Constructions of $B_h[g]$-sequences

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1. Introduction. A (finite) $B_h[g]$-sequence is a sequence $\vec{a} = (a_0, a_1, a_2, \ldots, a_j)$ of integers such that no integer has more than $g$ representations as sums of $h$ summands from $\vec{a}$. A survey of results on $B_h[1]$-sequences (up to 1966) is given in Chapter II of [5]. Recent papers on $B_h[g]$-sequences include [4]. Without loss of generality we may assume that

$$0 = a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_j.$$

Assuming that a sum contains the summand $a_j$ $x_j$ times, the sum may be written as $\sum_{j=0}^I x_j a_j$ where $\sum_{j=0}^I x_j = h$. Since $a_0 = 0$, we have $\sum_{j=0}^I x_j a_j = \sum_{j=1}^I x_j a_j$. Further $\sum_{j=1}^I x_j = h - x_0 \leq h$. Hence we get the following precise definition of a $B(g, h, J)$-sequence ($B_h[g]$-sequence with $J + 1$ elements): Let

$$C(h, J) = \{ \vec{x} = (x_1, x_2, \ldots, x_J) | x_j \text{ non-negative integers and } \sum_{j=1}^J x_j \leq h \}.$$

Further, let

$$D_k = D_k(\vec{a}) = \{ \vec{x} \in C(h, J) | \sum_{j=1}^J x_j a_j = k \}.$$

The sequence $\vec{a} = (a_0, a_1, a_2, \ldots, a_j)$ is a $B(g, h, J)$-sequence if $|D_k| \leq g$ for all integers $k \geq 0$.

Let

$$N(g, h, J) = \min \{ a_j | (a_0, a_1, a_2, \ldots, a_j) \text{ is a } B(g, h, J)\text{-sequence} \}.$$ 

A $B(g, h, J)$-sequence $(a_0, a_1, a_2, \ldots, a_j)$ where $a_j = N(g, h, J)$ is called optimal.

The main emphasis in the published literature has been on the behaviour of $N(g, h, J)$ for fixed $g$ and $h$ and varying $J$, in particular the asymptotic behaviour when $J \to \infty$. In this paper we consider mainly the situation with varying $h$, in particular the asymptotic behaviour when $h \to \infty$.

It is easy to see that $(0, 1, h+1, (h+1)^2, \ldots, (h+1)^{J-1})$ is a $B(1, h, J)$-sequence. Hence

$$N(g, h, J) \leq N(1, h, J) \leq (h+1)^{J-1}.$$
Let

\[ \zeta(g, J) = \liminf_{h \to \infty} \frac{N(g, h, J)}{h^{j-1}}, \quad \tilde{c}(g, J) = \limsup_{k \to \infty} \frac{N(g, h, J)}{h^{k-1}}. \]

Clearly, (1) implies that \( \tilde{c}(g, J) \leq 1. \)

2. Known bounds. Krückeberg [6] proved both lower and upper bounds on \( N(1, h, J) \). His lower bound generalizes to the following.

THEOREM 1. For all \( g, h, J \) we have

\[ N(g, h, J) \geq \frac{1}{h} \left\{ \frac{1}{g} \left( \frac{J+h}{h} \right) - 1 \right\}. \]

In particular \( \zeta(g, J) \geq 1/(gJ) \).

Proof. Let \( (a_0, a_1, \ldots, a_J) \) be a \( B(g, h, J) \)-sequence. If \( (x_1, x_2, \ldots, x_J) \in C(h, J) \), then \( 0 \leq \sum_{j=1}^{J} x_j a_j \leq h a_J \). Hence \( |C(h, J)| \leq g h a_J \) and so

\[ a_J \geq \frac{1}{h} \left( \frac{1}{g} |C(h, J)| - 1 \right) = \frac{1}{h} \left( \frac{1}{g} \left( \frac{J+h}{h} \right) - 1 \right). \]

Chen [2] gave stronger bounds on \( N(1, h, J) \):

\[ N(1, h, J) \geq \frac{1}{h} \left( \sum_{k=0}^{J} \frac{J+k}{k} \right) \left( \frac{1}{2} \right) \]

if \( h \) is even,

\[ N(1, h, J) \geq \frac{1}{h} \left( \sum_{k=0}^{J} \frac{J+k}{k} \right) \left( \frac{1}{2} \right) \]

if \( h \) is odd.

