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The equation $xyz = x+y+z = 1$ in integers of a quartic field

by

ANDREW BREMNER (Tempe, Az.)

1. Cassels [2] and Sansone and Cassels [4] showed that there are no rational solutions of the Diophantine equation

$$xyz = x+y+z = 1. \tag{1}$$

Small [5] studies the equation over finite fields, and Mollin *et al.* [3] investigate the equation over quadratic number fields, finding the finitely many such fields K in which there does exist a solution for integer units u_1, u_2, u_3 of K of the equation

$$u_1 u_2 u_3 = u_1+u_2+u_3.$$

Bremner [1] has determined all cubic fields whose ring of integers contains a solution to (1).

Here, we resolve completely the question of finding all quartic number fields whose ring of integers contains a solution to (1). There are two infinite families of such fields. The result is as follows.

THEOREM 1. *Let K be a quartic number field with ring of integers $\mathfrak{O}_K$. Then the equation*

$$xyz = x+y+z = 1$$

is solvable for $x, y, z \in \mathfrak{O}_K$ in precisely the following instances:

(i) *The infinite family $K = \mathbf{Q}(\theta)$,*

$$\theta^4+(u^2-u+2)\theta^3+2u\theta^2+(u+1)\theta+1 = 0, \quad u \in \mathbf{Z},\ u \neq 1,$$

with solution up to permutation

$$x = -\theta^3-(u^2-u+2)\theta^2-2u\theta-(u+1),$$

$$(u-1)y = -\theta^3-(u^2-u+1)\theta^2+(u^2-3u+1)\theta+(u-2),$$

$$(u-1)z = u\theta^3+(u^3-u^2+2u-1)\theta^2+(u^2+u-1)\theta+u^2.$$

(ii) *The infinite family $K = \mathbf{Q}(\phi)$,*

$$\phi^4+(u^2-u-2)\phi^3+2\phi^2-(u+1)\phi+1=0, \quad u\in \mathbf{Z},\ u\neq \pm 1, 3$$

with solution up to permutation

$$x=-\phi^3-(u^2-u-2)\phi^2-2\phi+(u+1),$$
$$(u-1)y=-\phi^3-(u^2-u-1)\phi^2-(u^2-u+1)\phi+u,$$
$$(u-1)z=u\phi^3+(u^3-u^2-2u+1)\phi^2+(u^2+u-1)\phi-u^2.$$

2. Let K be a number field with ring of integers $\mathfrak{O}_K$; suppose that x, y, z are units of $\mathfrak{O}_K$ satisfying (1). Since

$$\mathrm{Norm}_{K/\mathbf{Q}}(x)\,\mathrm{Norm}_{K/\mathbf{Q}}(y)\,\mathrm{Norm}_{K/\mathbf{Q}}(z)=1$$

we may suppose, without loss of generality, that

(2) $$\mathrm{Norm}_{K/\mathbf{Q}}(x)=1.$$

Now

$$x+y+1/xy=1$$

so that

$$y^2(x)+y(x^2-x)+1=0$$

and

(3) $$2y=-x+1+x\sqrt{1-\frac{2}{x}+\frac{1}{x^2}-\frac{4}{x^3}}.$$

Put

(4) $$1/x=X$$

where X is also a unit of $\mathfrak{O}_K$, with, from (2),

(5) $$\mathrm{Norm}_{K/\mathbf{Q}}(X)=1.$$

It follows from (3) that

$$1-2X+X^2-4X^3=W^2, \quad W\in\mathfrak{O}_K$$

so that $P=(X, W)$ is a point defined over $\mathfrak{O}_K$ on the elliptic curve

(6) $$E\colon\ 1-2x+x^2-4x^3=y^2.$$

Conversely, it is clear that a point $P=(X, W)$, defined over $\mathfrak{O}_K$, on the curve (6), with X a unit of $\mathfrak{O}_K$, gives rise via the transformations (3), (4) to a unit solution x, y, z of the original equation (1).

Now the result of Cassels [2] and Sansone and Cassels [4] is equivalent to the rational rank of (6) being equal to 0. Further, a simple calculation shows that the rational torsion group on (6) is cyclic of order 3, with generator (0, 1).

