varrho\\ |\gamma|\leq T_1\\ \sigma\leq\beta\leq\sigma+\eta}} |X_y(2\varrho)|^4\Big)^{1/4}$$

$$\ll N(\sigma, T)^{3/4}(X^2+T_1)^{1/4}X^{2\sigma-1/2}(\log X)$$

by Lemma 4 and (2.6),

$$\ll T_1^{3(1-\sigma)/2}(X^2+T_1)^{1/4}X^{2\sigma-1/2}(\log X)$$

by (2.3). Since $T_1 < X^{5/4}$, this expression is an increasing function of σ. Hence the maximum is

(3.6) $$\ll X^2(\log X)^{-1}\exp(-(\log X)^{1/4})$$

by (2.2).

For $\sigma < 4/5$ we apply the Cauchy–Schwarz inequality to obtain

$$S_y(\sigma) \leq N(\sigma, T_1)^{1/2}\Big(\sum_{\substack{\varrho\\ |\gamma|\leq T_1\\ \sigma\leq\beta\leq\sigma+\eta}} |X_y(2\varrho)|^2\Big)^{1/2}$$

$$\ll N(\sigma, T_1)^{1/2}(X+T_1)^{1/2}X^{2\sigma-1/2}(\log X)$$

by Lemma 2 and (2.6),

$$\ll (\log X)^{23}\exp\left((\log X)\left(2\sigma-\frac{1}{2}+\left(\frac{5}{4}-\varepsilon\right)\left(\frac{1}{2}+\frac{A(\sigma)}{2}\right)\right)\right)$$

using (2.4) and $T_1 \leq X^{5/4-\varepsilon}$. Therefore we must demonstrate that

(3.7) $$2\sigma-\frac{1}{2}+\left(\frac{5}{4}-\varepsilon\right)\left(\frac{1+A(\sigma)}{2}\right) < 2$$

for $\sigma \leq 4/5$. First we consider $\sigma \geq 3/4$, for which (2.5) gives

$$A(\sigma) = \frac{2}{3\sigma-1}-1.$$

Since the second derivative of the left hand side of (3.7) (with respect to σ) is then positive, its maximum is obtained at $\sigma = 3/4$ or $= 4/5$. The value at $\sigma = 3/4$ is

(3.8) $$2-4\varepsilon/5,$$

while the value at $\sigma = 4/5$ does not exceed 1.993.

For $1/2 \leq \sigma \leq 3/4$ (2.5) gives

$$A(\sigma) = 3-\frac{3}{2-\sigma}.$$

It is then clear that the left hand side of (3.7) is an increasing function of σ for $\sigma \leq 3/4$. The maximum is therefore attained at $\sigma = 3/4$, which produces the same value given by (3.8). For $\sigma < 1/2$ we take the trivial bound $A(\sigma) = 1$ and this completes the proof of (3.3) for $j = 1$.

To tackle S_2 we write

(3.9) $$S_y(\sigma, T) = \frac{1}{T}\sum_{\substack{T\leq|\gamma|\leq 2T\\ \sigma\leq\beta\leq\sigma+\eta}} |X(2\varrho+1)|.$$

Then

(3.10) $$S_2 \ll (\log X)^2 \max_{\substack{0\leq y\leq 1\\ 0\leq\sigma\leq 1\\ T_1\leq T\leq X^2}} S_y(\sigma, T).$$

Assuming that X is sufficiently large, the upper bound obtained for $S_y(\sigma, T)$ using the density results of Lemma 1 is a decreasing function of T for $1/2+\mu < \sigma < 1-\mu$ for any $\mu > 0$ (say $\mu = 1/100$). The maximum over T is therefore attained for $T = T_1$ for σ in this range, the value obtained being the same as for $S_y(\sigma)$. For $\sigma < 1/2+\mu$ or $\sigma > 1-\mu$ the required bound to establish (3.3) is readily obtained. This completes the proof of Theorem 2.