From this we can derive

\[ \zeta(1, J) \geq \frac{1}{J+1} \left( \frac{2J}{J+1} \right). \]

The best known lower bounds on \( N(g, 2, J) \) are due to Chen and Kővári [3].

Recently, Hajela [4] proved a lower bound on \( N(g, h, J) \) using trigonometrical polynomials. His formulation of the bound is such that it can not be immediately compared to the bound in Theorem 1. However, we will show that Hajela's bound is weaker than the bound in Theorem 1.

Let \( F = J+1 \) and \( n = N(g, h, J+1) \). Then Hajela's result is

\[ F \leq \frac{g^{1/h}(h!)^{1/k}}{(3n)^{1/k}} \inf_{m \in \mathbb{N}} \left( \frac{(2m^2+1+m)^{1/3}(2m^2)}{2-1/m} \right), \]

or equivalently

\[ F^h \leq \frac{2^{2a} g h!}{3n} \frac{2m^2+1+m}{(2-1/m)^{2a}} = \frac{g h!}{3n} \frac{2m^2+1+m}{(1-1/(2m))^{2a}} \]

for all integers \( m \geq 1 \).

From Theorem 1 we get

\[ n \geq 1 + \frac{1}{h} \left( \frac{J+k}{h} \right) - 1 \geq 1 + \frac{1}{h} \left( \frac{(J+1)^h}{h} \right) - 1 \]

and so

\[ F^h \leq \frac{1}{h} \left( \frac{1}{g} \left( \frac{J+k}{h} \right) - 1 \right) \]

\[ = \frac{1}{h} \left( \frac{1}{g} \left( \frac{J+k}{h} \right) - 1 \right) \]

and so

\[ F \leq gh! h^{n}. \]

We will show that (5) is stronger than (4). First, we note that for all integers \( m \geq 1 \) we have

\[ \frac{1}{m} \left( \frac{1}{2m} \right)^{2a} \leq \frac{1}{h} \left( \frac{1}{2m+1} \right) \leq \frac{1}{h} \left( \frac{1}{2h+1} \right) \]

and so

\[ h \leq \frac{m}{e(1-1/(2m))^{2a}}. \]

Hence, from (5) we get

\[ F \leq \frac{gh! n}{e(1-1/(2m))^{2a}} \]

for all \( m \geq 1 \), which gives an improvement over Hajela's bound (4) by a factor

\[ e^{-3/18}. \]

Bose and Chowla [1] proved that if \( g \) is a prime power, then

\[ N(1, h, q-1) \leq q^{h-1} \quad \text{and} \quad N(1, h, q) \leq (q^{h+1}-1)/(q-1). \]

These upper bounds are quite weak when \( h \) is large compared to \( J \). E.g. they give \( N(1, h, 2) \leq 2^{h+1}-1 \), whereas (1) gives \( N(1, h, 2) \leq h+1 \).

3. Some new constructions. Our first construction is a simple observation.

CONSTRUCTION 1. If \( (a_0, a_1, a_2, \ldots, a_J) \) is a \( B(g, h, J) \)-sequence, then so is

\[ (0, a_j - a_{j-1}, a_j - a_{j-2}, \ldots, a_j - a_1, a_j). \]

Proof. Define \( x^* \) by

\[ x^*_j = x_{j-1}^* \quad \text{for} \quad 1 \leq j \leq J-1, \]

\[ x^*_J = h - \sum_{j=1}^{J} x_j. \]

We note that if \( x^* \in C(h, J) \), then \( x^* \in C(h, J) \). Further,

\[ \sum_{j=1}^{J} x_j (a_j - a_{j-1}) = k \quad \text{if and only if} \quad \sum_{j=1}^{J} x_j^* a_j = h a_j - k. \]
THEOREM 2. For \( g \geq h + 1 \) we have \( N(g, h, 1) = 0 \), and \((0, 0)\) is the optimal \( B(g, h, 1)\)-sequence.

For \( g \leq h \) we have \( N(g, h, 1) = 1 \), and \((0, 1)\) is the optimal \( B(g, h, 1)\)-sequence.

Proof. Trivial.

