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q-difference equations and Ramanujan–Selberg continued fractions

by

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1. Introduction. By studying the q-difference equation

$$X_n = X_{n-1} + q^n X_{n-2},$$

Schur [11], [12] examined the famous Rogers–Ramanujan continued fraction

$$K(q) = 1 + \frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\cdots, \tag{1.1}$$

where q is a primitive mth root of unity. Namely, he established the following theorem:

THEOREM. *Let q be a primitive m-th root of unity. If m is a multiple of 5, then $K(q)$ diverges. When m is not a multiple of 5, let $\lambda = \left(\frac{m}{5}\right)$, the Legendre symbol. Furthermore, let ϱ denote the least positive residue of m modulo 5. Then for $m \not\equiv 0 \pmod 5$,*

$$K(q) = q^{(1-\lambda\varrho m)/5} K(\lambda).$$

Note that it is elementary that

$$K(1) = (\sqrt{5}+1)/2, \quad K(-1) = (\sqrt{5}-1)/2.$$

Recently, G. Andrews *et al.* [5] have proved that if $0 < |q| < 1$, then $1/K(q^{-1})$ oscillates between

$$1 - \frac{q}{1+}\,\frac{q^2}{1-}\,\frac{q^3}{1+}\,\frac{q^4}{1-}\cdots$$

and

$$\frac{q}{1+}\,\frac{q^4}{1+}\,\frac{q^8}{1+}\,\frac{q^{12}}{1+}\cdots.$$

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Let

$$S_1(q) := 1 + \frac{q}{1+} \frac{q+q^2}{1} \frac{q^3}{+1+} \frac{q^2+q^4}{1} + \cdots, \tag{1.2}$$

$$S_2(q) := 1 + \frac{q+q^2}{1} \frac{q^4}{+1+} \frac{q^3+q^6}{1} \frac{q^8}{+1+} \cdots, \tag{1.3}$$

$$S_3(q) := 1 + \frac{q+q^2}{1} \, + \frac{q^2+q^4}{1} \, + \frac{q^3+q^6}{1} \, + \frac{q^4+q^8}{1} \, + \cdots. \tag{1.4}$$

Set

$$(a;x)_0 = 1,$$
$$(a;x)_n = (1-a)(1-ax)\cdots(1-ax^{n-1}), \quad n \geqslant 1,$$
$$(a;x)_\infty = \lim_{n\to\infty} (a;x)_n, \quad |x| < 1.$$

The following three formulas were stated in Ramanujan's *Notebooks* ([10], p. 290 and p. 373).

THEOREM 1. *If* $|q| < 1$, *then*

$$S_1(q) = \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}. \tag{1.5}$$

THEOREM 2. *If* $|q| < 1$, *then*

$$S_2(q) = \frac{(q^3;q^8)_\infty (q^5;q^8)_\infty}{(q;q^8)_\infty (q^7;q^8)_\infty}. \tag{1.6}$$

THEOREM 3. *If* $|q| < 1$, *then*

$$S_3(q) = \frac{(q^3;q^6)_\infty^2}{(q;q^6)_\infty (q^5;q^6)_\infty} = \frac{(q^3;q^6)_\infty^3}{(q;q^2)_\infty}. \tag{1.7}$$

Theorems 1 and 2 were first proved in print by Selberg [13]. Other proofs have been given by Ramanathan [9], and Andrews *et al.* [6]. Theorem 3 was first proved in print by Watson [15]. Other proofs have been given by Selberg [13], and Andrews [2].

It is natural to examine the continued fractions (1.2), (1.3), and (1.4) when q is a root of unity and $|q| > 1$. These problems will be solved in Sections 4 and 5, respectively.

In Section 2, we shall give a simple and uniform proof of Theorems 1–3 by using Heine's continued fraction formula.

In Section 3, we shall study the following q-difference equations:

$$\begin{cases} X_{2n+1} = X_{2n}+q^{2n+1}X_{2n-1}, \\ X_{2n} = X_{2n-1}+(q^{2n}+q^n)X_{2n-2}, \end{cases} \tag{1.8}$$

$$\begin{cases} X_{2n+1} = X_{2n}+(q^{4n+2}+q^{2n+1})X_{2n-1}, \\ X_{2n} = X_{2n-1}+q^{4n}X_{2n-2}, \end{cases} \tag{1.9}$$

$$X_n = X_{n-1}+(q^n+q^{2n})X_{n-2}. \tag{1.10}$$

The solutions for these q-difference equations will be used in examining the continued fractions (1.2)–(1.4), when q is a root of unity and when $|q| > 1$.

2. Proof of Theorems 1–3. The basic hypergeometric function may be defined by

$${}_2\varphi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q; z\right] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n z^n}{(c;q)_n (q;q)_n}.$$

The well-known Heine [7] continued fraction formula is given in the following theorem (cf. [1]).

THEOREM. *For ${}_2\varphi_1$ defined above,*

$$\frac{{}_2\varphi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q; z\right]}{{}_2\varphi_1\left[\begin{matrix} a, bq \\ cq \end{matrix}; q; z\right]} = 1+\frac{a_1}{1}\,\frac{a_2}{+\,1}\,\frac{a_3}{+\,1}\,+\cdots, \tag{2.1}$$

where

$$a_{2n} = -\frac{zq^{n-1}(1-bq^n)(a-cq^n)}{(1-cq^{2n-1})(1-cq^{2n})}, \quad n \geqslant 1,$$

(2.2)

$$a_{2n+1} = -\frac{zq^n(1-aq^n)(b-cq^n)}{(1-cq^{2n})(1-cq^{2n+1})}, \quad n \geqslant 0.$$

Proof of Theorem 1. In (2.1), we set $c = 0$, $b = -1$, and $z = -q/a$. Then let $a \to \infty$. By (2.2), $a_{2n} = q^{2n}+q^n$ and $a_{2n+1} = q^{2n+1}$. Therefore the right-hand side of (2.1) reduces to $S_1(q)$. Since

$$\lim_{a\to\infty} (a;q)_n a^{-n} = \lim_{a\to\infty} \prod_{j=0}^{n-1} (a^{-1}-q^j) = (-1)^n q^{n(n-1)/2},$$

we have

$$\lim_{a\to\infty} {}_2\varphi_1\left[\begin{matrix} a, -1 \\ 0 \end{matrix}; q; \frac{-q}{a}\right] = \sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n(n+1)/2}}{(q;q)_n}, \tag{2.3}$$

(2.4) $$\lim_{a\to\infty} {}_2\varphi_1\left[\begin{matrix} a, -q \\ 0 \end{matrix}; q; \frac{-q}{a}\right] = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+1)/2}}{(q;q)_n}.$$

But (cf. [3], Corollary 2.7) if $|q| < 1$, we have

(2.5) $$\sum_{n=0}^{\infty} \frac{(aq;q)_n q^{n(n+1)/2}}{(q;q)_n} = (aq;q^2)_\infty (-q;q)_\infty.$$

By (2.5), we have

$$\sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n(n+1)/2}}{(q;q)_n} = (-q;q^2)_\infty (-q;q)_\infty,$$

$$\sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+1)/2}}{(q;q)_n} = (-q^2;q^2)_\infty (-q;q)_\infty.$$

Thus, Theorem 1 follows.

Proof of Theorem 2. In (2.1) and (2.2), we replace q by q^2, and set $c = 0$, $a = -q$, and $z = -q/b$. Let $b \to \infty$. Then $a_{2n} = q^{4n}$ and $a_{2n+1} = q^{2n+1} + q^{4n+2}$. Thus the right-hand side of (2.1) reduces to $S_2(q)$. On the other hand, it is easy to see that

(2.6) $$\lim_{b\to\infty} {}_2\varphi_1\left[\begin{matrix} -q, b \\ 0 \end{matrix}; q^2; \frac{-q}{b}\right] = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n},$$

(2.7) $$\lim_{b\to\infty} {}_2\varphi_1\left[\begin{matrix} -q, bq^2 \\ 0 \end{matrix}; q^2; \frac{-q}{b}\right] = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}.$$

By using the Gollnitz–Gordon identity (cf. [3], p. 116)

(2.8) $$\sum_{n_1,\ldots,n_{k-1}\geqslant 0} \frac{(-q;q^2)_{N_1} q^{N_1^2+\ldots+N_{k-1}^2}}{(q^2;q^2)_{n_1}\cdots(q^2;q^2)_{n_{k-1}}} = \prod_{\substack{n=1 \\ n\not\equiv 2 \pmod 4 \\ n\not\equiv 0,\, \pm(2k-1) \pmod{4k}}}^{\infty} (1-q^n)^{-1},$$

where $N_j = n_j + \ldots + n_{k-1}$, for case $k = 2$ we find that

(2.9) $$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \prod_{\substack{n=1 \\ n\equiv 1,4,7 \pmod 8}}^{\infty} (1-q^n)^{-1}.$$

And from Slater's identity ([14], p. 155) we find that

(2.10) $$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \frac{(q;q^8)_\infty (q^8;q^8)_\infty (-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

But

$$(q;q^2)_\infty = (q;q^8)_\infty (q^3;q^8)_\infty (q^5;q^8)_\infty (q^7;q^8)_\infty,$$

$$(q;q^2)_\infty (-q;q^2)_\infty = (q^2;q^4)_\infty,$$

$$(q^2;q^2)_\infty = (q^2;q^4)_\infty (q^4;q^8)_\infty (q^8;q^8)_\infty.$$

Thus, from (2.10),

$$\text{(2.11)} \qquad \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \frac{(q;q^2)_\infty (-q;q^2)_\infty (q^8;q^8)_\infty}{(q^2;q^2)_\infty (q^3;q^8)_\infty (q^5;q^8)_\infty}$$

$$= (q^3;q^8)_\infty^{-1} (q^4;q^8)_\infty^{-1} (q^5;q^8)_\infty^{-1}.$$

Therefore, by (2.9), (2.11), and (2.1),

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}} = \frac{(q^3;q^8)_\infty (q^5;q^8)_\infty}{(q;q^8)_\infty (q^7;q^8)_\infty}.$$

Theorem 2 follows immediately.