4. Proof of Theorem 3. First we require a more powerful version of Lemma 4 which can be obtained when $a_n \equiv 1$.

LEMMA 5. *Given the hypotheses of Lemma 2, without the a_n, and with $T \leq |t| \leq 2T$, where $T \gg N$, we have*

(4.1) $$\sum_{(\sigma,T)\in B}\Big|\sum_{n\sim N}(n+\theta)^{it-\sigma}\Big|^4 \ll (\delta^{-1}+\log N)(\log T)^3\,T N^{2-4\sigma_0}.$$

Proof. We note that if $T > N^2$ then (4.1) follows immediately from (2.13). By the method used in Lemma 4, (4.1) follows from the inequality

(4.2) $$\int_T^{2T}\Big|\sum_{n\sim N}(n+\theta)^{it-\sigma}\Big|^4 dt \ll (\log T)^3\,T N^{2-4\sigma}.$$

By partial summation it suffices to prove (4.2) for $\sigma = 1/2$ with the summation range altered to $N \leq n \leq M$ where $M < 2N$. From the approximate functional equation for the Hurwitz zeta-function, if $N^2 > t$, we obtain

$$\sum_{N\leq n\leq M}(n+\theta)^{it-1/2} = \chi(1/2-it)\sum_{t/(2\pi M)\leq m\leq t/(2\pi N)} e(-r\theta)m^{-it-1/2}+O(1)$$

where $e(x) = \exp(2\pi i x)$ (see Lemma 1 of [4], a proof may be found in [14]). Here $|\chi(1/2-it)| = 1$. Thus

$$\int_T^{2T} \Big|\sum_{N \leqslant n \leqslant M} (n+\theta)^{it-1/2}\Big|^4 dt$$
$$\ll \int_T^{2T} \Big|\sum_{t/(2\pi M) \leqslant m \leqslant t/(2\pi N)} e(-r\theta) m^{-it-1/2}\Big|^4 dt + O(T)$$
$$\leqslant \left(\frac{N}{T}\right)^2 \sum_{T/(4\pi N) \leqslant m_j \leqslant T/(\pi N)} \max_{T \leqslant A < B \leqslant 2T} \left|\int_A^B \left(\frac{m_1 m_2}{m_3 m_4}\right)^{it} dt\right| + O(T)$$
$$\leqslant \left(\frac{N}{T}\right)^2 \sum_{T/(4\pi N) \leqslant m_j \leqslant T/(\pi N)} \min(T, |\log(m_1 m_2) - \log(m_3 m_4)|^{-1}) + O(T)$$
$$\ll T(\log T)^3$$

after some standard calculations. This establishes (4.2) for $\sigma = 1/2$ as desired.

Proof of Theorem 3. Without loss of generality we may suppose that $\theta \in [0, 1)$. We write $K = \lambda^{-1}$. We start in a similar fashion to Theorem 2 by writing

(4.3) $$S(X, \delta, \lambda) = \sum_{X \leqslant m < 2X} \sum_{(m+\theta)^K < n \leqslant (m+\theta+\delta)^K} \Lambda(n).$$

We must then show that, for $\delta > X^{-E(\lambda)}$,

(4.4) $$S(X, \delta, \lambda) = \delta(\psi((2X)^K) - \psi(X^K)) + O(\delta X^{K(1-\varepsilon)} + \delta^{2/3} X^{K-KE(\lambda)/3} (\log X)^{40}).$$

Write

$$T_1 = X/K, \quad T_2 = X\delta^{-1}, \quad T_3 = X^K.$$

An application of (2.1) gives

(4.5) $$S(X, \delta, \lambda) = \sum_{X \leqslant m < 2X} ((m+\theta+\delta)^K - (m+\theta)^K) + O((\log X)^2) - \sum_{\substack{\varrho \\ |\gamma| \leqslant T_3}} \sum_{X \leqslant m < 2X} \frac{(m+\theta+\delta)^{K\varrho} - (m+\theta)^{K\varrho}}{\varrho}.$$

Now the first term on the right hand side of (4.5) is

(4.6) $$\delta((2X)^K - X^K) + O(\delta X^{K-1}).$$

The sum over zeros in the final term of (4.5) with $|\gamma| \leqslant T_1$ is

(4.7) $$K \int_\theta^{\theta+\delta} \sum_{\substack{\varrho \\ |\gamma| \leqslant T_1}} X_y(K\varrho) dy.$$