3. We first describe points on E, given by equation (6), that are defined over a quadratic number field.

It is easy to see that if a point P on E has coordinates in a quadratic number field k, then k is of type $\boldsymbol{Q}(\sqrt{1-2\alpha+\alpha^2-4\alpha^3})$, for some (non-zero) $\alpha \in \boldsymbol{Q}$. For denoting the conjugate point under quadratic conjugation by $\bar{P}$, then $P^* = P-\bar{P}$ is reversed under conjugation, and so is of type $P^* = (\alpha, \beta\sqrt{d})$, with $k = \boldsymbol{Q}(\sqrt{d})$, $\alpha, \beta \in \boldsymbol{Q}$. But then $d\beta^2 = 1-2\alpha+\alpha^2-4\alpha^3$, as required. More precisely, we have the following.

LEMMA 1. *Let $P \in E$ with coordinates of $P = (x_P, y_P)$ in a quadratic number field.*

Then either

$$x_P = t \in \boldsymbol{Q}-\{0\}, \quad y_P = \sqrt{1-2t+t^2-4t^3} \tag{7}$$

or

$$x_P^2+t(t-1)x_P+t = 0 \quad \textit{for some } t \in \boldsymbol{Q}-\{0\} \tag{8}$$

with $\pm y_P = (2t-1)x_P+1$.

Proof. Take the straight line l (defined over $\boldsymbol{Q}$) through P and $\bar{P}$. Then l meets E in a third rational point Q.

If $Q = o$, the point at infinity, then l is of type $x-x_P = 0$, and (7) follows immediately.

Otherwise, the only possibilities for Q are $(0, \pm 1)$, and by replacing P by $-P$ if necessary, we may suppose that $Q = (0, 1)$. Then l is of the form $y = mx+1$, $m \in \boldsymbol{Q}$, whence the points of intersection are given by

$$1-2x+x^2-4x^3 = m^2x^2+2mx+1,$$

so that x_P is a root of

$$4x^2+(m^2-1)x+2(m+1) = 0.$$

Putting $m = 2t-1$ gives

$$x_P^2+t(t-1)x_P+t = 0, \quad y_P = (2t-1)x_P+1,$$

as required.

COROLLARY. *If x is a unit of its quadratic field, then either* (i) $\pm P = (1, 2i)$, $(-1, 2\sqrt{2})$ *or* (ii) $\pm P = (i, 1+i)$, $(-i, 1-i)$, $(-1+\sqrt{2}, -4+3\sqrt{2})$, $(-1-\sqrt{2}, -4-3\sqrt{2})$.

Proof. If x is a unit of $\boldsymbol{Z}$, then $x = \pm 1$ and (i) follows from (7).

If x is a unit of a quadratic field, then Norm $(x) = \pm 1$, so necessarily in (8), $t = \pm 1$ and the result follows.

We now give a similar argument which deals with the cubic case, leading to the result of Bremner [1] in a neater manner.

For suppose that $P \in E$, with the coordinates of $P = (x_P, y_P)$ lying in a cubic number field.

Take a parabola through P and its two $\boldsymbol{Q}$-conjugates, with equation

$$dy = px^2 + qx + r, \quad d, p, q, r \in \boldsymbol{Z},\ d \neq 0,\ (d, p, q, r) = 1.$$

This parabola meets E in six points, two of which occur at $\mathfrak{o}$, and three of which form a conjugate set over $\boldsymbol{Q}$. The remaining point Q of intersection is thus rational.

If $Q = \mathfrak{o}$, then $p = 0$ so that x_P satisfies

$$d^2(1 - 2x_P + x_P^2 - 4x_P^3) = (qx_P + r)^2,$$

i.e.

(9) $$4d^2 x_P^3 + (q^2 - d^2)x_P^2 + (2qr + 2d^2)x_P + (r^2 - d^2) = 0.$$

If $Q \neq \mathfrak{o}$, then as before, we may suppose that $Q = (0, 1)$. Then $r = d$, and x_P satisfies

$$d^2(1 - 2x + x^2 - 4x^3) = (px^2 + qx + d)^2.$$

Removing the root at 0, x_P thus satisfies the residual cubic equation

(10) $$p^2 x_P^3 + (2pq + 4d^2)x_P^2 + (q^2 + 2pd - d^2)x_P + (2qd + 2d^2) = 0.$$

If now x_P is to be a unit of the cubic number field, then in the first instance, at (9) above, it follows that $r^2 - d^2 = \pm 4d^2$, which is impossible. So the second instance must hold, and from (10),

(11a) $$\pm p^2 = 2qd + 2d^2,$$

(11b) $$p^2 \mid 2pq + 4d^2,$$

(11c) $$p^2 \mid q^2 + 2pd - d^2.$$

Now $(d, p) = 1$, otherwise (c) leads to a contradiction of $(d, p, q, r) = 1$. Then (a) forces $d = 1$ and $2 \mid p$; and (b) forces $p \mid 4$ and $4 \nmid p$. Consequently $p = \pm 2$, and (a) gives $q = 1, -3$. Hence the only solutions are $(p, q, r, d) = (\pm 2, 1, 1, 1)$, $(\pm 2, -3, 1, 1)$. The former pair returns $X = x_P$ of norm -1, contrary to (5). And the latter pair returns the cubic fields given in Bremner [1].