Proof of Theorem 3. In (2.1) and (2.2), we replace q by q^2, and set $a = -q$, $b = -1$, $c = 0$, and $z = q$. Then $a_{2n} = q^{2n} + q^{4n}$ and $a_{2n+1} = q^{2n+1} + q^{4n+2}$. Thus the right-hand side of (2.1) reduces to $S_3(q)$, while the left-hand side is the quotient

$$\text{(2.12)} \qquad {}_2\varphi_1\left[\begin{matrix} -q, -1 \\ 0 \end{matrix}; q^2, q\right] \Big/ {}_2\varphi_1\left[\begin{matrix} -q, q^2 \\ 0 \end{matrix}; q^2; q\right].$$

To evaluate (2.12), we shall use Watson's theorem on the general basic hypergeometric function ${}_r\varphi_s$, which is defined by

$${}_r\varphi_s\left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q; z\right] := \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n (-1)^{n(1+s-r)} q^{(1+s-r)n(n-1)/2} z^n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n}.$$

THEOREM (Watson's q-analog of Whipple's theorem [2]). *If N is a non-negative integer, then*

$$\text{(2.13)} \quad {}_8\varphi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, b_2, c_2, q^{-N} \\ \sqrt{a}, -\sqrt{a}, aq/b_1, aq/c_1, aq/b_2, aq/c_2, aq^{1+N} \end{matrix}; q; a^2 q^{2+N}/b_1 c_1 b_2 c_2\right]$$

$$= \frac{(aq;q)_N (q/b_2 c_2;q)_N}{(aq/b_2;q)_N (aq/c_2;q)_N} {}_4\varphi_3\left[\begin{matrix} aq/b_1 c_1, b_2, c_2, q^{-N} \\ aq/b_1, aq/c_1, b_2 c_2 q^{-N}/a \end{matrix}; q; q\right].$$

In (2.13), we replace q by q^2, set $b_2 = -q$, $c_2 = -1$, let a tend to 1, and let b_1, c_1 and N tend to infinity, N, of course, passing through integral values only. Then it is not difficult to see that

$${}_4\varphi_3\left[\begin{matrix} aq^2/b_1 c_1, -q, -1, q^{-2N} \\ aq^2/b_1, aq^2/c_1, q^{-2N+1}/a \end{matrix}; q^2; q^2\right] \to {}_2\varphi_1\left[\begin{matrix} -q, -1 \\ 0 \end{matrix}; q^2; q\right],$$

because

$$\lim_{\substack{N\to\infty \\ a\to 1}} \frac{(q^{-2N};q^2)_n}{(q^{-2N+1}/a;q^2)_n} = q^{-n}.$$

On the other hand, the left-hand side tends to

$$1+2\sum_{n=1}^{\infty}(-1)^n q^{3n^2} = \sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2},$$

because as b_1, c_1, $N\to\infty$, the general term

$$\frac{(a;q^2)_n(aq^4;q^4)_n(-q;q^2)_n(-1;q^2)_n(b_1;q^2)_n b_1^{-n}(c_1;q^2)_n c_1^{-n}(q^{-2N};q^2)_n q^{2nN}a^{2n}q^{3n}}{(a;q^4)_n(-aq;q^2)_n(-aq^2;q^2)_n(q^2;q^2)_n(aq^2/b_1;q^2)_n(aq^2/c_1;q^2)_n(aq^{2N+2};q^2)_n}$$

tends to $2(-1)^n q^{3n^2}$, for $n\geqslant 1$. Thus, by (2.13),

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2} = \frac{(q^2;q^2)_\infty(q;q^2)_\infty}{(-q;q^2)_\infty(-q^2;q^2)_\infty}\,{}_2\varphi_1\left[\begin{matrix}-q,-1\\0\end{matrix};q^2;q\right]. \tag{2.14}$$

Similarly, we replace q by q^2, and set $a=q^2$, $b_2=-q$, $c_2=-q^2$ in (2.13), and let b_1, c_1 and N tend to infinity. Then

$${}_4\varphi_3\left[\begin{matrix}q^4/b_1c_1,-q,-q^2,q^{-2N}\\q^4/b_1,q^4/c_1,q^{-2N+1}\end{matrix};q^2;q^2\right]\to{}_2\varphi_1\left[\begin{matrix}-q,-q^2\\0\end{matrix};q^2;q\right],$$

while the left-hand side now reduces to

$$\sum_{n=0}^{\infty}\frac{(-1)^n(1-q^{2n+1})}{1-q}q^{3n^2+2n} = \frac{1}{1-q}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+2n},$$

because as b_1, c_1, and $N\to\infty$, the general term

$$\frac{(q^2;q^2)_n(q^6;q^4)_n(-q^2;q^2)_n(-q;q^2)_n(b_1;q^2)_n b_1^{-n}(c_1;q^2)_n c^{-n}(q^{-2N};q^2)_n q^{2nN}q^{5n}}{(q^2;q^4)_n(-q^3;q^2)_n(-q^2;q^2)_n(q^2;q^2)_n(q^4/b_1;q^2)_n(q^4/c_1;q^2)_n(q^{2N+4};q^2)_n}$$

then tends to

$$(-1)^n\frac{(1-q^{4n+2})(1+q)}{(1-q^2)(1+q^{2n+1})}q^{3(n^2-n)}q^{5n} = (-1)^n\frac{(1-q^{2n+1})}{1-q}q^{3n^2+2n}.$$

Thus from (2.13),

$$\frac{1}{1-q}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2} = \frac{(q^4;q^2)_\infty(q;q^2)_\infty}{(-q^3;q^2)_\infty(-q^2;q^2)_\infty}\,{}_2\varphi_1\left[\begin{matrix}-q,-q^2\\0\end{matrix};q^2;q\right]. \tag{2.15}$$

By using Jacobi's Triple Product (cf. [3]), we deduce that

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+2n} = (q^6;q^6)_\infty(q^3;q^6)_\infty^2,$$

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+2n} = (q^6;q^6)_\infty(q;q^6)_\infty(q^5;q^6)_\infty.$$

Thus

$$\frac{{}_2\varphi_1\left[\begin{matrix}-q,\,-1\\ 0\end{matrix};\,q^2;\,q\right]}{{}_2\varphi_1\left[\begin{matrix}-q,\,-q^2\\ 0\end{matrix};\,q^2;\,q\right]} = \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2+2n}} = \frac{(q^3;q^6)_\infty^2}{(q;q^6)_\infty\,(q^5;q^6)_\infty}.$$

This establishes Theorem 3.

3. Difference equations. For the statements and proofs of our theorems we need some simple facts about Gaussian polynomials which are defined by

$$\text{(3.1)}\qquad \begin{bmatrix}A\\B\end{bmatrix}_q := \begin{cases}\dfrac{(1-q^A)(1-q^{A-1})\dots(1-q^{A-B+1})}{(1-q^B)(1-q^{B-1})\dots(1-q)}, & \text{if } A \geqslant B \geqslant 0,\\ 0, & \text{otherwise,}\end{cases}$$

where A is a non-negative integer and B is any integer.

These polynomials satisfy the relations

$$\text{(3.2)}\qquad \begin{bmatrix}A\\B\end{bmatrix}_q = \begin{bmatrix}A-1\\B-1\end{bmatrix}_q + q^B\begin{bmatrix}A-1\\B\end{bmatrix}_q,$$

$$\text{(3.3)}\qquad \begin{bmatrix}A\\B\end{bmatrix}_q = \begin{bmatrix}A-1\\B\end{bmatrix}_q + q^{A-B}\begin{bmatrix}A-1\\B-1\end{bmatrix}_q.$$

A linear second order q-difference equation is

$$\text{(3.4)}\qquad a_n X_n + b_n X_{n-1} + c_n X_{n-2} = 0,$$

where a_n, b_n, c_n are functions of q. Let $X_n(a, b, q)$ denote the solution of (3.4) with $X_{-1}(q) = a$ and $X_0(q) = b$. Obviously $X_n(a, b, q)$ is then determined uniquely. Now let

$$\text{(3.5)}\qquad P_n = X_n(1, 1, q), \qquad Q_n = X_n(0, 1, q), \qquad n \geqslant -1.$$

Then a general solution of (3.5) can be written in the form $S(q)P_n + R(q)Q_n$ where $S(q)$ and $R(q)$ are certain functions of q.

We shall give explicit formulas for P_n and Q_n ($n \geqslant 1$) in (1.8)–(1.10).

We begin with (1.8).

THEOREM 4. *Let*

$$\text{(3.6)}\qquad f(\lambda) = \begin{cases}\lambda(3\lambda+1)/2, & \textit{if } \lambda \equiv 0 \pmod 2,\\ \lambda(3\lambda-1)/2, & \textit{if } \lambda \equiv 1 \pmod 2,\end{cases}$$

$$\text{(3.7)}\qquad h(\lambda) = \begin{cases}3\lambda^2/2 + 7\lambda/2 + 2, & \textit{if } \lambda \equiv 0 \pmod 2,\\ 3\lambda^2/2 + 5\lambda/2 + 1, & \textit{if } \lambda \equiv 1 \pmod 2,\end{cases}$$

(3.8) $$g(\lambda)=\begin{cases}3\lambda^2/2+\lambda, & \text{if } \lambda\equiv 0 \pmod 2,\\ 3\lambda^2/2+2\lambda+1/2, & \text{if } \lambda\equiv 1 \pmod 2,\end{cases}$$

(3.9) $$v(\lambda)=\begin{cases}0, & \text{if } \lambda\equiv 0 \pmod 2,\\ 1, & \text{if } \lambda\equiv 1 \pmod 2.\end{cases}$$

Then the solutions P_n, Q_n of the equations

$$\begin{cases}X_{2n+1}=X_{2n}+q^{2n+1}X_{2n-1},\\ X_{2n}=X_{2n-1}+(q^{2n}+q^n)X_{2n-2},\end{cases}$$

are given by

(3.10)
$$P_{2n-1}=\sum_{\lambda=0}^{\infty}(-1)^{[\lambda/2]}\left(q^{f(\lambda)}\begin{bmatrix}2n\\ n-2\lambda\end{bmatrix}_q-q^{h(\lambda)}\begin{bmatrix}2n\\ n-2\lambda-2\end{bmatrix}_q\right),$$
$$P_{2n}=\sum_{\lambda=0}^{\infty}(-1)^{[\lambda/2]}\left(q^{f(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda+v(\lambda)\end{bmatrix}_q-q^{h(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda-2+v(\lambda)\end{bmatrix}_q\right),$$
$$Q_{2n-1}=\sum_{\lambda=0}^{\infty}(-1)^{[(\lambda+1)/2]}q^{g(\lambda)}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_q,$$
$$Q_{2n}=\sum_{\lambda=0}^{\infty}(-1)^{[(\lambda+1)/2]}q^{g(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda-v(\lambda)\end{bmatrix}_q.$$

Proof. Let P^*_{2n-1}, P^*_{2n}, Q^*_{2n-1} and Q^*_{2n} denote the right-hand sides of (3.10) respectively. It is enough to show that $P^*_n=X_n(1,1,q)$ and $Q^*_n=X_n(0,1,q)$. By (3.10), we find that $P^*_{-1}=1$, $P^*_0=1$, $Q^*_{-1}=0$ and $Q^*_0=1$. From the definition of P^*_{2n+1},

$$P^*_{2n+1}=\sum_{\lambda=0}^{\infty}(-1)^{[\lambda/2]}\left(q^{f(\lambda)}\begin{bmatrix}2n+2\\ n+1-2\lambda\end{bmatrix}_q-q^{h(\lambda)}\begin{bmatrix}2n+2\\ n-1-2\lambda\end{bmatrix}_q\right)$$
$$=\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\left(q^{f(\lambda)}\begin{bmatrix}2n+2\\ n+1-2\lambda+v(\lambda)\end{bmatrix}_q-q^{h(\lambda)}\begin{bmatrix}2n+2\\ n-1-2\lambda+v(\lambda)\end{bmatrix}_q\right)$$
$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda-1)/2}\left(q^{f(\lambda)}\begin{bmatrix}2n+2\\ n-2\lambda+v(\lambda)\end{bmatrix}_q-q^{h(\lambda)}\begin{bmatrix}2n+2\\ n-2\lambda-2+v(\lambda)\end{bmatrix}_q\right).$$