Now, for $|\mathrm{Im}\, K\varrho| \leqslant X$, by Lemma 4.10 of [15] we have

(4.8) $$X_y(K\varrho) = \frac{(2X+y)^{K\varrho} - (X+y)^{K\varrho}}{K\varrho} + O(X^{K\beta-1}).$$

By Lemma 1

$$\sum_{\substack{\varrho \\ |\gamma| \leqslant T_1}} X^{K\beta-1} \ll (\log X)^{45} \max_{\substack{0 \leqslant \sigma \leqslant 1 \\ N(\sigma, T_1) \neq 0}} X^{A(\sigma)+K\sigma-1}$$

with $A(\sigma) \leqslant 12(1-\sigma)/5$ for all σ. Since $K > 12/5$ the above expression is an increasing function of σ. Hence

$$\sum_{\substack{\varrho \\ |\gamma| \leqslant T_1}} X^{K\beta-1} \ll X^{K-1}$$

using (2.2). Thus, in view of (4.8), the terms (4.6) and (4.7) together contribute

(4.9) $$\int_\theta^{\theta+\delta} ((2X+y)^K - (X+y)^K) dy + O(\delta X^{K-1}) - \int_\theta^{\theta+\delta} \sum_{\substack{\varrho \\ |\gamma| \leqslant T_1}} \frac{(2X+y)^{K\varrho} - (X+y)^{K\varrho}}{\varrho} dy$$
$$= \int_\theta^{\theta+\delta} (\psi((2X+y)^K) - \psi((X+y)^K) + O(X^{K-1}(\log X)^2)) dy \quad \text{(by (2.1))}$$
$$= \delta(\psi((2X)^K) - \psi(X^K)) + O(\delta X^{K-1}(\log X)^2)$$

since $\psi((R+1)^K) = \psi(R^K) + O(R^{K-1})$. Thus (4.9) gives the main term of (4.4) with an error of a suitable size since $\varepsilon < K^{-1}$.

We now consider the sum over zeros with $T_1 < |\gamma| \leqslant T_2$ in (4.5). We write

(4.10) $$S_y(\sigma) = \sum_{\substack{\varrho \\ T_1 < |\gamma| \leqslant T_2 \\ \sigma \leqslant \beta < \sigma+\eta}} |X_y(K\varrho)|.$$

Our present aim is to establish that

(4.11) $$S_y(\sigma) \ll (\log X)^{-1} (X^{K(1-\varepsilon)} + (\log X)^{40} \delta^{-1/3} X^{K(1-E(\lambda)/3)}).$$

First we consider $\sigma \geqslant 4/5$. Here we bound $S_y(\sigma)$ by

$$S_y(\sigma) \leqslant N(\sigma, T_2) \max_{T_1 \leqslant t \leqslant X^5} \Big|\sum_{X \leqslant n < 2X} (n+\theta)^{K\beta-1+it}\Big|.$$

By applying the results of Chapter 5 in [15] to the above sum (Theorem 5.14 with $l = 6$ for example) we conclude that, for $\varepsilon = 1/160$,

$$S_y(\sigma) \ll N(\sigma, T_2) X^{K(\sigma-\varepsilon)} (\log X)^{-1}.$$

Now, by (2.3),

$$N(\sigma, T_2) \ll T_2^{2(1-\sigma)} \leqslant X^{2(1+E(\lambda))(1-\sigma)}.$$

Since $2(1+E(\lambda)) \leqslant K$, we conclude that

(4.12) $$S_y(\sigma) \ll X^{K(1-\varepsilon)} (\log X)^{-1},$$

which is a suitable estimate.