4. Suppose now that $P \in E$ with coordinates of $P = (x_P, y_P)$ lying in a quartic number field; further, we assume x_P is a unit of the field. Take the cubic curve Γ through P and its three $\boldsymbol{Q}$-conjugates, with equation

(12) $$dy = px^3 + qx^2 + rx + s, \quad d, p, q, r, s \in \boldsymbol{Z},\ d \neq 0,\ (d, p, q, r, s) = 1.$$

Then Γ meets E in nine points. Three of these occur at $\mathfrak{o}$, and four form a $\boldsymbol{Q}$-rational set. The remaining two points of intersection are thus defined (as a pair) over $\boldsymbol{Q}$.

(i) Suppose that one of these points is o. Then $p = 0$, and there is a further zero at o. The quartic satisfied by x_P is

$$d^2(1-2x_P+x_P^2-4x_P^3) = (qx_P^2+rx_P+s)^2,$$

i.e.

$$q^2 x_P^4+(2qr+4d^2)x_P^3+(r^2+2qs-d^2)x_P^2+(2rs+2d^2)x_P+(s^2-d^2) = 0.$$

The requirement that x_P be a unit, with $\mathrm{Norm}(x_P) = 1$, implies

(13a) $$q^2 = s^2-d^2,$$

(13b) $$q^2 \mid 2qr+4d^2,$$

(13c) $$q^2 \mid r^2+2qs-d^2,$$

(13d) $$q^2 \mid 2rs+2d^2.$$

Now $(q, d) = 1$, otherwise (a), (c) contradict $(d, q, r, s) = 1$. Then (b) forces $q \mid 4$; and since $d \neq 0$, (a) gives $q = 4$, $(s, d) = (\pm 5, \pm 3)$. But now (b) cannot hold. Consequently no unit solutions x_P arise in this instance.

(ii) Suppose the residual pair is individually rational. It cannot be the pair $\{(0, 1), (0, -1)\}$ for then $d = s$, $d = -s$, forcing $d = 0$. In virtue of (i), we can thus assume it is a double root at one of the points $(0, \pm 1)$. In any event, $d^2 = s^2$ and $-2d^2 = 2rs$, so that $d = \pm s$, $r = -s$.

The quartic satisfied by x_P is

$$s^2(1-2x+x^2-4x^3) = (px^3+qx^2-sx+s)^2,$$

i.e.

$$p^2 x^4+2pqx^3+(q^2-2ps)x^2+(2ps-2qs+4s^2)x+2qs = 0.$$

The requirement that x_P be a unit of norm 1 gives

(14a) $$p^2 = 2qs,$$

(14b) $$p^2 \mid 2pq,$$

(14c) $$p^2 \mid q^2-2ps,$$

(14d) $$p^2 \mid 2ps-2qs+4s^2.$$

Now $(p, s) = 1$ otherwise (c) gives $(p, q, s) \neq 1$ where $(d, p, q, r, s) \neq 1$. So (a) gives $s = 1$, $p^2 = 2q$. Now (d) gives $p \mid 4$, $4 \nmid p$, and since $2 \mid p$, then $p = 2\varepsilon$ $(\varepsilon = \pm 1)$, $q = 2$. Thus x_P satisfies the quartic

(15) $$x^4+2\varepsilon x^3+(1-\varepsilon)x^2+\varepsilon x+1 = 0.$$

Denoting a root of (15) by Θ, we recover the following unit solution of (1):

$$(16)\qquad \begin{aligned} x &= -\Theta^3-2\varepsilon\Theta^2+(\varepsilon-1)\Theta-\varepsilon,\\ y &= \varepsilon\Theta^2+\Theta,\\ z &= \Theta^3+\varepsilon\Theta^2-\varepsilon\Theta+(1+\varepsilon).\end{aligned}$$

(iii) If neither (i) nor (ii) happens, then necessarily the residual pair is not individually defined over $\boldsymbol{Q}$.

