ACTA ARITHMETICA
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# On Entry 8 of Chapter 15 of Ramanujan's Notebook II

by

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**1.** Sections 1–7 of Chapter 15 of Ramanujan's second notebook [9] are mainly devoted to asymptotic expansions of series [3]. At the beginning of Section 8 Ramanujan indicates a method of calculating the error in the asymptotic expansion

$$\sum_{n\geq 1}(e^{\tau n^2}-1)^{-1} = R(\tau)+o(1) \quad (\tau\to 0+) \tag{1.1}$$

where

$$R(\tau) = \frac{\pi^2}{6\tau}+\frac{1}{2}\sqrt{\frac{\pi}{\tau}}\,\zeta\left(\frac{1}{2}\right)+\frac{1}{4}. \tag{1.2}$$

He gives the hint that the identities ([4], [6])

$$\frac{1}{2}\int_0^\infty \{\vartheta_3(0|\tau)-1\}\cos a\tau\, d\tau = \sum_{n\geq 1}\frac{n^2}{a^2+n^4} \tag{1.3}$$

$$= \frac{\pi}{2\sqrt{2a}}\,\frac{\sinh(\pi\sqrt{2a})-\sin(\pi\sqrt{2a})}{\cosh(\pi\sqrt{2a})-\cos(\pi\sqrt{2a})},$$

where $a>0$ and $\vartheta_3(0|\tau)$ is Jacobi's elliptic theta function of zero argument, can be used to prove the following exact formula extending (1.1):

ENTRY 8: *If* $\tau>0$, *then*

$$\text{(A)}\quad \sum_{n\geq 1}(e^{\tau n^2}-1)^{-1}$$

$$= R(\tau)+\sqrt{\frac{\pi}{2\tau}}\sum_{n\geq 1} n^{-1/2}\left\{\frac{\cos(\pi/4+2\pi\sqrt{\pi n/\tau})-e^{-2\pi\sqrt{\pi n/\tau}}\cos(\pi/4)}{\cosh(2\pi\sqrt{\pi n/\tau})-\cos(2\pi\sqrt{\pi n/\tau})}\right\}.$$

That such an exact formula exists is 'truly remarkable' ([4], p. 124). B. C. Berndt ([2], pp. 50–51) called (A) 'one of the most interesting and incredible formulas in the notebooks' representing an analogue of the well-known transformation property of $\vartheta_3(0|\tau)$.

Entry 8 was proved by B. C. Berndt and R. J. Evans in [4]. Their method of proof is mainly based on Poisson's summation formula, the transformation of $\vartheta_3(0|\tau)$ and the equations (1.3).

It may be surprising that there is an elegant alternative exact formula of type (A) which we will prove and generalize in the present small note.

2. Note that (A) can be rewritten in the form ([4], p. 125)

$$\text{(A')}\quad \sum_{n\geq 1}(e^{n^2\tau}-1)^{-1} = R(\tau)+\tfrac{1}{2}\sqrt{\frac{\pi}{\tau}}\sum_{n\geq 1} n^{-1/2}\left\{\frac{\sinh(2\pi\sqrt{n\pi/\tau})-\sin(2\pi\sqrt{n\pi/\tau})}{\cosh(2\pi\sqrt{n\pi/\tau})-\cos(2\pi\sqrt{n\pi/\tau})}-1\right\}.$$

We prove: If $\tau>0$, then

$$\text{(B)}\quad \sum_{n\geq 1}(e^{n^2\tau}-1)^{-1} = R(\tau)+\tfrac{1}{4}\sqrt{\frac{\tau}{\pi}}\sum_{n\geq 1} n^{-3/2}\left\{\frac{\sinh(\sqrt{\pi\tau/n})-\sin(\sqrt{\pi\tau/n})}{\cosh(\sqrt{\pi\tau/n})-\cos(\sqrt{\pi\tau/n})}\right\}.$$