THEOREM 3. For \( g \geq (h+2)(h+1)/2 \) we have \( N(g, h, 2) = 0 \), and \((0, 0, 0)\) is the optimal \( B(g, h, 2)\)-sequence.

For \( h < g < (h+2)(h+1)/2 \) we have \( N(g, h, 2) = 1 \), and \((0, 0, 1)\) and \((0, 1, 1)\) are the optimal \( B(g, h, 2)\)-sequences.

For \( g \leq h \) we have \( N(g, h, 2) = \left\lceil \frac{(h+1)/g}{g} \right\rceil \), and the optimal \( B(g, h, 2)\)-sequences are \((0,a,\left\lfloor (h+1)/g \right\rfloor)\) for all \( a \) such that \( 1 \leq a < \left\lceil \frac{(h+1)/g}{g} \right\rceil \) and \( \gcd(a, \left\lfloor (h+1)/g \right\rfloor) = 1 \).

Proof. The part with \( g > h \) is trivial. Consider \( g \leq h \). First, we show that \( N(g, h, 2) > h/g \). Consider \((0,a,b)\) where \( 0 \leq a \leq b \leq h/g \). For \( 0 \leq a \leq g \) we have \( ab + (ga - aa) = a(b-a) + ga \leq g(b-a) + ga = gb < h \),

and \((ab + (ga - aa)) = gb \). Hence \((ab + (ga - aa)) \in D_{ab}\) for all \( 0 \leq a \leq g \) and so \( D_{ab} \leq g+1 \). Therefore \((0,a,b)\) is not a \( B(g, h, 2)\)-sequence. This proves that \( N(g, h, 2) \geq h/g \), i.e., \( N(g, h, 2) \geq \left\lceil \frac{(h+1)/g}{g} \right\rceil \).

Next, consider \((0,a,b)\) where \( b = \left\lfloor (h+1)/g \right\rfloor \) and \( \gcd(a, b) = 1 \). Consider \( D_k \) for some \( k \) such that \( D_k \) is non-empty. Let \((y_1, y_2) \in D_k\) be such that \( y_1 \leq x_1 \) for all \((x_1, x_2) \in D_k\). Let \((x_1, x_2) \in D_k\).

Then

\[ x_1 + x_2 a + x_2 b = x_2 a + x_2 b. \]

In particular \( x_1 a \equiv y_1 a \pmod{b} \). Hence \( y_1 = y_1 \pmod{b} \), and so \( x_1 = y_1 + ab \), where \( a \geq 0 \) by the minimality of \((y_1, y_2)\). Substituting in (8), we get \( x_2 = y_2 - aa \). Further, \( h \geq x_1 \geq ab \), and so \( a \leq h/b < g \). Hence \( D_k \leq \{(y_1 + ab, y_2 - aa) \mid 0 \leq a \leq g \} \) and so \( D_{ab} \leq g \).

Finally, consider \((0,a,b)\) where \( b = \left\lfloor (h+1)/g \right\rfloor \) and \( d = \gcd(a, b) > 1 \). Similarly to the first part of the proof we get

\[ \left( \frac{b}{d^2} \right) \in D_{ab} \]

for \( 0 \leq a \leq g \). Hence \((0,a,b)\) is not a \( B(g, h, 2)\)-sequence.

CONSTRUCTION 3. The following sequences are \( B(1,h,3)\)-sequences.

(i) If \( h \) is even, \( 1 \leq a \leq h \), and \( \gcd(a, h+1) = 1 \):

\[ \left( 0,a, \frac{h(h+1)}{2} + a, \frac{h(h+1)}{2} + h + 1 \right); \]
(ii) If \( h \) is odd:
\[
\left( 0, \frac{h(h+1)}{2} + h + 2, \frac{h(h+1)}{2} + h + 2 \right),
\]
\[
\left( 0, h+1, \frac{h(h+1)}{2} + h + 2, \frac{h(h+1)}{2} + h + 2 \right);
\]
(iii) If \( h \equiv 3 \pmod{4}:
\]
\[
\left( 0, \frac{h-1}{2} \left( h+1 \right)^2, \frac{h+1}{2} + h + 2, \frac{h+1}{2} + h + 2 \right),
\]
\[
\left( 0, \frac{h+3}{2} \left( h+1 \right)^2, \frac{h}{2} + h + 2, \frac{h}{2} + h + 2 \right).
\]