By Lemma 1, the intersection therefore contains either a double point at $x = t$, $t \in \boldsymbol{Q}-\{0\}$, or contains the pair of points corresponding to the roots of $x^2+t(t-1)x+t = 0$, $t \in \boldsymbol{Q}-\{0\}$. In particular, the sextic polynomial representing the intersection of the curves (6) and (12) either contains a repeated root at $x = t$, or contains the quadratic factor $x^2+t(t-1)x+t$. Note now that $p \neq 0$.

The intersection is given by

$$(17)\qquad d^2(1-2x+x^2-4x^3) = (px^3+qx^2+rx+s)^2.$$

A rational root $x = t$ implies $1-2t+t^2-4t^3$ is square, so that perforce $t = 0$, a contradiction.

Consequently, on writing (17) as a sextic equation for x, then it possesses $x^2+t(t-1)x+t$, $t \in \boldsymbol{Q}-\{0\}$, as a quadratic factor. Since we are assuming x_p is a unit of norm 1, the residual quartic is of type $x^4+ax^3+bx^2+cx+1$, $a, b, c \in \boldsymbol{Z}$. It follows that

$$(18)\qquad \begin{aligned} p^2x^6+2pqx^5+(q^2+2pr)x^4+(2qr+2ps+4d^2)x^3+(r^2+2qs-d^2)x^2\\ +(2rs+2d^2)x+(s^2-d^2)\\ = p^2[x^2+t(t-1)x+t][x^4+ax^3+bx^2+cx+1].\end{aligned}$$

By Gauss' Lemma, $p^2[x^2+t(t-1)x+t]$ has integer coefficients, so that

$$(19)\qquad t = u/p, \quad u \in \boldsymbol{Z}, \quad u \neq 0, \quad (u, p) = 1.$$

The right-hand side at (18) is $[p^2x^2+u(u-p)x+up][x^4+ax^3+bx^2+cx+1]$, and equating coefficients of powers of x,

$$(20a)\qquad 2pq = p^2a+u(u-p),$$

$$(20b)\qquad q^2+2pr = p^2b+u(u-p)a+up,$$

$$(20c)\qquad 2qr+2ps+4d^2 = p^2c+u(u-p)b+upa,$$

$$(20d)\qquad r^2+2qs-d^2 = p^2+u(u-p)c+upb,$$

$$(20e)\qquad 2rs+2d^2 = u(u-p)+upc,$$

$$(20f)\qquad s^2-d^2 = up.$$

If π is an odd prime factor of p, then it follows easily that $p \equiv q \equiv r \equiv s \equiv d \equiv 0 \bmod \pi$. Thus $\pm p$ must be a power of 2. Suppose next $p \equiv 0 \bmod 4$. Then sequentially, (a) implies $u \equiv 0 \bmod 4$, (b) implies $q \equiv 0 \bmod 4$, (c) implies $d \equiv 0 \bmod 2$, (d) and (e) imply $r \equiv s \equiv 0 \bmod 2$, contradicting $(p, q, r, s, d) \neq 1$.

Since we can assume without loss of generality that $p > 0$, then $p = 1$ or 2. However, (a) implies $p \mid u^2$, and from (19) we have $(p, u) = 1$. Necessarily therefore $p = 1$.

Substitute into equations (20), and use (f) to eliminate d^2:

(21a) $$2q = a+u(u-1),$$

(21b) $$q^2+2r = b+u(u-1)a+u,$$

(21c) $$2qr+2s = c+u(u-1)b+ua+4u-4s^2,$$

(21d) $$r^2+2qs = 1+u(u-1)c+ub-u+s^2,$$

(21e) $$2rs = u(u+1)+uc-2s^2.$$

Use (21a) to eliminate q, and, after multiplying (b), (c), (d) by s, s, s^2, respectively, use (21e) to eliminate r:

(22a) $$4sb = sa^2-2u(u-1)sa+4uc+4u(u+1)+(u^4-2u^3+u^2-4u)s-8s^2,$$

(22b) $$\big(u(u+1)-2us-2s^2\big)a+uac+\big(u^2(u-1)-2s\big)c-2u(u-1)sb$$
$$+u^2(u^2-1)-8us+(-2u^2+2u+4)s^2+8s^3 = 0,$$

(22c) $$u^2c^2+2u^2(u+1-2s^2)c-4us^2b+4s^3a$$
$$+u^2(u+1)^2-4(u^2+1)s^2+4u(u-1)s^3 = 0.$$