**Remark.** Obviously (A') and (B) yield a new deep transformation formula for the infinite series on the right.

**Proof of (B).** Instead of the first equality in (1.3) we use ([11], p. 24)

$$\Gamma(s)\zeta(s)=\int_0^\infty x^{s-1}\left\{\frac{1}{e^x-1}-\frac{1}{x}+\frac{1}{2}\right\}dx,$$

valid for $-1<\operatorname{Re}s=\sigma<0$. Then the partial fractional expansion

$$(e^x-1)^{-1}=\frac{1}{x}-\frac{1}{2}+2x\sum_{n\geq 1}\{4\pi^2n^2+x^2\}^{-1}\qquad (x\in\mathbf{R}^+)$$

and Mellin's inversion theorem yield

$$2\sum_{n\geq 1}x\{4\pi^2n^2+x^2\}^{-1}=\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\zeta(s)x^{-s}ds$$

where $x\in\mathbf{R}^+$, $(c)$ denoting the line $(c-i\infty, c+i\infty)$, $c\in(-1,0)$. Thus introducing the real parameter $\tau\in\mathbf{R}^+$

$$\text{(2.1)}\quad 2\tau\sum_{n\geq 1}x^2\{4\pi^2n^2+x^4\tau^2\}^{-1}=\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\zeta(s)x^{-2s}\tau^{-s}ds.$$

Observe the absolute convergence of $\sum_{k\geq 1}k^{2s}$ for $\operatorname{Re}s<-1/2$. Then we get by (2.1)

$$\text{(2.2)}\quad \omega(\tau)=2\tau\sum_{m\geq 1}\sum_{n\geq 1}\frac{m^2}{(4\pi^2n^2)m^4+\tau^2}=I_c(\tau)$$



with

$$\text{(2.3)}\quad I_c(\tau)=\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\zeta(s)\zeta(2s)\tau^{-s}ds$$

where $c\in(-1,-1/2)$.

The inversion of summation and integration is justified by the asymptotic equality

$$\text{(2.4)}\quad |\Gamma(\sigma+it)|\sim(2\pi)^{1/2}e^{-\pi|t|/2}|t|^{\sigma-1/2}\qquad (|t|\to\infty)$$

uniformly in any finite $\sigma$-interval of $\mathbf{R}$, and by

$$\text{(2.5)}\quad \zeta(s)=O(|t|^{\mu})\qquad (|t|\to\infty)$$

in any half plane $\sigma\geq\sigma_0$ with constant $\mu=\mu(\sigma_0)$.

Equation (2.2) is the key. By absolute convergence we can invert the order of summation on the left of (2.2) to obtain with the right equality of (1.3)

$$\text{(2.6)}\quad \omega(\tau)=2\tau\sum_{n\geq 1}\frac{1}{4\pi^2n^2}\sum_{m\geq 1}\frac{m^2}{m^4+(\tau/2\pi n)^2} =\frac{1}{4}\sqrt{\frac{\tau}{\pi}}\sum_{n\geq 1}n^{-3/2}\left\{\frac{\sinh(\sqrt{\pi\tau/n})-\sin(\sqrt{\pi\tau/n})}{\cosh(\sqrt{\pi\tau/n})-\cos(\sqrt{\pi\tau/n})}\right\}.$$

Now consider $I_c(\tau)$. Note that the integrand

$$\text{(2.7)}\quad \psi(s)=\Gamma(s)\zeta(s)\zeta(2s)\tau^{-s}\qquad (\tau>0)$$

is meromorphic in $\mathbf{C}$ with the only simple poles at $s=0$, $1/2$, $1$. The simple poles of $\Gamma(s)$ at $a=-n$ $(n\in\mathbf{N})$ are cancelled by the simple zeros of $\zeta(s)$ at $s=-2n$ $(n\in\mathbf{N})$. The sum of residues is (1.2). Thus by (2.4) and (2.5) the Phragmén–Lindelöf principle and Cauchy's theorem yield for (2.3)

$$\text{(2.8)}\quad I_{c'}(\tau)=R(\tau)+I_c(\tau)$$

where $c'>1$, $c\in(-1,-1/2)$.

But for $c'>1$, $x\in\mathbf{R}^+$ we have the inverse Mellin transform

$$\text{(2.9)}\quad (e^x-1)^{-1}=\frac{1}{2\pi i}\int_{(c')}\Gamma(s)\zeta(s)x^{-s}ds.$$

Hence by (2.4) and (2.7) we get for $\tau>0$, $c'>1$

$$\text{(2.10)}\quad \sum_{n\geq 1}(e^{n^2\tau}-1)^{-1}=\frac{1}{2\pi i}\int_{(c')}\psi(s)ds=I_{c'}(\tau).$$

Thus formula (B) is proved.