For the range $3/4 \leqslant \sigma < 4/5$ we apply Hölder's inequality and (4.1) to (4.10) to obtain

(4.13) $$S_y(\sigma) \ll T_2^{3A(\sigma)/4}(\log X)^{35}(T_2 X^{4K\sigma-2})^{1/4}.$$

For (4.11) to hold we therefore need to demonstrate that

(4.14) $$X^{K\sigma-1/2}T_2^{(1+3A(\sigma))/4} \ll T_2^{1/3}X^{K(1-E(\lambda))/3-1/3}.$$

Rearranging (4.14) gives, after taking the 12th power, the equivalent inequality:

(4.15) $$T_2^{9A(\sigma)-1}X^{12K\sigma} \ll X^{12K+2-4KE(\lambda)}.$$

The left hand side of (4.15) is an increasing function of T_2 so it suffices to prove (4.15) with

$$T_2 = X^{5K/14+5/7}.$$

Thus the left hand side of (4.15) is X raised to the power

(4.16) $$(9A(\sigma)-1)(5K/14+5/7)+12K\sigma.$$

From (2.5) the second derivative of (4.16) with respect to σ is positive, so the maximum occurs at $\sigma = 3/4$ or $\sigma = 4/5$. At $\sigma = 3/4$ (4.16) is

(4.17) $$(74K+22)/7.$$

A quick calculation reveals that this is the same power of X occurring on the right hand side of (4.15). The value of (4.16) at $\sigma = 4/5$ is

(4.18) $$100/49+2602K/245,$$

which is less than (4.17) when $K < 45/2$, which is certainly the case here.

For the region $1/2 \leqslant \sigma \leqslant 3/4$ it is clear that $S_y(\sigma)$ is an increasing function of σ. The evaluation performed above for $\sigma = 3/4$ thus provides a suitable bound for the maximum in this region. As expected, the region $\sigma < 1/2$ poses no additional difficulties. A combination of all our results then establishes (4.11) as desired.

The final step required to complete the proof of Theorem 3 is to obtain a satisfactory estimate for

(4.19) $$\frac{1}{T}\sum_{\substack{T\leqslant|\gamma|\leqslant 2T\\ \sigma\leqslant\beta<\sigma+\eta}}^{\varrho} |X_y(K\varrho+1)|,$$

with $T_2 \leqslant T \leqslant T_3$. It is clear that the estimate obtained for $S_y(\sigma)$ holds for the sum in (4.19) as well. This establishes (4.4) and so completes the proof of Theorem 3.

We now indicate briefly how to prove Theorem 4. The proof begins as previously, but we restrict m to $[X, X+X(\log X)^{-B})$ in place of $[X, 2X)$. We also replace $(m+\delta)^K-m^K$ with $(m+\delta m/X)^K-m^K$. The effect of these changes is to reduce all the Dirichlet polynomials which arise once (2.1) has been applied to their common form $\sum_m m^s$. We do this because no results appear to be in the literature for

$$\int_0^T \Big|\sum_m (m+y)^{it}\Big|^{2h} dt$$

when $h \geqslant 3$. To prove Theorem 4 no use is made of Lemma 5 (the reflection principle it contains is inefficient for $\lambda < 1/5$), but instead Hölder's inequality is used to give

$$\sum_{(\sigma,t)\in\mathcal{B}} \Big|\sum_{m\leqslant M} m^{it+\sigma}\Big| \leqslant |\mathcal{B}|^{1-1/2h}\Big(\sum_{(\sigma,t)\in\mathcal{B}} \Big|\sum_{m\leqslant M} m^{it+\sigma}\Big|^{2h}\Big)^{1/2h}.$$

Lemma 2 can then be applied since

$$\Big|\sum_{m\leqslant M} m^{it+\sigma}\Big|^{2h} = \Big|\sum_{m\leqslant M^h} a_m m^{it+\sigma}\Big|^2$$

for certain a_m. Calculations of a similar nature to those used in Theorems 2 and 3 give a satisfactory estimate for the error terms using the value of $E(\lambda)$ stated in (1.8). The restriction $m\in\mathcal{A}$ can be added by estimating the error terms for $\sigma \geqslant 4/5$ using the technique employed in Theorem 2. Since Lemma 5 is not used, the method of Theorem 4 works for $\sigma < 4/5$ with the added restriction.