By (3.2) and (3.3) we have

$$P^*_{2n+1}=\sum_{\substack{\lambda=0\\ \lambda\equiv 0(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\left(q^{f(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda+v(\lambda)\end{bmatrix}_q+q^{f(\lambda)+n+1-2\lambda}\begin{bmatrix}2n+1\\ n-2\lambda+1+v(\lambda)\end{bmatrix}_q\right.$$
$$\left.-q^{h(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda-2+v(\lambda)\end{bmatrix}_q-q^{h(\lambda)+n-1-2\lambda}\begin{bmatrix}2n+1\\ n-2\lambda-1+v(\lambda)\end{bmatrix}_q\right)$$
$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda-1)/2}\left(q^{f(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda+v(\lambda)\end{bmatrix}_q\right.$$

$$+q^{f(\lambda)+n+2\lambda+1}\begin{bmatrix}2n+1\\ n-2\lambda-1+v(\lambda)\end{bmatrix}_q$$

$$-q^{h(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda-2+v(\lambda)\end{bmatrix}_q-q^{h(\lambda)+n+3+2\lambda}\begin{bmatrix}2n+1\\ n-2\lambda-3+v(\lambda)\end{bmatrix}_q\Bigg)$$

$$=P^*_{2n}+\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\Bigg(q^{f(\lambda)+n+1-2\lambda}\begin{bmatrix}2n+1\\ n-2\lambda+1\end{bmatrix}_q$$

$$-q^{h(\lambda)+n-1-2\lambda}\begin{bmatrix}2n+1\\ n-2\lambda-1\end{bmatrix}_q\Bigg)$$

$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda-1)/2}\Bigg(q^{f(\lambda)+n+2\lambda+1}\begin{bmatrix}2n+1\\ n-2\lambda\end{bmatrix}_q$$

$$-q^{h(\lambda)+n+2\lambda+3}\begin{bmatrix}2n+1\\ n-2\lambda-2\end{bmatrix}_q\Bigg).$$

Using (3.3) and (3.2) again, we find that

$$P^*_{2n+1}=P^*_{2n}+\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\Bigg(q^{f(\lambda)+n+1-2\lambda}\begin{bmatrix}2n\\ n-2\lambda+1\end{bmatrix}_q$$

$$+q^{f(\lambda)+2n+1}\begin{bmatrix}2n\\ n-2\lambda\end{bmatrix}_q$$

$$-q^{h(\lambda)+n-1-2\lambda}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_q-q^{h(\lambda)+2n+1}\begin{bmatrix}2n\\ n-2\lambda-2\end{bmatrix}_q\Bigg)$$

$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda-1)/2}\Bigg(q^{f(\lambda)+n+2\lambda+1}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_q$$

$$+q^{f(\lambda)+2n+1}\begin{bmatrix}2n\\ n-2\lambda\end{bmatrix}_q$$

$$-q^{h(\lambda)+n+2\lambda+3}\begin{bmatrix}2n\\ n-2\lambda-3\end{bmatrix}_q-q^{h(\lambda)+2n+1}\begin{bmatrix}2n\\ n-2\lambda-2\end{bmatrix}_q\Bigg)$$

$$=P^*_{2n}+q^{2n+1}P^*_{2n-1}+R_{2n+1},$$

where

$$(3.11)\qquad R_{2n+1}=\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\Bigg(q^{f(\lambda)+n+1-2\lambda}\begin{bmatrix}2n\\ n-2\lambda+1\end{bmatrix}_q$$

$$-q^{h(\lambda)+n-1-2\lambda}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_q\Bigg)$$

$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda-1)/2}\left(q^{f(\lambda)+n+2\lambda+1}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_q-q^{h(\lambda)+n+2\lambda+3}\begin{bmatrix}2n\\ n-2\lambda-3\end{bmatrix}_q\right).$$

In the first summation of (3.11), the first term ($\lambda=0$) is

$$q^{f(0)+n+1}\begin{bmatrix}2n\\ n+1\end{bmatrix}_q-q^{h(0)+n-1}\begin{bmatrix}2n\\ n-1\end{bmatrix}_q=q^{n+1}\begin{bmatrix}2n\\ n+1\end{bmatrix}_q-q^{n+1}\begin{bmatrix}2n\\ n+1\end{bmatrix}_q=0,$$

because

$$\begin{bmatrix}m\\ k\end{bmatrix}_q=\begin{bmatrix}m\\ m-k\end{bmatrix}_q. \tag{3.12}$$

Replacing λ by $\lambda-1$ in the second summation of (3.11), we find that the second summation of (3.11) reduces to

$$\sum_{\substack{\lambda=2\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2-1}\left(q^{f(\lambda-1)+n+2\lambda-1}\begin{bmatrix}2n\\ n-2\lambda+1\end{bmatrix}_q-q^{h(\lambda-1)+n+2\lambda+1}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_q\right).$$

For λ even, by (3.7), we find that $f(\lambda-1)+n+2\lambda-1=f(\lambda)+n+1-2\lambda$, and by (3.7), $h(\lambda-1)+n+2\lambda+1=h(\lambda)+n-2\lambda-1$. Then $R_{2n+1}=0$. Consequently,

$$P^*_{2n+1}=P^*_{2n}+q^{2n+1}P^*_{2n-1}.$$

Similarly,

$$P^*_{2n}=\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\left(q^{f(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda\end{bmatrix}_q-q^{h(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda-2\end{bmatrix}_q\right)$$
$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda-1)/2}\left(q^{f(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda+1\end{bmatrix}_q-q^{h(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda-1\end{bmatrix}_q\right)$$
$$=\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\left(q^{f(\lambda)}\begin{bmatrix}2n\\ n-2\lambda\end{bmatrix}_q+q^{f(\lambda)+n+1+2\lambda}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_q\right.$$
$$\left.-q^{h(\lambda)}\begin{bmatrix}2n\\ n-2\lambda-2\end{bmatrix}_q-q^{h(\lambda)+n+3+2\lambda}\begin{bmatrix}2n\\ n-2\lambda-3\end{bmatrix}_q\right)$$
$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda-1)/2}\left(q^{f(\lambda)}\begin{bmatrix}2n\\ n-2\lambda\end{bmatrix}_q+q^{f(\lambda)+n-2\lambda+1}\begin{bmatrix}2n\\ n-2\lambda+1\end{bmatrix}_q\right.$$
$$\left.-q^{h(\lambda)}\begin{bmatrix}2n\\ n-2\lambda-2\end{bmatrix}_q-q^{h(\lambda)+n-2\lambda-1}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_q\right)$$
$$=P^*_{2n-1}+\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\left(q^{f(\lambda)+n+1+2\lambda}\begin{bmatrix}2n-1\\ n-2\lambda-2\end{bmatrix}_q+q^{h(\lambda)+2n}\begin{bmatrix}2n-1\\ n-2\lambda-1\end{bmatrix}_q\right.$$

$$-q^{h(\lambda)+n+3+2\lambda}\begin{bmatrix}2n-1\\n-2\lambda-4\end{bmatrix}_q - q^{h(\lambda)+2n}\begin{bmatrix}2n-1\\n-2\lambda-3\end{bmatrix}_q\Bigg)$$

$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1 \pmod 2}}^{\infty}(-1)^{(\lambda-1)/2}\left(q^{f(\lambda)+n-2\lambda+1}\begin{bmatrix}2n-1\\n-2\lambda+1\end{bmatrix}_q + q^{f(\lambda)+2n}\begin{bmatrix}2n-1\\n-2\lambda\end{bmatrix}_q\right.$$

$$\left.-q^{h(\lambda)+n-2\lambda-1}\begin{bmatrix}2n-1\\n-2\lambda-1\end{bmatrix}_q - q^{h(\lambda)+2n}\begin{bmatrix}2n-1\\n-2\lambda-2\end{bmatrix}_q\right)$$

$$= P^*_{2n-1}+q^{2n}P^*_{2n-2}+q^n S_{2n},$$

where

$$S_{2n} = \sum_{\substack{\lambda=0\\ \lambda\equiv 0 \pmod 2}}^{\infty}(-1)^{\lambda/2}\left(q^{f(\lambda)+2\lambda+1}\begin{bmatrix}2n-1\\n-2\lambda-2\end{bmatrix}_q - q^{h(\lambda)+2\lambda+3}\begin{bmatrix}2n-1\\n-2\lambda-4\end{bmatrix}_q\right)$$
$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1 \pmod 2}}^{\infty}(-1)^{(\lambda-1)/2}\left(q^{f(\lambda)-2\lambda+1}\begin{bmatrix}2n-1\\n-2\lambda+1\end{bmatrix}_q - q^{h(\lambda)-2\lambda-1}\begin{bmatrix}2n-1\\n-2\lambda-1\end{bmatrix}_q\right). \tag{3.13}$$

Replacing λ by $\lambda-1$ in the first summation, and λ by $\lambda+1$ in the second summation of (3.13), we find that $S_{2n} = P^*_{2n-2}$, because

$$f(\lambda) = \begin{cases} f(\lambda-1)+2\lambda-1, & \text{if } \lambda\equiv 1 \pmod 2,\\ f(\lambda+1)-2\lambda-1, & \text{if } \lambda\equiv 0 \pmod 2,\end{cases}$$

$$h(\lambda) = \begin{cases} h(\lambda-1)+2\lambda+1, & \text{if } \lambda\equiv 1 \pmod 2,\\ h(\lambda+1)-2\lambda-3, & \text{if } \lambda\equiv 0 \pmod 2.\end{cases}$$

Therefore

$$P^*_{2n} = P^*_{2n-1}+(q^{2n}+q^n)P^*_{2n-2}.$$

Next,

$$Q^*_{2n+1} = \sum_{\lambda=0}^{\infty}(-1)^{[(\lambda+1)/2]}q^{g(\lambda)}\begin{bmatrix}2n+2\\n-2\lambda\end{bmatrix}_q$$

$$= \sum_{\substack{\lambda=0\\ \lambda\equiv 0 \pmod 2}}^{\infty}(-1)^{\lambda/2}\left(q^{g(\lambda)}\begin{bmatrix}2n+1\\n-2\lambda\end{bmatrix}_q + q^{g(\lambda)+n+2\lambda+2}\begin{bmatrix}2n+1\\n-2\lambda-1\end{bmatrix}_q\right)$$

$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1 \pmod 2}}^{\infty}(-1)^{(\lambda+1)/2}\left(q^{g(\lambda)}\begin{bmatrix}2n+1\\n-2\lambda-1\end{bmatrix}_q + q^{g(\lambda)+n-2\lambda}\begin{bmatrix}2n+1\\n-2\lambda\end{bmatrix}_q\right)$$

$$= Q^*_{2n}+\sum_{\substack{\lambda=0\\ \lambda\equiv 0 \pmod 2}}^{\infty}(-1)^{\lambda/2}q^{g(\lambda)+n+2\lambda+2}\begin{bmatrix}2n\\n-2\lambda-1\end{bmatrix}_q$$

$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda+1)/2}\,q^{g(\lambda)+n-2\lambda}\begin{bmatrix}2n+1\\ n-2\lambda\end{bmatrix}_q$$

$$=Q^*_{2n}+\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\left(q^{g(\lambda)+n+2\lambda+2}\begin{bmatrix}2n\\ n-2\lambda-2\end{bmatrix}_q\right.$$

$$\left.+q^{g(\lambda)+2n+1}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_q\right)$$

$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda+1)/2}\left(q^{g(\lambda)+n-2\lambda}\begin{bmatrix}2n\\ n-2\lambda\end{bmatrix}_q+q^{g(\lambda)+2n+1}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_q\right)$$

$$=Q^*_{2n}+q^{2n+1}\,Q^*_{2n-1}+T_{2n+1},$$

where

$$T_{2n+1}=\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\,q^{g(\lambda)+n+2\lambda+2}\begin{bmatrix}2n\\ n-2\lambda-2\end{bmatrix}_q+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda+1)/2}\,q^{g(\lambda)+n-2\lambda}\begin{bmatrix}2n\\ n-2\lambda\end{bmatrix}_q. \tag{3.14}$$