**3.** We turn to a generalization of (B). Let $\Omega$ denote the vector space of all arithmetical functions $f\colon N \to C$. For $f, g \in \Omega$ and $q, n \in N$ define ([1], §8.3)

$$s_q(n) = \sum_{d|(n,q)} f(d) g(q/d). \tag{3.1}$$

Then $s_q(n)$ is periodic mod $q$ and represents a generalization of Ramanujan's exponential sum [10]

$$c_q(n) = \sum_{\substack{1 \le h \le q \\ (h,q)=1}} \exp\{2\pi i n h/q\} = \sum_{d|(n,q)} d\mu(q/d). \tag{3.2}$$

We prove: If $\tau > 0$ and $s_q(n)$ is defined by (3.1), then

$$\text{(C)} \quad \sum_{n \ge 1} s_q(n)(e^{n^2\tau} - 1)^{-1} = \sum_{d|q} f(d) g(q/d) R(d^2\tau)$$
$$+ \frac{1}{4}\sqrt{\frac{\tau}{\pi}} \sum_{d|q} f(d) g(q/d) d \sum_{n \ge 1} n^{-3/2} \left\{ \frac{\sinh(d\sqrt{\tau\pi/n}) - \sin(d\sqrt{\tau\pi/n})}{\cosh(d\sqrt{\tau\pi/n}) - \cos(d\sqrt{\tau\pi/n})} \right\}.$$

Remarks. (i) Take $q = f(1) = g(1) = 1$ in (3.1). Thus (C) contains (B).
(ii) (C) may be regarded as the absolutely convergent generalized Mellin–Ramanujan expansion for the function on the right of (C) (see [7], [8]).

Proof of (C). We require a generalization of the Ramanujan–Crum expansion ([11], pp. 10–11; [5])

$$\sum_{n \ge 1} c_q(n) n^{-s} = \zeta(s) \sum_{d|q} \mu(q/d) d^{1-s} \quad (\operatorname{Re} s > 1). \tag{3.3}$$

This is given by ([8], (3.2))

$$\sum_{n \ge 1} s_q(n) n^{-s} = \zeta(s) \sum_{d|q} f(d) g(q/d) d^{-s} \quad (\operatorname{Re} s > 1). \tag{3.4}$$

Since $s_q(n)$ is periodic mod $q$ the series on the left of (3.4) converges absolutely for $\operatorname{Re} s > 1$. Thus (2.7), (2.9) and (3.4) yield for $q \in N$ and $c' > 1$

$$\sum_{n \ge 1} s_q(n)(e^{n^2\tau} - 1)^{-1} = \frac{1}{2\pi i} \int_{(c')} \psi(s) \sum_{d|q} f(d) g(q/d) d^{-2s} ds \tag{3.5}$$
$$= \sum_{d|q} f(d) g(q/d) I_{c'}(d^2\tau).$$

By (2.8) we get

$$\sum_{n \ge 1} s_q(n)(e^{n^2\tau} - 1)^{-1} = \sum_{d|q} f(d) g(q/d) \{R(d^2\tau) + I_c(d^2\tau)\} \tag{3.6}$$

with $c \in (-1, -1/2)$ and $R$ defined by (1.2).



By (2.2) and (2.6) we have

$$\sum_{d|q} f(d) g(q/d) \omega(d^2\tau) \tag{3.7}$$
$$= \frac{1}{4}\sqrt{\frac{\tau}{\pi}} \sum_{d|q} f(d) g(q/d) d \sum_{n \ge 1} n^{-3/2} \left\{ \frac{\sinh(d\sqrt{\tau\pi/n}) - \sin(d\sqrt{\tau\pi/n})}{\cosh(d\sqrt{\tau\pi/n}) - \cos(d\sqrt{\tau\pi/n})} \right\}.$$

Thus by (3.5)–(3.7) formula (C) follows immediately.

From the great variety of possible choices of the functions $f, g \in \Omega$ in (C) we here only mention briefly the interesting case when $s_q(n)$ reduces to Ramanujan's exponential sum (3.2), i.e. when $f(d) = d$ and $g(d) = \mu(d)$ in (3.1). Observe that Euler's totient function $\varphi(\cdot)$ is given in terms of the Möbius $\mu$-function by ([1], p. 26)