Use (22a) to eliminate b: there results

(23a) $$-(u-1)sA^2+2uAC+4(3u-1)s(s^2-u) = 0,$$

(23b) $$-s^2A^2+u^3C^2-4s^2(u^3-1)(s^2-u) = 0$$

where

(24) $$A = ua-\big(u^2(u-1)+2s\big),$$

(25) $$C = c+(u+1-2s-2s^2).$$

Multiply (23b) by A^2, and use (a) to eliminate C (recall from (20f) that $d^2 = s^2-u$):

(26) $$s^2(A^2-4d^2)\big((-u^3+2u^2-u+4)A^2+4u(3u-1)^2d^2\big) = 0.$$

If $s^2(A^2-4d^2) \neq 0$ then it follows that

$$(u^3-2u^2+u-4)A^2 = 4u(3u-1)^2d^2$$

so that

$$\left(1-\frac{2}{u}+\frac{1}{u^2}-\frac{4}{u^3}\right)A^2 = \left(\frac{2(3u-1)d}{u}\right)^2,$$

which in virtue of the remarks of the final paragraph of Section 2, forces $A = 0 = 2(3u-1)d$, a contradiction. Accordingly, $s^2(A^2-4d^2) = 0$ and we have either I: $A = 2d\varepsilon$ ($\varepsilon = \pm 1$), or II: $s = 0$.

Case I. If $A = 2d\varepsilon$, then (23) gives $C = -2sd\varepsilon$. Solving (24) for a,

(27) $$a = 2/(s-d\varepsilon)+u(u-1)$$

and from (21a),

(28) $$q = 1/(s-d\varepsilon)+u(u-1).$$

Thus

(29) $$s-d\varepsilon = \alpha = \pm 1$$

so that

(30) $$s+d\varepsilon = \alpha u$$

whence solving (29), (30) for s, d,

(31) $$s = \alpha(u+1)/2, \quad d\varepsilon = \alpha(u-1)/2.$$

Solving (25) for c, (21b) for b, and (21e) for r, gives the following parametrization:

(32) $$\begin{aligned} p &= 1, \\ q &= u(u-1)+\alpha, \\ r &= (\alpha/2+1)u-\alpha/2, \\ s &= \alpha(u+1)/2, \\ d\varepsilon &= \alpha(u-1)/2, \\ a &= u(u-1)+2\alpha, \\ b &= (\alpha+1)u+(1-\alpha), \\ c &= \alpha(u+1). \end{aligned}$$

Taking $\alpha = +1$, $\alpha = -1$, gives the respective triples

(33) $$(a,b,c) = (u^2-u+2, 2u, u+1), \quad (u^2-u-2, 2, -u-1).$$

An elementary exercise shows that the quartic of which x_p is a root, namely $x^4+ax^3+bx^2+cx+1$, is irreducible precisely for $u \neq 1$ in the first instance at (33), and $u \neq \pm 1, 3$ in the second instance at (33).

It is now possible to recover a solution of the original equation (1), via the transformations (3), (4).

The first case at (33) leads to the following.

Let $u \in \mathbf{Z}$, $u \neq 0, 1$, and define

(34) $$\theta^4+(u^2-u+2)\,\theta^3+2\,u\,\theta^2+(u+1)\,\theta+1 = 0.$$

Then

(35) $$\begin{aligned} x &= -\theta^3-(u^2-u+2)\,\theta^2-2\,u\,\theta-(u+1),\\ (u-1)\,y &= -\theta^3-(u^2-u+1)\,\theta^2+(u^2-3\,u+1)\,\theta+(u-2),\\ (u-1)\,z &= u\,\theta^3+(u^3-u^2+2\,u-1)\,\theta^2+(u^2+u-1)\,\theta+u^2. \end{aligned}$$

Notice again that the quadratic equation giving (3) ensures that y (and hence z) is both an algebraic integer and a unit, so that despite the appearance of denominators at (35), there is an automatic guarantee of integrality. (It can in fact be shown that the equation satisfied by y is

$$y^4-(u+3)\,y^3+4\,uy^2-(u^2+u-2)\,y-1 = 0;$$

the equation for z is left as an exercise.)