Replacing λ by $\lambda-1$ in the first summation of (3.14), we easily see that $T_{2n+1}=0$, because $g(\lambda-1)+n+2(\lambda-1)+2=g(\lambda)+n-2\lambda$, if $\lambda\equiv 1 \pmod 2$. Thus

$$Q^*_{2n+1}=Q^*_{2n}+q^{2n+1}\,Q^*_{2n-1}.$$

At last, we have

$$Q^*_{2n}=\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\,q^{g(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda\end{bmatrix}_q+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda+1)/2}\,q^{g(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda-1\end{bmatrix}_q$$

$$=\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\left(q^{g(\lambda)}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_q+q^{g(\lambda)+n-2\lambda}\begin{bmatrix}2n\\ n-2\lambda\end{bmatrix}_q\right)$$

$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda+1)/2}\left(q^{g(\lambda)}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_q+q^{g(\lambda)+n+2\lambda+2}\begin{bmatrix}2n\\ n-2\lambda-2\end{bmatrix}_q\right)$$

$$=Q^*_{2n}+\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{\lambda/2}\left(q^{g(\lambda)+n-2\lambda}\begin{bmatrix}2n-1\\ n-2\lambda\end{bmatrix}_q+q^{g(\lambda)+2n}\begin{bmatrix}2n-1\\ n-2\lambda-1\end{bmatrix}_q\right)$$

$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}(-1)^{(\lambda+1)/2}\left(q^{g(\lambda)+n+2\lambda+2}\begin{bmatrix}2n-1\\ n-2\lambda-3\end{bmatrix}_q+q^{g(\lambda)+2n}\begin{bmatrix}2n-1\\ n-2\lambda-2\end{bmatrix}_q\right)$$

$$=Q^*_{2n-1}+q^{2n}\,Q^*_{2n-2}+q^n\,V_{2n},$$

where

$$(3.15)\quad V_{2n} = \sum_{\substack{\lambda=2 \\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty} (-1)^{\lambda/2}\, q^{g(\lambda)-2\lambda} \begin{bmatrix} 2n-1 \\ n-2\lambda \end{bmatrix}_q + \sum_{\substack{\lambda=1 \\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty} (-1)^{(\lambda+1)/2}\, q^{g(\lambda)+2\lambda+2} \begin{bmatrix} 2n-1 \\ n-2\lambda-3 \end{bmatrix}_q + \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_q .$$

Replacing λ by $\lambda+1$ in the first summation and λ by $\lambda-1$ in the second summation of (3.15), we find that

$$V_{2n} = \sum_{\substack{\lambda=1 \\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty} (-1)^{(\lambda+1)/2}\, q^{g(\lambda+1)-2(\lambda+1)} \begin{bmatrix} 2n-1 \\ n-2\lambda-2 \end{bmatrix}_q + \sum_{\substack{\lambda=2 \\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty} (-1)^{\lambda/2}\, q^{g(\lambda-1)+2\lambda} \begin{bmatrix} 2n-1 \\ n-2\lambda-1 \end{bmatrix}_q + \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}_q = Q^*_{2n-2}$$

because $g(\lambda+1)-2(\lambda+1) = g(\lambda)$, if $\lambda \equiv 1 \pmod 2$, and $g(\lambda-1) = g(\lambda)$, if $\lambda \equiv 0 \pmod 2$. Thus

$$Q^*_{2n} = Q^*_{2n-1} + (q^{2n}+q^n)\, Q^*_{2n-2}.$$

The theorem has been proved completely.

THEOREM 5. *Let*

$$(3.16)\qquad f(\lambda) = \begin{cases} 3\lambda^2/4+\lambda/4, & \textit{if } \lambda \equiv 0,\,1 \pmod 4, \\ 3\lambda^2/4+5\lambda/4+1/2, & \textit{if } \lambda \equiv 2,\,3 \pmod 4, \end{cases}$$

$$(3.17)\qquad g(\lambda) = \begin{cases} 3\lambda^2/4+5\lambda/4, & \textit{if } \lambda \equiv 0,\,1 \pmod 4, \\ 3\lambda^2/4+\lambda/4-1/2, & \textit{if } \lambda \equiv 2,\,3 \pmod 4, \end{cases}$$

$$(3.18)\qquad u(\lambda) = \begin{cases} 0, & \textit{if } \lambda \equiv 0,\,1 \pmod 4, \\ 1, & \textit{if } \lambda \equiv 2,\,3 \pmod 4, \end{cases}$$

$$(3.19)\qquad \delta(\lambda) = (-1)^{[(\lambda+2)/4]} = \begin{cases} 1, & \textit{if } \lambda \equiv 0,1,6,7 \pmod 8, \\ -1, & \textit{if } \lambda \equiv 2,3,4,5 \pmod 4, \end{cases}$$

$$(3.20)\qquad \varepsilon(\lambda) = (-1)^{[(3\lambda+3)/4]} = \begin{cases} 1, & \textit{if } \lambda \equiv 0,2,5,7 \pmod 8, \\ -1, & \textit{if } \lambda \equiv 1,3,4,6 \pmod 8. \end{cases}$$

Then the solutions P_n, Q_n of the equations

$$\begin{cases} X_{2n+1} = X_{2n} + (q^{4n+2}+q^{2n+1})\, X_{2n-1}, \\ X_{2n} = X_{2n-1} + q^{4n}\, X_{2n-2}, \end{cases}$$

are given by

$$P_{2n-1} = \sum_{\lambda=0}^{\infty} \delta(\lambda)\, q^{f(\lambda)} \begin{bmatrix} 2n \\ n-\lambda-u(\lambda) \end{bmatrix}_{q^2},$$

$$P_{2n} = \sum_{\lambda=0}^{\infty} \delta(\lambda)\, q^{f(\lambda)} \begin{bmatrix} 2n+1 \\ n-\lambda \end{bmatrix}_{q^2},$$

(3.21)

$$Q_{2n-1} = \sum_{\lambda=0}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)} \begin{bmatrix} 2n \\ n-1-\lambda+u(\lambda) \end{bmatrix}_{q^2},$$

$$Q_{2n} = \sum_{\lambda=0}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)} \begin{bmatrix} 2n+1 \\ n-\lambda \end{bmatrix}_{q^2}.$$

Proof. We use the same notation and remarks as in the beginning of the proof of Theorem 4. Then

$$\begin{aligned} P_{2n+1}^* &= \sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda)\, q^{f(\lambda)} \begin{bmatrix} 2n+2 \\ n+1-\lambda \end{bmatrix}_{q^2} + \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda)\, q^{f(\lambda)} \begin{bmatrix} 2n+2 \\ n-\lambda \end{bmatrix}_{q^2} \\ &= \sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda) \left(q^{f(\lambda)} \begin{bmatrix} 2n+1 \\ n-\lambda \end{bmatrix}_{q^2} + q^{f(\lambda)+2n+2-2\lambda} \begin{bmatrix} 2n+1 \\ n+1-\lambda \end{bmatrix}_{q^2} \right) \\ &\quad + \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda) \left(q^{f(\lambda)} \begin{bmatrix} 2n+1 \\ n-\lambda \end{bmatrix}_{q^2} + q^{f(\lambda)+2n+4+2\lambda} \begin{bmatrix} 2n+1 \\ n-1-\lambda \end{bmatrix}_{q^2} \right) \\ &= P_{2n}^* + \sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda) \left(q^{f(\lambda)+2n+2-2\lambda} \begin{bmatrix} 2n \\ n+1-\lambda \end{bmatrix}_{q^2} + q^{f(\lambda)+4n+2} \begin{bmatrix} 2n \\ n-\lambda \end{bmatrix}_{q^2} \right) \\ &\quad + \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda) \left(q^{f(\lambda)+2n+4+2\lambda} \begin{bmatrix} 2n \\ n-\lambda-2 \end{bmatrix}_{q^2} + q^{f(\lambda)+4n+2} \begin{bmatrix} 2n \\ n-\lambda-1 \end{bmatrix}_{q^2} \right) \\ &= P_{2n}^* + q^{4n+2} P_{2n-1}^* + q^{2n+1} R_{2n+1}, \end{aligned}$$

where

(3.22)
$$\begin{aligned} R_{2n+1} = {} & \sum_{\substack{\lambda=4 \\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda)\, q^{f(\lambda)+1-2\lambda} \begin{bmatrix} 2n \\ n+1-\lambda \end{bmatrix}_{q^2} \\ & + \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda)\, q^{f(\lambda)+3+2\lambda} \begin{bmatrix} 2n \\ n-\lambda-2 \end{bmatrix}_{q^2} \\ & + q \begin{bmatrix} 2n \\ n+1 \end{bmatrix}_{q^2} + \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}. \end{aligned}$$

Replacing λ by $\lambda+2$ in the first summation and λ by $\lambda-2$ in the second summation of (3.22), we have

$$R_{2n+1} = \sum_{\substack{\lambda=2\\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda+2)\, q^{f(\lambda+2)+1-2(\lambda+2)} \begin{bmatrix} 2n \\ n-1-\lambda \end{bmatrix}_{q^2}$$

$$+ \sum_{\substack{\lambda=4\\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda-2)\, q^{f(\lambda-2)+3+2(\lambda-2)} \begin{bmatrix} 2n \\ n-\lambda \end{bmatrix}_{q^2} + \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2} + q \begin{bmatrix} 2n \\ n-1 \end{bmatrix}_{q^2}.$$

By (3.19) and (3.16), it is easy to verify that

$$\delta(\lambda+2) = \delta(\lambda) \quad \text{and} \quad f(\lambda+2)+1-2(\lambda+2) = f(\lambda), \quad \text{if } \lambda \equiv 2, 3\,(\mathrm{mod}\,4),$$
$$\delta(\lambda-2) = \delta(\lambda) \quad \text{and} \quad f(\lambda-2)+2(\lambda-2)+3 = f(\lambda), \quad \text{if } \lambda \equiv 0, 1\,(\mathrm{mod}\,4).$$

Therefore we deduce that $R_{2n+1} = P^*_{2n-1}$ and

$$P^*_{2n+1} = P^*_{2n} + (q^{4n+2} + q^{2n+1})\, P^*_{2n-1}.$$

Similarly,

$$P^*_{2n} = \sum_{\substack{\lambda=0\\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda)\left(q^{f(\lambda)} \begin{bmatrix} 2n \\ n-\lambda \end{bmatrix}_{q^2} + q^{f(\lambda)+2n+2+2\lambda} \begin{bmatrix} 2n \\ n-\lambda-1 \end{bmatrix}_{q^2} \right)$$

$$+ \sum_{\substack{\lambda=2\\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda)\left(q^{f(\lambda)} \begin{bmatrix} 2n \\ n-\lambda-1 \end{bmatrix}_{q^2} + q^{f(\lambda)+2n-2\lambda} \begin{bmatrix} 2n \\ n-\lambda \end{bmatrix}_{q^2} \right)$$

$$= P^*_{2n-1} + \sum_{\substack{\lambda=0\\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda)\left(q^{f(\lambda)+2n+2+2\lambda} \begin{bmatrix} 2n-1 \\ n-\lambda-2 \end{bmatrix}_{q^2} + q^{f(\lambda)+4n} \begin{bmatrix} 2n-1 \\ n-\lambda-1 \end{bmatrix}_{q^2} \right)$$

$$+ \sum_{\substack{\lambda=2\\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda)\left(q^{f(\lambda)+2n-2\lambda} \begin{bmatrix} 2n-\lambda \\ n-\lambda \end{bmatrix}_{q^2} + q^{f(\lambda)+4n} \begin{bmatrix} 2n-1 \\ n-\lambda-1 \end{bmatrix}_{q^2} \right)$$

$$= P^*_{2n-1} + q^{4n} P^*_{2n-2} + q^{2n} S_{2n},$$

where

(3.23)
$$S_{2n} = \sum_{\substack{\lambda=0\\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda)\, q^{f(\lambda)+2+2\lambda} \begin{bmatrix} 2n-1 \\ n-\lambda-2 \end{bmatrix}_{q^2}$$

$$+ \sum_{\substack{\lambda=2\\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty} \delta(\lambda)\, q^{f(\lambda)-2\lambda} \begin{bmatrix} 2n-1 \\ n-\lambda \end{bmatrix}_{q^2}.$$