Proof. To prove (i), suppose \((x_1, x_2, x_3, y_1, y_2, y_3) \in C(h, 3)\) and

\[
x_1a + x_2 \left( \frac{h(h+1)}{2} + a \right) + x_3 \left( \frac{h(h+1)}{2} + h + 1 \right)
\]
\[
= y_1a + y_2 \left( \frac{h(h+1)}{2} + a \right) + y_3 \left( \frac{h(h+1)}{2} + h + 1 \right).
\]

We have to show that \((x_1, x_2, x_3) = (y_1, y_2, y_3)\). From (9) we get

\[
x_1a + x_2a = y_1a + y_2a \pmod{h+1}.
\]

Since \(\gcd(a, h+1) = 1\), we get \(x_1 + x_2 \equiv y_1 + y_2 \pmod{h+1}\) and so

\[
x_1 + x_2 = y_1 + y_2.
\]

Combining (9) and (10) we get

\[
x_2 + x_3 \left( \frac{h+1}{2} \right) = y_2 + y_3 \left( \frac{h+1}{2} \right).
\]

In particular

\[
x_2 = y_2 \pmod{(h+2) + 1}.
\]

Without loss of generality we may assume that \(x_2 \geq y_2\). Suppose that \(x_2 > y_2\). Since \(x_2 \leq h\), (12) implies that \(x_2 = y_2 + h + 2\). By (10), \(y_1 = x_1 + x_2 + 1\), and by (11), \(y_3 = x_3 + h + 2\). Hence \(h \geq y_1 + y_3 = x_1 + x_3 + h + 1 \geq h + 1\), a contradiction. Hence \(x_2 = y_2\). By (10) and (11), \(x_1 = y_1\) and \(x_3 = y_3\). This proves (i).

To prove (ii) and (iii) we first note that each of the sequences in (ii) is obtained from the other by Construction 1 and similarly for (iii). Therefore, it is sufficient to show that one of (ii) and one of (iii) is a \(B(1, h, 3)\)-sequence.

We prove that the second of the two sequences in (ii) is a \(B(1, h, 3)\)-sequence. Suppose that \((x_1, x_2, x_3, y_1, y_2, y_3) \in C(h, 3)\) and

\[
x_1(h+1) + x_2 \left( \frac{h(h+1)}{2} + h + 1 \right) + x_3 \left( \frac{h(h+1)}{2} + h + 2 \right)
\]
\[
= y_1(h+1) + y_2 \left( \frac{h(h+1)}{2} + h + 1 \right) + y_3 \left( \frac{h(h+1)}{2} + h + 2 \right).
\]

Without loss of generality we may assume that \(x_3 \geq y_3\). From (13) we get

\[
x_3 \equiv y_3 \pmod{(h+1)/2}.
\]

We consider two cases.

Case I, \(x_3 = y_3\). From (13) we get

\[
2x_1 + x_2 (h+2) = 2y_1 + y_2 (h+2).
\]

Hence \(2x_1 = 2y_1 \pmod{h+2}\) and so \(x_1 \equiv y_1 \pmod{h+2}\). This implies that \(x_1 = y_1\). By (15), \(x_2 = y_2\).

Case II, \(x_3 = y_3 + (h+1)/2\). We will show that this is not possible. From (13) we get

\[
2x_1 + x_2 (h+2) + (h+1)/2 + h + 2 = 2y_1 + y_2 (h+2).
\]

In particular \(2x_1 + 1 \equiv 2y_1 \pmod{h+2}\) and so \(x_1 \equiv y_1 + (h+1)/2 \pmod{h+2}\). There are now two subcases:

Case II (i), \(x_1 = y_1 + (h+1)/2\). Then \(h \geq x_1 + x_3 = y_1 + y_3 + h + 1 \geq h + 1\), a contradiction.

Case II (ii), \(y_1 = x_1 + (h+3)/2\). Then (16) gives \(y_3 = x_3 + (h-1)/2\).

Hence \(h \geq y_1 + y_3 = x_1 + x_3 + h + 1 \geq h + 1\), again a contradiction.