References

[1] A. Balog, *On the fractional part of p^θ*, Arch. Math. (Basel) 40 (1983), 434–440.
[2] — *On the distribution of p^θ mod 1*, Acta Math. Hungar. 45 (1985), 179–199.
[3] R. J. Cook, *An upper bound for the sum of large differences between primes*, Proc. Amer. Math. Soc. 81 (1981), 33–40.
[4] G. Harman, *On the distribution of $\sqrt{p}$ modulo one*, Mathematika 30 (1983), 104–116.
[5] — *Metrical theorems on fractional parts of real sequences*, J. Reine Angew. Math. 396 (1989), 192–211.
[6] G. Harman and A. Balog, *On mean values of Dirichlet polynomials*, preprint, Arch. Math. (Basel), to appear.
[7] D. R. Heath-Brown, *The difference between consecutive primes II*, J. London Math. Soc. (2) 19 (1979), 207–220.
[8] — *The difference between consecutive primes III*, ibid. 20 (1979), 177–178.
[9] M. N. Huxley, *On the difference between consecutive primes*, Invent. Math. 15 (1972), 164–170.
[10] R. F. Kaufman, *The distribution of $\{\sqrt{p}\}$* (Russian), Mat. Zametki 26 (1979), 497–504.
[11] E. Landau, *Über einige Summen, die von den Nullstellen der Riemann'schen Zetafunction abhängen*, Acta Math. 35 (1912), 271–294.
[12] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Springer, Berlin 1971.
[13] H. L. Montgomery and R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. (2) 8 (1974), 73–82.

[14] N. Tchudakoff, *On Goldbach–Vinogradov's theorem*, Ann. of Math. 48 (1947), 515–545.
[15] E. C. Titchmarsh, *Theory of the Riemann Zeta-function* (revised by D. R. Heath-Brown), Oxford 1986.
[16] I. M. Vinogradov, *Special Variants of the Method of Trigonometric Sums* (Russian), Nauka, Moscow 1976.

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О полиномиальных сравнениях

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Для натуральных n и q через $\mathfrak{M}_n(q)$ обозначим множество всех многочленов

$$f(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbf{Z}[x]$$

с условием $(a_n, \ldots, a_0, q) = 1$, для $f(x) \in \mathbf{Z}[x]$ и натурального P через $\varrho(f, P, q)$ обозначим количество решений сравнения

$$f(x) \equiv 0 \pmod{q}, \quad 0 \leqslant x \leqslant P-1. \tag{1}$$

Положим

$$N_n(P, q) = \max_{f \in \mathfrak{M}_n(q)} \varrho(f, P, q); \quad N_n(q) = N_n(q, q).$$

Величина $N_n(q)$ исследовалась в ряде работ (см. [1], [5] и ссылки в них). В [1] была получена неулучшаемая оценка

$$N_n(q) \ll q^{1-1/n} \tag{2}$$

(постоянные в символе "$\ll$" здесь и далее зависят только от n и, возможно, $\varepsilon > 0$).

В работе [2], в связи с оценками некоторых тригонометрических сумм, оценивалась величина $N_n(P, q)$ для q равного степени простого числа. В [4] была доказана оценка

$$N_n(P, q) \ll Pq^{-1/n} + q^{1-1/n-\varrho_n+\varepsilon}, \tag{3}$$

где $\varrho_n = (n-1)/n(n^2-n+1)$, нетривиальная при $P \geqslant q^{1-1/n-\varrho_n}$.

Здесь получена оценка, нетривиальная при всех $P \leqslant q$.

Теорема. *При любом $\varepsilon > 0$ справедлива оценка*

$$N_n(P, q) \ll P^{\varepsilon}(P^{1-1/n-\theta_n} + Pq^{-1/n}), \quad \textit{где } \theta_n = (n-1)/n(n^3-n^2+1).$$

Доказательство. Выберем многочлен $f \in \mathfrak{M}_n(q)$, для которого $\varrho(f, P, q) = N_n(P, q)$, и пусть $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_\Omega$ — все простые делители q с учетом кратности. Положим $P_1 = [P/p_1] + 1$. Тогда число решений сравнения (1) не превосходит числа решений сравнения

$$f(z + p_1 x) \equiv 0 \pmod{q}, \quad 0 \leqslant z \leqslant p_1 - 1, 0 \leqslant x \leqslant P_1 - 1. \tag{4}$$