Replacing λ by $\lambda-2$ in the first summation of (3.23), we find that $S_{2n} = 0$, because by (3.19) and (3.16), if $\lambda \equiv 2, 3\ (\mathrm{mod}\,4)$, then $\delta(\lambda-2) = -\delta(\lambda)$, and $f(\lambda-2)-2+2\lambda = f(\lambda)-2\lambda$. Therefore,

$$P^*_{2n} = P^*_{2n-1} + q^{4n} P^*_{2n-2}.$$

Next,

$$
\begin{aligned}
Q_{2n+1}^* &= \sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1 \pmod 4}}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)} \begin{bmatrix} 2n+2 \\ n-\lambda \end{bmatrix}_{q^2} + \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3 \pmod 4}}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)} \begin{bmatrix} 2n+2 \\ n-\lambda+1 \end{bmatrix}_{q^2} \\
&= \sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1 \pmod 4}}^{\infty} \varepsilon(\lambda) \left(q^{g(\lambda)} \begin{bmatrix} 2n+1 \\ n-\lambda \end{bmatrix}_{q^2} + q^{g(\lambda)+2n+2\lambda+4} \begin{bmatrix} 2n+1 \\ n-\lambda-1 \end{bmatrix}_{q^2} \right) \\
&\quad + \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3 \pmod 4}}^{\infty} \varepsilon(\lambda) \left(q^{g(\lambda)} \begin{bmatrix} 2n+1 \\ n-\lambda \end{bmatrix}_{q^2} + q^{g(\lambda)+2n+2-2\lambda} \begin{bmatrix} 2n+1 \\ n-\lambda+1 \end{bmatrix}_{q^2} \right) \\
&= Q_{2n}^* + \sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1 \pmod 4}}^{\infty} \varepsilon(\lambda) \left(q^{g(\lambda)+2n+2\lambda+4} \begin{bmatrix} 2n \\ n-\lambda-2 \end{bmatrix}_{q^2} \right. \\
&\quad \left. + q^{g(\lambda)+4n+2} \begin{bmatrix} 2n \\ n-\lambda-1 \end{bmatrix}_{q^2} \right) \\
&\quad + \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3 \pmod 4}}^{\infty} \varepsilon(\lambda) \left(q^{g(\lambda)+2n+2-2\lambda} \begin{bmatrix} 2n \\ n-\lambda+1 \end{bmatrix}_{q^2} + q^{g(\lambda)+4n+2} \begin{bmatrix} 2n \\ n-\lambda \end{bmatrix}_{q^2} \right) \\
&= Q_{2n}^* + q^{4n+2} Q_{2n-1}^* + q^{2n+1} T_{2n+1},
\end{aligned}
$$

where

$$
\begin{aligned}
(3.24) \qquad T_{2n+1} = &\sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1 \pmod 4}}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)+2\lambda+3} \begin{bmatrix} 2n \\ n-\lambda-2 \end{bmatrix}_{q^2} \\
&+ \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3 \pmod 4}}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)+1-2\lambda} \begin{bmatrix} 2n \\ n-\lambda+1 \end{bmatrix}_{q^2}.
\end{aligned}
$$

Replacing λ by $\lambda-2$ in the first summation and λ by $\lambda+2$ in the second summation of (3.24), we see that $T_{2n+1} = Q_{2n-1}^*$, because by (3.20) and (3.17), it is easy to see that

$$
\begin{aligned}
\varepsilon(\lambda-2) = \varepsilon(\lambda) \quad &\text{and} \quad g(\lambda-2)+2\lambda-1 = g(\lambda), \quad \text{if } \lambda \equiv 2,3 \pmod 4, \\
\varepsilon(\lambda+2) = \varepsilon(\lambda) \quad &\text{and} \quad g(\lambda+2)-2\lambda-3 = g(\lambda), \quad \text{if } \lambda \equiv 0,1 \pmod 4.
\end{aligned}
$$

Therefore

$$
Q_{2n+1}^* = Q_{2n}^* + (q^{4n+2} + q^{2n+1})\, Q_{2n-1}^*.
$$

Finally,

$$
\begin{aligned}
Q_{2n}^* = &\sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1 \pmod 4}}^{\infty} \varepsilon(\lambda) \left(q^{g(\lambda)} \begin{bmatrix} 2n \\ n-\lambda-1 \end{bmatrix}_{q^2} + q^{g(\lambda)+2n-2\lambda} \begin{bmatrix} 2n \\ n-\lambda \end{bmatrix}_{q^2} \right) \\
&+ \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3 \pmod 4}}^{\infty} \varepsilon(\lambda) \left(q^{g(\lambda)} \begin{bmatrix} 2n \\ n-\lambda \end{bmatrix}_{q^2} + q^{g(\lambda)+2n+2+2\lambda} \begin{bmatrix} 2n \\ n-\lambda-1 \end{bmatrix}_{q^2} \right)
\end{aligned}
$$

$$= Q_{2n-1}^{*} + \sum_{\substack{\lambda=0 \\ \lambda \equiv 0,1 \pmod 4}}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)+2n-2\lambda} \begin{bmatrix} 2n \\ n-\lambda \end{bmatrix}_{q^2}$$

$$+ \sum_{\substack{\lambda=2 \\ \lambda \equiv 2,3 \pmod 4}}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)+2n+2\lambda+2} \begin{bmatrix} 2n \\ n-\lambda-1 \end{bmatrix}_{q^2}$$

$$= Q_{2n-1}^{*} + \sum_{\substack{\lambda=0 \\ \lambda \equiv 0,1 \pmod 4}}^{\infty} \varepsilon(\lambda) \left(q^{g(\lambda)+2n-2\lambda} \begin{bmatrix} 2n-1 \\ n-\lambda \end{bmatrix}_{q^2} + q^{g(\lambda)+4n} \begin{bmatrix} 2n-1 \\ n-1-\lambda \end{bmatrix}_{q^2} \right)$$

$$+ \sum_{\substack{\lambda=2 \\ \lambda \equiv 2,3 \pmod 4}}^{\infty} \varepsilon(\lambda) \left(q^{g(\lambda)+2n+2\lambda+2} \begin{bmatrix} 2n-1 \\ n-\lambda-2 \end{bmatrix}_{q^2} + q^{g(\lambda)+4n} \begin{bmatrix} 2n-1 \\ n-\lambda-1 \end{bmatrix}_{q^2} \right)$$

$$= Q_{2n-1}^{*} + q^{4n} Q_{2n-2}^{*} + q^{2n} V_{2n},$$

where

$$V_{2n} = \sum_{\substack{\lambda=0 \\ \lambda \equiv 0,1 \pmod 4}}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)-2\lambda} \begin{bmatrix} 2n-1 \\ n-\lambda \end{bmatrix}_{q^2} + \sum_{\substack{\lambda=2 \\ \lambda \equiv 2,3 \pmod 4}}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)+2\lambda+2} \begin{bmatrix} 2n-1 \\ n-\lambda-2 \end{bmatrix}_{q^2} \tag{3.25}$$

$$= \sum_{\substack{\lambda=4 \\ \lambda \equiv 0,1 \pmod 4}}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)-2\lambda} \begin{bmatrix} 2n-1 \\ n-\lambda \end{bmatrix}_{q^2} + \sum_{\substack{\lambda=2 \\ \lambda \equiv 2,3 \pmod 4}}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)+2\lambda+2} \begin{bmatrix} 2n-1 \\ n-\lambda-2 \end{bmatrix}_{q^2},$$

because, by (3.17) and (3.20),

$$\varepsilon(0)\, q^{g(0)} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2} + \varepsilon(1)\, q^{g(1)-2} \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}_{q^2} = 0.$$

Replacing λ by $\lambda+2$ in the first summation of (3.25), we find easily that $V_{2n} = 0$, because, if $\lambda \equiv 2, 3 \pmod 4$, it is easy to see that $\varepsilon(\lambda+2)+\varepsilon(\lambda) = 0$ and $g(\lambda+2)-2(\lambda+2) = g(\lambda)+2\lambda+2$. Consequently,

$$Q_{2n}^{*} = Q_{2n-1}^{*} + q^{4n} Q_{2n-2}^{*}.$$

Thus, we have proved this theorem.

THEOREM 6. *Let*

$$f(\lambda) = \begin{cases} 3\lambda^2/8+\lambda/2+1/8, & \text{if } \lambda \equiv 1 \pmod 2, \\ 3\lambda^2/8+\lambda/4, & \text{if } \lambda \equiv 0 \pmod 2; \end{cases} \tag{3.26}$$

$$g(\lambda) = 3\lambda(\lambda+1)/2, \tag{3.27}$$

$$a(\lambda) = \lambda-[\lambda/4], \tag{3.28}$$

$$b(\lambda) = \lambda - [(\lambda+2)/4], \tag{3.29}$$

$$c(\lambda) = \lambda + [\lambda/2] + 1, \tag{3.30}$$

$$d(\lambda) = \lambda + [(\lambda+1)/2]. \tag{3.31}$$

Then the solutions P_n, Q_n of the equation

$$X_n = X_{n-1} + (q^n + q^{2n}) X_{n-2}$$

are given by

$$\begin{aligned} P_{2n-1} &= \sum_{\lambda=0}^{\infty} q^{f(\lambda)} \begin{bmatrix} 2n \\ n-a(\lambda) \end{bmatrix}_{q^2}, \\ P_{2n} &= \sum_{\lambda=0}^{\infty} q^{f(\lambda)} \begin{bmatrix} 2n+1 \\ n-b(\lambda) \end{bmatrix}_{q^2}, \\ Q_{2n-1} &= \sum_{\lambda=0}^{\infty} q^{g(\lambda)} \begin{bmatrix} 2n \\ n-c(\lambda) \end{bmatrix}_{q^2}, \\ Q_{2n} &= \sum_{\lambda=0}^{\infty} q^{g(\lambda)} \begin{bmatrix} 2n+1 \\ n-d(\lambda) \end{bmatrix}_{q^2}. \end{aligned} \tag{3.32}$$

Proof. From (3.28)–(3.31), it is easy to see that

$$a(\lambda) = \begin{cases} b(\lambda), & \text{if } \lambda \equiv 0,1 \pmod 4, \\ b(\lambda)+1, & \text{if } \lambda \equiv 2,3 \pmod 4, \end{cases} \tag{3.33}$$

$$c(\lambda) = \begin{cases} d(\lambda)+1, & \text{if } \lambda \equiv 0 \pmod 2, \\ d(\lambda), & \text{if } \lambda \equiv 1 \pmod 2. \end{cases} \tag{3.34}$$