Finally, consider the first of the two sequences in (iii). Suppose that

\[
(x_1, x_2, x_3, y_1, y_2, y_3) \in C(h, 3)\]

and

\[
x_1(h+1) + x_2 \left( \frac{h(h+1)}{2} + h + 2 \right) + x_3 \left( \frac{h(h+1)}{2} + h + 2 \right)
\]
\[
= y_1(h+1) + y_2 \left( \frac{h(h+1)}{2} + h + 2 \right) + y_3 \left( \frac{h(h+1)}{2} + h + 2 \right).
\]

Then

\[
x_1 \left( \frac{h-1}{2} + x_3 + \frac{h+3}{2} \right) \equiv y_1 \left( \frac{h-1}{2} + y_3 + \frac{h+3}{2} \right) \pmod{h+1}.
\]

Without loss of generality we may assume that

\[
x_1 \left( \frac{h-1}{2} + x_3 + \frac{h+3}{2} \right) \geq y_1 \left( \frac{h-1}{2} + y_3 + \frac{h+3}{2} \right).
\]
Since
\[
0 \leq x_1 \frac{h-1}{2} + x_2 \frac{h+3}{2} - \frac{h+1}{2} < 2 \left( \frac{h+1}{2} \right)^2
\]
there are again two cases to consider.

Case I, \( x_1 \frac{h-1}{2} + x_2 \frac{h+3}{2} = y_1 \frac{h-1}{2} + y_2 \frac{h+3}{2} \). In this case,
\[
-2x_1 \equiv -2y_1 \pmod{\frac{h+3}{2}}.
\]

Since \((h+3)/2\) is odd, this implies that \(x_1 \equiv y_1 \pmod{(h+3)/2}\). Without loss of
generality we may assume that \(x_1 \geq y_1\).

Case I (i), \( x_1 = y_1 \). Then \(x_2 = y_2\), and so (17) implies \(x_2 = y_2\).

Case I (ii), \( x_1 = y_1 + (h+3)/2 \). Then \(y_2 = x_2 + (h-1)/2 \). From (17) we get
\(x_2 = y_2 + (h-1)/2\). Hence, \(x_1 + x_2 = y_1 + y_2 + h + 1 \geq h + 1\), a contradiction.

Case II, \( x_1 \frac{h+3}{2} + x_2 \frac{h+3}{2} = y_1 \frac{h+3}{2} + y_2 \frac{h+3}{2} \). Then
\[
\frac{h+3}{2} \leq x_1 \frac{h+1}{2} + x_2 \frac{h+3}{2}
\]
\[
= x_1 \frac{h+1}{2} + y_2 \frac{h+3}{2} + \frac{(h+1)^2}{2} \geq \frac{(h+1)^2}{2} = \frac{(h-1)h+3}{2} + 2.
\]

Hence \(x_1 + x_2 > h - 1\) and so \(x_1 + x_2 = h\). This implies that
\[
2x_3 = \frac{3h+1}{2} + y_1 \frac{h-1}{2} + y_2 \frac{h+3}{2}.
\]

Hence \(2x_3 \equiv 2 + 2y_2 \pmod{(h-1)/2}\) and so \(x_3 \equiv 1 + y_2 \pmod{(h-1)/2}\). From
(18) we get \(x_3 \geq (3h+1)/4\) and so \(x_3 = 1 + y_3 + (h-1)/2\). Combining this with
(18) we get
\[
0 = \frac{h-1}{2} (1 + y_1 + y_3) \geq \frac{h-1}{2},
\]
a contradiction.

**Theorem 4.** For all \(h\) we have
\[
N(1, h, 3) \leq \frac{h(h+1)}{2} + 2 \left[ \frac{h+1}{2} \right] + 1.
\]

In particular \(5/12 \leq c(1, 3) \leq \hat{c}(1, 3) \leq 1/2\).

**Proof.** The theorem follows directly from Construction 3 and (3).

A direct search has shown that for \(1 < h \leq 24\) we have
\[
N(1, h, 3) = \frac{h(h+1)}{2} + 2 \left[ \frac{h+1}{2} \right] + 1
\]
and Construction 3 gives all the optimal \(B(1, h, 3)\)-sequences. Whether this is
true for all \(h > 1\) is an open question.