$$\varphi(q) = \sum_{d|q} d\mu(q/d) \quad (q \in N).$$

Thus (C) yields the exact formula

$$\text{(D)} \quad \sum_{n \ge 1} c_q(n)(e^{\tau n^2} - 1)^{-1} = \frac{1}{4}\varphi(q) + \frac{1}{2}\sqrt{\frac{\pi}{\tau}} \zeta\left(\frac{1}{2}\right) \sum_{d|q} \mu(d) + \frac{\pi^2}{6\tau} \sum_{d|q} d^{-1} \mu\left(\frac{q}{d}\right)$$
$$+ \frac{1}{4}\sqrt{\frac{\tau}{\pi}} \sum_{d|q} d^{1/2} \mu\left(\frac{q}{d}\right) \sum_{n \ge 1} \left(\frac{d}{n}\right)^{3/2} \left\{ \frac{\sinh(d\sqrt{\pi\tau/n}) - \sin(d\sqrt{\pi\tau/n})}{\cosh(d\sqrt{\pi\tau/n}) - \cos(d\sqrt{\pi\tau/n})} \right\}.$$

This is an ordinary Mellin–Ramanujan expansion in the sense of [7]. For $q = 1$, (D) yields again (B). For $q = 2$ we have $c_2(n) = (-1)^n$. Since

$$\sum_{d|q} \mu(d) = \begin{cases} 1, & q = 1, \\ 0, & q > 1, \end{cases}$$

(D) reduces to the exact formula for the alternating case of (B):

$$\text{(B')} \quad \sum_{n \ge 1} (-1)^n (e^{\tau n^2} - 1)^{-1}$$
$$= \frac{1}{4} - \frac{\pi^2}{12\tau} + \frac{1}{4}\sqrt{\frac{\tau}{\pi}} \sum_{d|2} d^{1/2} \mu\left(\frac{2}{d}\right) \sum_{n \ge 1} \left(\frac{d}{n}\right)^{3/2} \left\{ \frac{\sinh(d\sqrt{\pi\tau/n}) - \sin(d\sqrt{\pi\tau/n})}{\cosh(d\sqrt{\pi\tau/n}) - \cos(d\sqrt{\pi\tau/n})} \right\}.$$

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# Constructions of $B_h[g]$-sequences

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**1. Introduction.** A (finite) $B_h[g]$-*sequence* is a sequence $\bar{a} = (a_0, a_1, a_2, \ldots, a_J)$ of integers such that no integer has more than $g$ representations as sums of $h$ summands from $\bar{a}$. A survey of results on $B_h[1]$-sequences (up to 1966) is given in Chapter II of [5]. Recent papers on $B_h[g]$-sequences include [4].

Without loss of generality we may assume that

$$0 = a_0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_J.$$

Assuming that a sum contains the summand $a_j$ $x_j$ times, the sum may be written as $\sum_{j=0}^{J} x_j a_j$ where $\sum_{j=0}^{J} x_j = h$. Since $a_0 = 0$, we have $\sum_{j=0}^{J} x_j a_j = \sum_{j=1}^{J} x_j a_j$. Further $\sum_{j=1}^{J} x_j = h - x_0 \leqslant h$. Hence we get the following precise definition of a $B(g,h,J)$-sequence ($B_h[g]$-sequence with $J+1$ elements): Let

$$C(h,\ J) = \{\bar{x} = (x_1, x_2, \ldots, x_J) |\ x_j \text{ non-negative integers and } \sum_{j=1}^{J} x_j \leqslant h\}.$$

Further, let

$$D_k = D_k(\bar{a}) = \{\bar{x} \in C(h, J) |\ \sum_{j=1}^{J} x_j a_j = k\}.$$

The sequence $\bar{a} = (a_0, a_1, a_2, \ldots, a_J)$ is a $B(g,h,J)$-*sequence* if $|D_k| \leqslant g$ for all integers $k \geqslant 0$.

Let

$$N(g,h,J) = \min\{a_J |\ (a_0, a_1, a_2, \ldots, a_J) \text{ is a } B(g,h,J)\text{-sequence}\}.$$

A $B(g,h,J)$-sequence $(a_0, a_1, a_2, \ldots, a_J)$ where $a_J = N(g,h,J)$ is called *optimal*.

The main emphasis in the published literature has been on the behaviour of $N(g,h,J)$ for fixed $g$ and $h$ and varying $J$, in particular the asymptotic behaviour when $J \to \infty$. In this paper we consider mainly the situation with varying $h$, in particular the asymptotic behaviour when $h \to \infty$.

It is easy to see that $(0, 1, h+1, (h+1)^2, \ldots, (h+1)^{J-1})$ is a $B(1,h,J)$-sequence. Hence

(1) $$N(g,h,J) \leqslant N(1,h,J) \leqslant (h+1)^{J-1}.$$