We begin with P_{2n+1}. By using the same notation as in the proof of Theorem 4, from (3.33), we have

$$\begin{aligned} P^*_{2n+1} &= \sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1 \pmod 4}}^{\infty} q^{f(\lambda)} \begin{bmatrix} 2n+2 \\ n+1-b(\lambda) \end{bmatrix}_{q^2} + \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3 \pmod 4}}^{\infty} q^{f(\lambda)} \begin{bmatrix} 2n+2 \\ n-b(\lambda) \end{bmatrix}_{q^2} \\ &= \sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1 \pmod 4}}^{\infty} q^{f(\lambda)} \left(\begin{bmatrix} 2n+1 \\ n-b(\lambda) \end{bmatrix}_{q^2} + q^{2n+2-2b(\lambda)} \begin{bmatrix} 2n+1 \\ n+1-b(\lambda) \end{bmatrix}_{q^2} \right) \\ &\quad + \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3 \pmod 4}}^{\infty} q^{g(\lambda)} \left(\begin{bmatrix} 2n+1 \\ n-b(\lambda) \end{bmatrix}_{q^2} + q^{2n+4+2b(\lambda)} \begin{bmatrix} 2n+1 \\ n-1-b(\lambda) \end{bmatrix}_{q^2} \right) \\ &= P^*_{2n} + \sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1 \pmod 4}}^{\infty} \left(q^{f(\lambda)+2n+2-2a(\lambda)} \begin{bmatrix} 2n \\ n+1-a(\lambda) \end{bmatrix}_{q^2} \right. \\ &\quad \left. + q^{f(\lambda)+4n+2} \begin{bmatrix} 2n \\ n-a(\lambda) \end{bmatrix}_{q^2} \right) \end{aligned}$$

$$+\sum_{\substack{\lambda=2\\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty}\left(q^{f(\lambda)+2n+2+2a(\lambda)}\begin{bmatrix}2n\\ n-1-a(\lambda)\end{bmatrix}_{q^2}+q^{f(\lambda)+4n+2}\begin{bmatrix}2n\\ n-a(\lambda)\end{bmatrix}_{q^2}\right)$$

$$=P^*_{2n}+q^{4n+2}P^*_{2n-1}+q^{2n+1}R_{2n+1},$$

where

$$R_{2n+1}=\sum_{\substack{\lambda=0\\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty}q^{f(\lambda)+1-2a(\lambda)}\begin{bmatrix}2n\\ n+1-a(\lambda)\end{bmatrix}_{q^2} \tag{3.35}$$

$$+\sum_{\substack{\lambda=2\\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty}q^{f(\lambda)+1+2a(\lambda)}\begin{bmatrix}2n\\ n-1-a(\lambda)\end{bmatrix}_{q^2}$$

$$=q\begin{bmatrix}2n\\ n+1\end{bmatrix}_{q^2}+\begin{bmatrix}2n\\ n\end{bmatrix}_{q^2}+\sum_{\substack{\lambda=4\\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty}q^{f(\lambda)+1-2a(\lambda)}\begin{bmatrix}2n\\ n+1-a(\lambda)\end{bmatrix}_{q^2}$$

$$+\sum_{\substack{\lambda=2\\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty}q^{f(\lambda)+1+2a(\lambda)}\begin{bmatrix}2n\\ n-1-a(\lambda)\end{bmatrix}_{q^2}.$$

Replacing λ by $\lambda+2$ in the first summation, and λ by $\lambda-2$ in the second summation of (3.35), we find that

$$R_{2n+1}=\begin{bmatrix}2n\\ n\end{bmatrix}_{q^2}+q\begin{bmatrix}2n\\ n-1\end{bmatrix}_{q^2}+\sum_{\substack{\lambda=2\\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty}q^{f(\lambda)}\begin{bmatrix}2n\\ n-a(\lambda)\end{bmatrix}_{q^2}$$

$$+\sum_{\substack{\lambda=4\\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty}q^{f(\lambda)}\begin{bmatrix}2n\\ n-a(\lambda)\end{bmatrix}_{q^2}=P^*_{2n-1},$$

because, by (3.26) and (3.28), we see that

$$f(\lambda+2)+1-2a(\lambda+2)=f(\lambda),\quad a(\lambda+2)-1=a(\lambda),\quad \text{if } \lambda\equiv 2,3 \pmod 4,$$

$$f(\lambda-2)+1+2a(\lambda-2)=f(\lambda),\quad a(\lambda-2)+1=a(\lambda),\quad \text{if } \lambda\equiv 0,1 \pmod 4.$$

Therefore

$$P^*_{2n+1}=P^*_{2n}+(q^{4n+2}+q^{2n+1})P^*_{2n-1}.$$

Also we have

$$P^*_{2n}=\sum_{\substack{\lambda=0\\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty}q^{f(\lambda)}\begin{bmatrix}2n+1\\ n-a(\lambda)\end{bmatrix}_{q^2}+\sum_{\substack{\lambda=2\\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty}q^{f(\lambda)}\begin{bmatrix}2n+1\\ n-a(\lambda)+1\end{bmatrix}_{q^2}$$

$$=\sum_{\substack{\lambda=0\\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty}\left(q^{f(\lambda)}\begin{bmatrix}2n\\ n-a(\lambda)\end{bmatrix}_{q^2}+q^{f(\lambda)+2n+2+2a(\lambda)}\begin{bmatrix}2n\\ n-a(\lambda)-1\end{bmatrix}_{q^2}\right)$$

$$+\sum_{\substack{\lambda=2\\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty}\left(q^{f(\lambda)}\begin{bmatrix}2n\\ n-a(\lambda)\end{bmatrix}_{q^2}+q^{f(\lambda)+2n+2-2a(\lambda)}\begin{bmatrix}2n\\ n-a(\lambda)+1\end{bmatrix}_{q^2}\right)$$

$$=P^*_{2n-1}+\sum_{\substack{\lambda=0\\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty}\left(q^{f(\lambda)+2n+2+2b(\lambda)}\begin{bmatrix}2n-1\\ n-b(\lambda)-2\end{bmatrix}_{q^2}+q^{f(\lambda)+4n}\begin{bmatrix}2n-1\\ n-1-b(\lambda)\end{bmatrix}_{q^2}\right)$$

$$+\sum_{\substack{\lambda=2\\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty}\left(q^{f(\lambda)+2n-2b(\lambda)}\begin{bmatrix}2n-1\\ n-b(\lambda)\end{bmatrix}_{q^2}+q^{f(\lambda)+4n}\begin{bmatrix}2n-1\\ n-1-b(\lambda)\end{bmatrix}_{q^2}\right)$$

$$=P^*_{2n-1}+q^{4n}P^*_{2n-2}+q^{2n}S_{2n},$$

where

$$S_{2n}=\sum_{\substack{\lambda=0\\ \lambda\equiv 0,1\,(\mathrm{mod}\,4)}}^{\infty}q^{f(\lambda)+2+2b(\lambda)}\begin{bmatrix}2n-1\\ n-b(\lambda)-2\end{bmatrix}_{q^2}+\sum_{\substack{\lambda=2\\ \lambda\equiv 2,3\,(\mathrm{mod}\,4)}}^{\infty}q^{f(\lambda)-2b(\lambda)}\begin{bmatrix}2n-1\\ n-b(\lambda)\end{bmatrix}_{q^2}. \tag{3.36}$$

Replacing λ by $\lambda-2$ in the first summation, and λ by $\lambda+2$ in the second summation of (3.36), we find that $S_{2n}=P^*_{2n-2}$, because, by (3.26) and (3.29), we have

$$f(\lambda-2)+2+2b(\lambda-2)=f(\lambda),\quad b(\lambda-2)+1=b(\lambda),\quad \text{if } \lambda\equiv 2,3\pmod 4,$$

$$f(\lambda+2)-2b(\lambda+2)=f(\lambda),\quad b(\lambda+2)-1=b(\lambda),\quad \text{if } \lambda\equiv 0,1\pmod 4.$$

Therefore

$$P^*_{2n}=P^*_{2n-1}+(q^{4n}+q^{2n})P^*_{2n-2}.$$

Similarly, we have, by (3.34),

$$Q^*_{2n+1}=\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}q^{g(\lambda)}\begin{bmatrix}2n+2\\ n-d(\lambda)\end{bmatrix}_{q^2}+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}q^{g(\lambda)}\begin{bmatrix}2n+2\\ n+1-d(\lambda)\end{bmatrix}_{q^2}$$

$$=\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}\left(q^{g(\lambda)}\begin{bmatrix}2n+1\\ n-d(\lambda)\end{bmatrix}_{q^2}+q^{g(\lambda)+2n+4+2d(\lambda)}\begin{bmatrix}2n+1\\ n-1-d(\lambda)\end{bmatrix}_{q^2}\right)$$

$$+\sum_{\substack{\lambda=1\\ \lambda\equiv 1\,(\mathrm{mod}\,2)}}^{\infty}\left(q^{g(\lambda)}\begin{bmatrix}2n+1\\ n-d(\lambda)\end{bmatrix}_{q^2}+q^{g(\lambda)+2n+2-2d(\lambda)}\begin{bmatrix}2n+1\\ n+1-d(\lambda)\end{bmatrix}_{q^2}\right)$$

$$=Q^*_{2n}+\sum_{\substack{\lambda=0\\ \lambda\equiv 0\,(\mathrm{mod}\,2)}}^{\infty}\left(q^{g(\lambda)+2n+2+2c(\lambda)}\begin{bmatrix}2n\\ n-c(\lambda)-1\end{bmatrix}_{q^2}\right.$$

$$\left.+q^{g(\lambda)+4n+2}\begin{bmatrix}2n\\ n-c(\lambda)\end{bmatrix}_{q^2}\right)$$

$$+ \sum_{\substack{\lambda=1 \\ \lambda \equiv 1 \pmod 2}}^{\infty} \left(q^{g(\lambda)+2n+2-2c(\lambda)} \begin{bmatrix} 2n \\ n+1-c(\lambda) \end{bmatrix}_q + q^{g(\lambda)+4n+2} \begin{bmatrix} 2n \\ n-c(\lambda) \end{bmatrix}_{q^2} \right)$$

$$= Q_{2n}^* + q^{4n+2} Q_{2n-1}^* + q^{2n+1} T_{2n+1},$$

where

(3.37)
$$T_{2n+1} = \sum_{\substack{\lambda=0 \\ \lambda \equiv 0 \pmod 2}}^{\infty} q^{g(\lambda)+1+2c(\lambda)} \begin{bmatrix} 2n \\ n-c(\lambda)-1 \end{bmatrix}_{q^2} + \sum_{\substack{\lambda=1 \\ \lambda \equiv 1 \pmod 2}}^{\infty} q^{g(\lambda)+1-2c(\lambda)} \begin{bmatrix} 2n \\ n+1-c(\lambda) \end{bmatrix}_{q^2}.$$

Replacing λ by $\lambda-1$ in the first summation and λ by $\lambda+1$ in the second summation of (3.37), we find that $T_{2n+1} = Q_{2n-1}^*$, because, by (3.27) and (3.30), we know that

$$g(\lambda-1)+1+2c(\lambda-1) = g(\lambda), \quad c(\lambda-1)+1 = c(\lambda), \quad \text{if } \lambda \equiv 1 \pmod 2,$$
$$g(\lambda+1)+1-2c(\lambda+1) = g(\lambda), \quad c(\lambda+1)-1 = c(\lambda), \quad \text{if } \lambda \equiv 0 \pmod 2.$$