**Construction 4.** If \(g \geq 2\) and \(m \geq 1\), then
\[
(0, m, (g+1)m^2 + gm, (g+1)m^2 + (g+1)m + 1)
\]
is a \(B(g, 2mg, 3)\)-sequence.

**Proof.** We order the elements of \(D_k\) as follows: \((y_1, y_2, y_3) < (x_1, x_2, x_3)\) if
\[
y_1 + y_2 + y_3 (m+1) < x_1 + x_2 + x_3 (m+1)
\]
or if
\[
y_1 + y_2 + y_3 (m+1) = x_1 + x_2 + x_3 (m+1) \quad \text{and} \quad y_1 < x_1.
\]
Let \((y_1, y_2, y_3)\) be the minimal element of \(D_k\) under this ordering. Let
\[
y_i = \epsilon_i m + \delta_i \quad \text{for} \quad i = 1, 2, 3 \quad \text{where} \quad 0 \leq \delta_i < m.
\]
Let \((x_1, x_2, x_3) \in D_k\). Then
(19) \[
x_1 + x_2 ((g+1)m^2 + gm) + x_3 ((g+1)m^2 + (g+1)m + 1)
\]
\[
= y_1 + y_2 ((g+1)m^2 + gm) + y_3 ((g+1)m^2 + (g+1)m + 1).
\]
In particular
\[
x_1 + x_2 (m+1) = y_1 + y_2 (m+1) \pmod{(g+1)m^2 + gm}.
\]
Hence
\[
x_1 + x_2 (m+1) = y_1 + y_2 (m+1) + \alpha ((g+1)m^2 + gm)
\]
where \(\alpha \geq 0\) by the minimality of \((y_1, y_2, y_3)\). Further
\[
\alpha ((g+1)m^2 + gm) \leq x_1 + x_2 (m+1) \leq 2mg(m+1) < 2 ((g+1)m^2 + gm).
\]
Hence, \(\alpha = 0\) or \(\alpha = 1\). Let
\[
x_2 = \left| \{(x_1, x_2, x_3) \in D_k \mid x_1 + x_2 (m+1) = y_1 + y_2 (m+1) + \alpha ((g+1)m^2 + gm)\} \right|.
\]
We have to show that \(n_0 + n_1 \leq g\).

First, we consider the case \(\alpha = 0\). By (20),
\[
x_1 = y_1 + \beta (m+1), \quad \beta \geq 0
\]
Hence \(x_1 = y_1 + \beta (m+1), \quad \beta \geq 0\).
by the minimality of \((y_1, y_2, y_3)\). Substituting in (20) and (19) we get

\[
\begin{align*}
  x_1 &= y_1 + \beta(m + 1), \\
  x_2 &= y_2 + \alpha m, \\
  x_3 &= y_3 - \beta m.
\end{align*}
\]

In particular, \(0 \leq x_3 = y_3 - \beta m = (e_3 - \beta)m + \delta_3\). Hence

\[
\beta \leq e_3.
\]

Further,

\[
2mg \geq x_1 + x_2 + x_3 = e_1m + \delta_1 + e_2m + \delta_2 + e_3m + \delta_3 - 2 + \gamma(m + 1).
\]

Rearranging, we get

\[
\beta \leq \frac{(2g - e_1 - e_2 - e_3 - m)(\delta_1 + \delta_2)}{2m + 1}.
\]

Define \(\phi\) by

\[
\phi = \begin{cases} 
1 & \text{if } e_3 \leq \frac{(2g - e_1 - e_2 - e_3 - m)(\delta_1 + \delta_2)}{2m + 1}, \\
0 & \text{otherwise.}
\end{cases}
\]

Combining (21) and (25)-(28) we get

\[
n_0 \leq e_3 + \phi.
\]

Next, consider the case \(\alpha = 1\). Similarly to the case \(\alpha = 0\) we get

\[
\begin{align*}
  x_1 &= y_1 - 1 + \gamma(m + 1), \\
  x_2 &= y_2 - (g + 1)m - 1 + \gamma m, \\
  x_3 &= y_3 + (g + 1)m - \gamma m.
\end{align*}
\]

We have \(0 \leq x_4 = e_1m + \delta_1 - 1 + \gamma(m + 1)\). Hence

\[
\gamma \geq (1 - e_1m - \delta_1)(m + 1).
\]