Similarly, the second case at (33) leads to the following:

Let $u \in \mathbf{Z}$, $u \neq 0, \pm 1, 3$; and define

(36) $$\phi^4+(u^2-u-2)\,\phi^3+2\,\phi^2-(u+1)\,\phi+1 = 0.$$

Then

(37) $$\begin{aligned} x &= -\phi^3-(u^2-u-2)\,\phi^2-2\,\phi+(u+1),\\ (u-1)\,y &= -\phi^3-(u^2-u-1)\,\phi^2-(u^2-u+1)\,\phi+u,\\ (u-1)\,z &= u\,\phi^3+(u^3-u^2-2\,u+1)\,\phi^2+(u^2+u-1)\,\phi-u^2; \end{aligned}$$

where, as above, y and z are both integers and units of $\mathbf{Q}(\phi)$.

Case II. If $s = 0$, then (21e) implies

(38) $$c = -u-1.$$

The equations (21) become:

(39a) $$2\,q = a+u\,(u-1),$$

(39b) $$q^2+2\,r = b+u\,(u-1)\,a+u,$$

(39c) $$2\,qr = u\,(u-1)\,b+ua+3\,u-1,$$

(39d) $$r^2 = ub-u^3+1,$$

with, from (20f),

(40) $$d^2 = -u.$$

Use (39a) to eliminate q in (39b):

(41) $$2\,r = b-\tfrac{1}{4}\,a^2+\tfrac{1}{2}\,u\,(u-1)\,a-\tfrac{1}{4}\,u^2\,(u-1)^2+u.$$

Use (39a), (41) to eliminate q, r in (39c):

(42) $$b\left(\tfrac{1}{2}a-\tfrac{1}{2}u(u-1)\right)=\tfrac{1}{8}a^3-\tfrac{1}{8}u(u-1)a^2+\left(-\tfrac{1}{8}u^2(u-1)^2+\tfrac{1}{2}u\right)a+\left(\tfrac{1}{8}u^3(u-1)^3-\tfrac{1}{2}u^2(u-1)+3u-1\right).$$

Then, using (41) to eliminate b,

(43) $$r\left(a-u(u-1)\right)=\tfrac{1}{4}u(u-1)a^2+\left(-\tfrac{1}{2}u^2(u-1)^2+u\right)a+\left(\tfrac{1}{4}u^3(u-1)^3-u^2(u-1)+3u-1\right).$$

Using (42), (43) in (39d), there results after simplification

(44) $$\left[(u^3-2u^2+u-4)\left(a^2-2u(u-1)a\right)+\left(u^3(u-1)^4-4u^2(u-1)^2+4(3u-1)^2\right)\right]\times\left[ua^2-2u^2(u-1)a+\left(u^3(u-1)^2+4\right)\right]=0.$$

If the first factor is to be zero, then

$$(u^3-2u^2+u-4)\left(a-u(u-1)\right)^2+4(3u-1)^2=0,$$

so in particular

(45) $$-u^3+2u^2-u+4=v^2,\qquad v\in \boldsymbol{Z}.$$

The elliptic curve represented by (45) has rank 1 (and there are integer points for $u=0, 1, -15$); however, in virtue of (40), we are only interested in the curve of genus 2 represented by

(46) $$d^6+2d^4+d^2+4=v^2.$$

But the only integer points on the curve (46) have $d=0$, for it may be written in the form

$$(d^3+d)^2+4=v^2$$

which clearly forces $d^3+d=0$, i.e. $d=0$. Since we assumed $d\neq 0$, only the second factor at (44) may be zero.

In this instance, then

$$ua^2-2u^2(u-1)a+u^3(u-1)^2+4=0,$$

that is,

(47) $$u\left(a-u(u-1)\right)^2+4=0.$$

Using (40),

$$d\left(a-u(u-1)\right)=2\varepsilon,\qquad \varepsilon=\pm 1,$$

so that

$$u\left(a-u(u-1)\right)=2d\varepsilon$$

and from (24),

$$A = 2d\varepsilon;$$

thus this instance has been covered by Case I.

Theorem 1 now follows from the solutions (16), (35) and (37) noting that the solutions at (16) are the particular cases of (35) and (37) at $u = 0$.

The discriminant of $Q(\theta)$ at (34) is equal to

$$-(u-1)^2(4u^7-5u^6-6u^5+77u^4-304u^3+725u^2-1006u+643);$$

and the discriminant of $Q(\phi)$ at (36) is

$$(u+1)^2(4u^7-35u^6+102u^5-69u^4-216u^3+499u^2-402u+117).$$

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DEPARTMENT OF MATHEMATICS
ARIZONA STATE UNIVERSITY
Tempe, AZ 85287-1804
USA

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