Consequently,

$$Q_{2n+1}^* = Q_{2n}^* + (q^{4n+2} + q^{2n+1}) Q_{2n-1}^*.$$

At last,

$$Q_{2n}^* = \sum_{\substack{\lambda=0 \\ \lambda \equiv 0 \pmod 2}}^{\infty} q^{g(\lambda)} \begin{bmatrix} 2n+1 \\ n-c(\lambda)+1 \end{bmatrix}_{q^2} + \sum_{\substack{\lambda=1 \\ \lambda \equiv 1 \pmod 2}}^{\infty} q^{g(\lambda)} \begin{bmatrix} 2n+1 \\ n-c(\lambda) \end{bmatrix}_{q^2}$$

$$= \sum_{\substack{\lambda=0 \\ \lambda \equiv 0 \pmod 2}}^{\infty} \left(q^{g(\lambda)} \begin{bmatrix} 2n \\ n-c(\lambda) \end{bmatrix}_{q^2} + q^{g(\lambda)+2n-2c(\lambda)+2} \begin{bmatrix} 2n \\ n-c(\lambda)+1 \end{bmatrix}_{q^2} \right)$$

$$+ \sum_{\substack{\lambda=1 \\ \lambda \equiv 1 \pmod 2}}^{\infty} \left(q^{g(\lambda)} \begin{bmatrix} 2n \\ n-c(\lambda) \end{bmatrix}_{q^2} + q^{g(\lambda)+2n+2+2c(\lambda)} \begin{bmatrix} 2n \\ n-c(\lambda)-1 \end{bmatrix}_{q^2} \right)$$

$$= Q_{2n-1}^* + \sum_{\substack{\lambda=0 \\ \lambda \equiv 0 \pmod 2}}^{\infty} \left(q^{g(\lambda)+2n-2d(\lambda)} \begin{bmatrix} 2n-1 \\ n-d(\lambda) \end{bmatrix}_{q^2} + q^{g(\lambda)+4n} \begin{bmatrix} 2n-1 \\ n-d(\lambda)-1 \end{bmatrix}_{q^2} \right)$$

$$+ \sum_{\substack{\lambda=1 \\ \lambda \equiv 1 \pmod 2}}^{\infty} \left(q^{g(\lambda)+2n+2+2d(\lambda)} \begin{bmatrix} 2n-1 \\ n-d(\lambda)-2 \end{bmatrix}_{q^2} + q^{g(\lambda)+4n} \begin{bmatrix} 2n-1 \\ n-d(\lambda)-1 \end{bmatrix}_{q^2} \right)$$

$$= Q_{2n-1}^* + q^{4n} Q_{2n-2}^* + q^{2n} V_{2n},$$

where

(3.38)
$$V_{2n} = \sum_{\substack{\lambda=0 \\ \lambda\equiv 0 \,(\mathrm{mod}\, 2)}}^{\infty} q^{g(\lambda)-2d(\lambda)} \begin{bmatrix} 2n-1 \\ n-d(\lambda) \end{bmatrix}_{q^2} + \sum_{\substack{\lambda=1 \\ \lambda\equiv 1 \,(\mathrm{mod}\, 2)}}^{\infty} q^{g(\lambda)+2+2d(\lambda)} \begin{bmatrix} 2n-1 \\ n-d(\lambda)-2 \end{bmatrix}_{q^2}$$
$$= \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2} + \sum_{\substack{\lambda=2 \\ \lambda\equiv 0 \,(\mathrm{mod}\, 2)}}^{\infty} q^{g(\lambda)-2d(\lambda)} \begin{bmatrix} 2n-1 \\ n-d(\lambda) \end{bmatrix}_{q^2} + \sum_{\substack{\lambda=1 \\ \lambda\equiv 1 \,(\mathrm{mod}\, 2)}}^{\infty} q^{g(\lambda)+2+2d(\lambda)} \begin{bmatrix} 2n \\ n-d(\lambda)-2 \end{bmatrix}_{q^2}.$$

Replacing λ by $\lambda+1$ in the first summation, and λ by $\lambda-1$ in the second summation of (3.38), we find that $V_{2n} = Q^*_{2n-2}$, because, by (3.27) and (3.31), it is easy to verify that

$$g(\lambda+1)-2d(\lambda+1) = g(\lambda), \qquad d(\lambda+1) = d(\lambda)+1, \qquad \text{if } \lambda \equiv 1 \pmod 2,$$
$$g(\lambda-1)+2+2d(\lambda-1) = g(\lambda), \qquad d(\lambda-1)+1 = d(\lambda), \qquad \text{if } \lambda \equiv 0 \pmod 2.$$

Consequently

$$Q^*_{2n} = Q^*_{2n-1} + (q^{4n}+q^{2n})\, Q^*_{2n-2}.$$

Thus we have finished the proof of the theorem.

4. Continued fractions I: q – a root of unity. Let

$$D(a_1, a_2, \ldots, a_n) = \begin{vmatrix} 1 & a_1 & & & 0 \\ -1 & 1 & a_2 & & \\ & -1 & 1 & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & a_n \\ & & & -1 & 1 \end{vmatrix},$$

$$D(a_1, a_2, \ldots) = \lim_{n\to\infty} D(a_1, a_2, \ldots, a_n),$$

(4.1)
$$C(a_1, a_2, \ldots, a_n) = 1 + \frac{a_1}{1} \, \frac{}{+} \, \frac{a_2}{1} \, \frac{}{+} \, \frac{a_3}{1} \, \frac{}{+} \cdots \frac{}{+} \, \frac{a_n}{1},$$

(4.2)
$$C(a_1, a_2, \ldots) = \lim_{n\to\infty} C(a_1, a_2, \ldots, a_n).$$

In his paper [11], Schur pointed out some simple facts which will be used in the proofs of our theorems.

1. The finite continued fraction can be written as

$$C(a_1, a_2, \ldots, a_n) = P_n/Q_n, \tag{4.3}$$

where

$$P_n = D(a_1, a_2, \ldots, a_n), \quad n \geqslant 1, \tag{4.4}$$

$$Q_n = \begin{cases} 1, & \text{if } n = 1, \\ D(a_2, a_3, \ldots, a_n), & \text{if } n \geqslant 2. \end{cases} \tag{4.5}$$

Then an infinite continued fraction $C(a_1, a_2, \ldots)$ converges to l if and only if

$$\lim_{n\to\infty} \frac{P_n}{Q_n} = \lim_{n\to\infty} \frac{D(a_1, a_2, \ldots, a_n)}{D(a_2, \ldots, a_n)} = l.$$

2. Let

$$D_l^{(k)} = \begin{cases} D(a_k, a_{k+1}, \ldots, a_l), & \text{if } k \leqslant l, \\ 1, & \text{if } k > l. \end{cases}$$

Then

$$D(a_1, a_2, \ldots, a_n) = D(a_n, a_{n-1}, \ldots, a_1), \tag{4.6}$$

$$D(a_1, a_2, \ldots, a_n) = D(a_1, \ldots, a_{n-1}) + a_n D(a_1, \ldots, a_{n-2}), \tag{4.7}$$

$$D_n^{(1)} = D_{m-1}^{(1)} D_n^{(m+1)} + a_m D_{m-2}^{(1)} D_n^{(m+2)} \quad \text{for } 1 \leqslant m \leqslant n. \tag{4.8}$$

Let $P_{-1} = P_0 = 1$, $Q_{-1} = 0$, and $Q_0 = 1$. Then, from (4.3), (4.4) and (4.7), we find that for $n \geqslant 1$

$$P_n = P_{n-1} + a_n P_{n-2} \quad \text{and} \quad Q_n = Q_{n-1} + a_n Q_{n-2}.$$

Therefore, P_n, Q_n are the solutions of the difference equation

$$X_n = X_{n-1} + a_n X_{n-2},$$

or more specifically, if a_n is a function of q, then

$$P_n = X_n(1, 1, q) \quad \text{and} \quad Q_n = X_n(0, 1, q).$$

Thus P_n, Q_n are the same as in Section 3.

For our theorems, we also need some facts about Gaussian polynomials which can be verified easily. In this section, we always assume that q is a primitive mth root of unity. It is easy to see that

$$\begin{bmatrix} m \\ l \end{bmatrix}_q = 0, \quad \text{if } 0 < l < m, \tag{4.9}$$

$$\begin{bmatrix} m+1 \\ l \end{bmatrix}_q = \begin{cases} 0, & \text{if } 0 < l < m, \\ 1, & \text{if } l = 0 \text{ or } m, \end{cases} \tag{4.10}$$

$$\begin{bmatrix} 2m \\ l \end{bmatrix}_q = \begin{cases} 0, & \text{if } 0 < l < 2m, \text{ and } l \neq m, \\ 2, & \text{if } l = m, \end{cases} \tag{4.11}$$

$$\begin{bmatrix} 2m+1 \\ l \end{bmatrix}_q = \begin{cases} 0, & \text{if } 1 < l < 2m, \text{ and } l \neq m, \text{ and } m+1, \\ 1, & \text{if } l = 1 \text{ or } 2m, \\ 2, & \text{if } l = m \text{ or } m+1. \end{cases} \tag{4.12}$$

If m is even, then

$$\begin{bmatrix} m \\ l \end{bmatrix}_{q^2} = \begin{cases} 0, & \text{if } 0 < l < m, \quad l \neq m/2, \\ 2, & \text{if } l = m/2, \end{cases} \tag{4.13}$$

$$\begin{bmatrix} m+1 \\ l \end{bmatrix}_{q^2} = \begin{cases} 0, & \text{if } 1 < l < m, \ l \neq m/2, \text{ and } m/2+1, \\ 1, & \text{if } l = 1 \text{ or } m, \\ 2, & \text{if } l = m/2 \text{ or } m/2+1. \end{cases} \tag{4.14}$$

By the definition of Gaussian polynomials, for any x, and a non-negative integer n,

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_x = \begin{bmatrix} n \\ n \end{bmatrix}_x = 1. \tag{4.15}$$

Now we are ready to discuss the Ramanujan–Selberg continued fractions when q is a root of unity.