Further, \(0 \leq x_2 = (e_2 - g - 1 + \gamma)m + \delta_2 - 1\) and so

\[
\gamma \geq g + 1 - e_2 + \frac{1}{m}(1 - \delta_2).
\]

Define \(\psi\) by

\[
\psi = \begin{cases} 
1 & \text{if } \delta_2 = 0, \\
0 & \text{if } \delta_2 > 0.
\end{cases}
\]

Then, by (34),

\[
\gamma \geq g + 1 - e_2 + \psi.
\]
Since \( \phi = 1 \) we have, by (27),
\[
(2m + 1) \varepsilon_3 \leq 2m g - m \varepsilon_2 - 1 + \psi
\]
\[
= 2m g - m(2g - 1 + \psi - \varepsilon_3) - 1 + \psi.
\]
Rearranging we get \((m + 1) \varepsilon_3 \leq (m - 1)(1 - \psi)\). Hence \( \varepsilon_3 = 0 \) and so by (29)
\[
\tag{51}
\gamma \leq 1.
\]
Next, since \( \delta_1 = 0 \), we get, by (33),
\[
\tag{52}
\gamma \geq 1.
\]
Further, substituting (50) in (38) we get
\[
\tag{53}
\gamma \leq 1 - \psi + 2\psi/(m + 1) \leq 1.
\]
Combining this with (52) we get \( n_1 \leq 1 \). Hence, by (51),
\[
\tag{54}
n_0 + n_1 \leq 2 \leq g.
\]
This completes the proof of Construction 4. From the construction we get

**Theorem 5.** For \( g \geq 2 \) and all \( h \) we have
\[
N(g, h, J) \leq (g + 1)(m^2 + m + 1)
\]
where \( m = \lceil h/(2g) \rceil \). In particular
\[
\frac{1}{6g} \leq \varepsilon(g, 3) \leq \varepsilon(g, 3) \leq \frac{1}{4g} \cdot \frac{g + 1}{g}.
\]

**Remark.** From Theorems 4 and 5 we get \( \varepsilon(g, 3) \leq (1/g) \varepsilon(1, 3) \) for \( g \geq 6 \).

For \( g = 2 \) Construction 4 gives the sequence \( (0, m, 3m^2 + 2m, 3m^2 + 3m + 1) \).

For \( m \leq 10 \) this is optimal, and this and the one obtained from it by Construction 1 are the only optimal \( B(2, 2m, 3) \)-sequences. We conjecture that this is true for all \( m \).

For \( g \geq 3 \) it appears that the sequence in Construction 4 is not optimal. E.g., for \( 1 \leq m \leq 4 \) an optimal \( B(6, 12m, 3) \)-sequence is \( (0, m, 7m^2 + 3m - 1, 7m^2 + 4m) \), and so \( N(6, 12m, 3) = 7m^2 + 4m \), whereas Construction 4 gives \( (0, m, 7m^2 + 6m, 7m^2 + 7m + 1) \).

Based on the limited numerical data available, it appears that \( N(g, 2m^2, 3) \sim (g + 1) m^2 \) when \( g \) is fixed and \( m \to \infty \).

For \( m = 1 \), Construction 4 gives the \( B(g, 2g, 3) \)-sequence \( (0, 1, 2g + 1, 2g + 3) \).

However, we can prove that \( (0, 1, g + 3, g + 5) \) is a \( B(g, 2g, 3) \)-sequence for all \( g \).

Hence \( N(g, 2g, 3) \leq g + 5 \).

Moreover, for \( g \leq 50 \), \( N(g, 2g, 3) = g + 5 \), and we conjecture that this is true in general.

**Theorem 6.** For all \( g \geq 1 \), \( h > 1 \), \( J \), \( K \) we have
\[
N(g, h, J + K) \leq (h - 1)N(g, h, J + 1)N(1, h, K)
\]
and \( \varepsilon(g, J + K) \leq \varepsilon(g, J)\varepsilon(1, K) \).

**Proof.** Let \( (0, a_1, \ldots, a_J) \) be a \( B(g, h, J) \)-sequence with \( a_j = N(g, h, J) \) and \( (0, b_1, \ldots, b_J) \) a \( B(1, h, K) \)-sequence with \( b_j = N(1, h, K) \). We cannot have \( b_1 = 1 \) and \( b_J - b_{J-1} = 1 \) simultaneously since this would imply
\[
1 \cdot b_1 + 1 \cdot b_{J-1} = 1 \cdot b_J
\]
which is impossible for a \( B(1, h, J) \)-sequence. Therefore, Construction 1 shows that we may assume that \( b_j > 1 \). From Construction 2, with \( A = (h - 1)a_j + 1 \), it follows that
\[
N(g, h, J + K) \leq \left((h - 1)N(g, h, J) + 1\right)N(1, h, K).
\]
In particular
\[
N(g, h, J + K) \leq hN(g, h, J)N(1, h, K),
\]
which implies that
\[
\limsup_{h \to \infty} \frac{N(g, h, J + K)}{h^{1/k - 1}} \leq \limsup_{h \to \infty} \frac{N(g, h, J)}{h^{1/k - 1}} \limsup_{h \to \infty} \frac{N(1, h, K)}{h^{1/k - 1}},
\]

i.e., \( \varepsilon(g, J + K) \leq \varepsilon(g, J)\varepsilon(1, K) \).

**Theorem 7.** For all \( g_1 \geq 1 \), \( g_2 > 1 \), \( h > 1 \), \( J \) we have
\[
N(g_1 g_2, h, J + 3) \leq (h - 1)N(g_1, h, J + 1)N(g_2 + 1, (m^2 + m + 1)
\]
where \( m = \lceil h/(2g) \rceil \).

Further
\[
\varepsilon(g_1 g_2, J + 3) \leq \varepsilon(g_1, J)\varepsilon(g_2, J + 1).
\]

**Proof.** Combining Constructions 1 and 4 we get the \( B(g, 2g_2, 3) \)-sequence
\[
(0, m + 1, (g + 1)m^2 + gm + 1, (g + 1)(m^2 + m + 1)).
\]
This is in particular a \( B(g, h, 3) \)-sequence, and \( m + 1 > 1 \). The Theorem follows from Construction 2.

References
Polynomials whose powers are sparse

by

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Erdős [Erd] defines \( Q(N) \) as the least possible number of nonzero coefficients ("the number of terms") in the square of a polynomial \( f(x) \) with exactly \( N \) nonzero real coefficients. Erdős proves the existence of positive constants \( C_1, C_2 \) such that

\[
Q(N) < C_1 N^{1-C_2}.
\]

Verdenius [Ver] extends this result in two directions. He works with complete polynomials \( f \), that is,

\[
f(x) = \sum_{i=0}^{N-1} d_i x^i, \quad d_i \neq 0, \quad 0 \leq i \leq N-1.
\]

He also establishes a similar inequality for cubes. Letting \( Q_3(N) \) denote the least possible number of terms in the \( k \)th power of a complete real polynomial of degree \( N-1 \), Verdenius gives positive constants \( C_{1,2}, C_{1,3} \) such that for any integer \( N \geq 1 \),

\[
Q_3(N) < C_{1,2} N^{0.81071...}, \quad Q_3(N) < C_{1,3} N^{0.99934...}.
\]

In the present note we extend this result to \( k \)th powers for each integer \( k \geq 2 \). Our main theorem is:

**Theorem 1.** Given an integer \( k \geq 2 \), there are positive constants \( C_{1,k}, C_{2,k} \) such that for any integer \( N \geq 1 \),

\[
Q_k(N) < C_{1,k} N^{1-C_{2,k}}.
\]

**Remark.** Schinzel [Sch] has studied a similar problem for fields of prime characteristic \( p \). For any integer \( k \) not a power of \( p \), he obtains polynomials with arbitrarily many terms, whose \( k \)th power has at most \( 2k \) terms. He also obtains lower bounds.

Two consequences of Theorem 1:

**Theorem 2.** Given an integer \( k \geq 2 \), there are positive constants \( C_{1,k}, C_{2,j,k} \), \( 2 \leq j \leq k \), such that for any integer \( N \geq 1 \) there is a complete polynomial \( f(x) \in \mathbb{R}[x] \) of degree \( N-1 \) such that the number of terms in each power \( f^j(x), 2 \leq j \leq k \), is bounded by \( C_{1,k} N^{1-C_{2,j,k}} \).