THEOREM 7. *Let*

$$S_1(q) = 1 + \frac{q}{1+} \frac{q+q^2}{1} \frac{q^3}{+1+} \frac{q^2+q^4}{1} + \ldots,$$

where q is a primitive m-th root of unity. Then, if $m \equiv 0 \pmod 2$, $S_1(q)$ diverges; if $m \equiv 1 \pmod 2$, $S_1(q)$ converges. Furthermore, for odd m, and $\varrho = \left(\frac{m}{4}\right)$, the Legendre symbol,

$$S_1(q) = (-1)^{(m-\varrho)/4} \sqrt{2}\, q^{(m-\varrho)^2/8}.$$

Proof. In this case, by (4.1), (4.2), $S_1(q) = C(a_1, a_2, \ldots)$ with $a_{2n-1} = q^{2n-1}$ and $a_{2n} = q^n + q^{2n}$. Noting $a_{2m} = 2$, $a_n = a_{n-2m}$, and $D_n^{(2m+1)} = D_{n-2m}^{(1)}$, by (4.8), we find that, for $n \geqslant 2m$,

$$\begin{aligned} P_n &= P_{2m-1} P_{n-2m} + 2 P_{2m-2} Q_{n-2m}, \\ Q_n &= Q_{2m-1} P_{n-2m} + 2 Q_{2m-2} Q_{n-2m}. \end{aligned} \tag{4.16}$$

In particular, letting $n = 2m$ in (4.16), we have

$$P_{2m} = P_{2m-1} + 2P_{2m-2}, \quad Q_{2m} = Q_{2m-1} + 2Q_{2m-2}. \tag{4.17}$$

In order to discuss the continued fraction, we shall study P_{2m-1}, P_{2m-2}, Q_{2m-1} and Q_{2m-2} by using Theorem 4. By (3.10), we have

$$P_{2m-1} = \sum_{\lambda=0}^{\infty} (-1)^{[\lambda/2]} \left(q^{f(\lambda)} \begin{bmatrix} 2m \\ m-2\lambda \end{bmatrix}_q - q^{h(\lambda)} \begin{bmatrix} 2m \\ m-2\lambda-2 \end{bmatrix}_q \right).$$

From (4.11) and (4.15), it is not difficult to see that if $m \equiv 1 \pmod 4$ or $m \equiv 3 \pmod 4$, then

$$P_{2m-1} = (-1)^0 q^{f(0)} \begin{bmatrix} 2m \\ m \end{bmatrix}_q = 2;$$

if $m \equiv 2 \pmod 4$, then

$$P_{2m-1} = 2 + (-1)^{(m-2)/4} q^{(3m^2-2m)/8} - (-1)^{(m-2)/4} q^{(3m^2+2m)/8} = 2-1-1 = 0;$$

and if $m \equiv 0 \pmod 4$, then

$$P_{2m-1} = 2 + (-1)^{m/4} q^{(3m^2+2m)/8} - (-1)^{m/4} q^{(3m^2-2m)/8} = 2 + q^{m/4} - q^{-3m/4} = 2.$$

Similarly, by (3.10) we have

$$P_{2m} = \sum_{\lambda=0}^{\infty} (-1)^{[\lambda/2]} \left(q^{f(\lambda)} \begin{bmatrix} 2m+1 \\ m-2\lambda+v(\lambda) \end{bmatrix}_q - q^{h(\lambda)} \begin{bmatrix} 2m+1 \\ m-2\lambda-2+v(\lambda) \end{bmatrix}_q \right).$$

By (4.12), (4.15), and (3.9), (3.6), (3.7), we can see that if $m \equiv 1 \pmod 4$, then

$$\begin{aligned} P_{2m} &= 2 + (-1)^{(m-1)/4} q^{(3m^2-4m+1)/8} + (-1)^{(m-1)/4} q^{(3m^2+4m+1)/8} \\ &= 2 + 2(-1)^{(m-1)/4} q^{(m-1)^2/8}; \end{aligned}$$

if $m \equiv 2 \pmod 4$, then

$$\dot{P}_{2m} = 2 + (-1)^{(m-2)/4} q^{(3m^2-2m)/8} - (-1)^{(m-2)/4} q^{(3m^2+2m)/8} = 2-1-1 = 0;$$

if $m \equiv 3 \pmod 4$, then

$$\begin{aligned} P_{2m} &= 2 - (-1)^{(m-3)/4} q^{(3m^2-4m+1)/8} - (-1)^{[(m-1)/4]} q^{(3m^2+4m+1)/8} \\ &= 2 - 2(-1)^{(m-3)/4} q^{(m+1)^2/8}; \end{aligned}$$

and if $m \equiv 0 \pmod 4$, then

$$P_{2m} = 2 + (-1)^{m/4} q^{(3m^2+2m)/8} - (-1)^{[(m-2)/4]} q^{(3m^2-4m)/8} = 2 + q^{m/4} - q^{-3m/4} = 2.$$

By (3.10), we also have

$$Q_{2m-1} = \sum_{\lambda=0}^{\infty} (-1)^{[(\lambda+1)/2]} q^{g(\lambda)} \begin{bmatrix} 2m \\ m-2\lambda-1 \end{bmatrix}_q.$$

From (4.13)–(4.15), and (3.8), we find that if $m \equiv 1 \pmod 4$, then

$$Q_{2m-1} = (-1)^{[(m+1)/4]} q^{(3m^2-2m-1)/8} = (-1)^{(m-1)/4} q^{(m^2-1)/8};$$

if $m \equiv 0,\ 2 \pmod 4$, then

$$Q_{2m-1} = 0;$$

and if $m \equiv 3 \pmod 4$ then

$$Q_{2m-1} = (-1)^{(m+1)/4} q^{(3m^2+2m-1)/8} = (-1)^{(m+1)/4} q^{(m^2-1)/8}.$$

And we have

$$Q_{2m} = \sum_{\lambda=0}^{\infty} (-1)^{[(\lambda+1)/2]} q^{g(\lambda)} \begin{bmatrix} 2m+1 \\ m-2\lambda-v(\lambda) \end{bmatrix}_q.$$

By (4.16)–(4.18), and (3.8), (3.9), we find that if $m \equiv 1 \pmod 4$, then

$$Q_{2m} = 2+(-1)^{[(m+1)/4]} q^{(3m^2-2m-1)/8} = 2+(-1)^{(m-1)/4} q^{(m^2-1)/8};$$

if $m \equiv 2 \pmod 4$, then

$$Q_{2m} = 2;$$

if $m \equiv 3 \pmod 4$, then

$$Q_{2m} = 2+(-1)^{(m+1)/4} q^{(3m^2+2m-1)/8} = 2+(-1)^{(m+1)/4} q^{(m^2-1)/8};$$

and if $m \equiv 0 \pmod 4$, then

$$Q_{2m} = 2+(-1)^{m/4} q^{(3m^2-4m)/8} + (-1)^{[(m+2)/4]} q^{(3m^2+4m)/8} = 2-1-1 = 0.$$

Summarizing the results above and using (4.17), we get the following tables:

Table 1

	P_{2m}	P_{2m-1}	P_{2m-2}
$m \equiv 0 \pmod 4$	2	2	0
$m \equiv 1 \pmod 4$	$2+2(-1)^{(m-1)/4} q^{(m-1)^2/8}$	2	$(-1)^{(m-1)/4} q^{(m-1)^2/8}$
$m \equiv 2 \pmod 4$	0	0	0
$m \equiv 3 \pmod 4$	$2+2(-1)^{(m+1)/4} q^{(m+1)^2/8}$	2	$(-1)^{(m+1)/4} q^{(m+1)^2/8}$

Table 2

	Q_{2m}	Q_{2m-1}	Q_{2m-2}
$m \equiv 0 \pmod 4$	0	0	0
$m \equiv 1 \pmod 4$	$2+(-1)^{(m-1)/4} q^{(m^2-1)/8}$	$(-1)^{(m-1)/4} q^{(m^2-1)/8}$	1
$m \equiv 2 \pmod 4$	2	0	1
$m \equiv 3 \pmod 4$	$2+(-1)^{(m+1)/4} q^{(m^2-1)/8}$	$(-1)^{(m+1)/4} q^{(m^2-1)/8}$	1

Now we are ready to study the convergence of $S_1(q)$. Using (4.16) and Tables 1 and 2, we shall discuss the following cases.

(i) If $m \equiv 0 \pmod 4$, then $P_{2mt+r} = 2^t P_r$ and $Q_{2mt+r} = 0$ for $t \geqslant 1$, $0 \leqslant r < 2m$. Therefore, we easily see that $S_1(q)$ diverges.

(ii) If $m \equiv 2 \pmod 4$, then $P_{2mt+r} = 0$ and $Q_{2mt+r} = 2^t Q_r$ for $t \geqslant 1$, $0 \leqslant r < 2$. But $Q_{2m-1} = 0$, $Q_{2m-2} = 1$. Therefore $S_1(q)$ diverges.

(iii) If $m \equiv 1, 3 \pmod 4$, let

(4.18) $$a = (-1)^{(m-\varrho)/4} q^{(m-\varrho)^2/8}, \quad \text{so that} \quad a^{-1} = (-1)^{(m-\varrho)/4} q^{(m^2-1)/8},$$

where $\varrho = \left(\frac{m}{4}\right)$, the Legendre symbol. We can easily see that $P_{2m-2} = a$, and $Q_{2m-1} = a^{-1}$. By using (4.16) repeatedly with $n = 2mt+r$, where t, r are non-negative integers, and $r < 2m$, we find that

(4.19) $$P_{2mt+r} = A_t P_r + a B_t Q_r, \quad Q_{2mt+r} = a^{-1} C_t P_r + D_t Q_r.$$

It is clear that A_t, B_t, C_t and D_t in (4.19) are integers and uniquely determined. From (4.19) and (4.16), we find that

(4.20) $$\begin{aligned} A_t &= 2A_{t-1} + 2C_{t-1}, & B_t &= 2B_{t-1} + 2D_{t-1}, \\ C_t &= A_{t-1} + 2C_{t-1}, & D_t &= B_{t-1} + 2D_{t-1}. \end{aligned}$$

For our purpose, we need the following simple lemma.

Lemma 1. *Under the assumption above,*

(4.21) $$A_t = D_t \quad \text{and} \quad B_t = 2C_t.$$

Proof. We prove this lemma by induction on t.

It is trivially true for $t = 1$. Assuming the lemma is true for $t-1$, we find that, from (4.20),

$$A_t = 2A_{t-1} + 2C_{t-1} = 2D_{t-1} + B_{t-1} = D_t,$$
$$B_t = 2B_{t-1} + 2D_{t-1} = 4C_{t-1} + 2A_{t-1} = 2C_t.$$

Thus Lemma 1 is proved.

From this lemma, we find that

$$\frac{B_t}{A_t} = 1 + \cfrac{1}{1 + \cfrac{2}{\cfrac{B_{t-1}}{A_{t-1}}}} \to \sqrt{2}, \quad \text{as } t \to +\infty,$$

and

$$\lim_{t\to\infty} \frac{P_{2mt+r}}{Q_{2mt+r}} = \lim_{t\to 1} \frac{P_r + aQ_r \frac{B_t}{A_t}}{a^{-1} \frac{B_t}{2A_t} P_r + Q_r} = \frac{P_r + \sqrt{2}\, a Q_r}{a^{-1} \frac{\sqrt{2}}{2} P_r + Q_r} = \sqrt{2}\, a.$$

This result is independent of r, and $S_1(q)$ converges to $\sqrt{2}\,a$. Thus, we have proved Theorem 7 completely.

Next we shall study the continued fraction $S_2(q)$.

THEOREM 8. *Let*

$$S_2(q) = 1+\frac{q+q^2}{1}\,\frac{q^4}{+1+}\,\frac{q^3+q^6}{1}\,\frac{q^8}{+1+}\ldots,$$

where q is a primitive m-th root of unity. Then $S_2(q)$ diverges if and only if $m \equiv 0 \pmod 8$. Furthermore, if m is not a multiple of 8, we have

$$S_2(q) = \begin{cases} (1+\sqrt{2})\,q^{(m+1)/2}, & \text{if } m \equiv \pm 1 \pmod 8, \\ q^{(-m+2)/4}, & \text{if } m \equiv 2 \pmod 8, \\ q^{(m+2)/4}, & \text{if } m \equiv -2 \pmod 8, \\ (1-\sqrt{2})\,q^{(m+1)/2}, & \text{if } m \equiv \pm 3 \pmod 8, \\ c/|c|,\ c = q^{(-m+4)/8}+q^{(m+4)/8}, & \text{if } m \equiv 4 \pmod 8. \end{cases}$$

Proof. First, we shall study P_{2m-1}, P_{2m-2}, Q_{2m-1}, and Q_{2m-2} by using Theorem 5. In this case, $a_{2n-1} = q^{2n-1}+q^{4n-2}$ and $a_{2n} = q^{4n}$. Therefore, $a_{2m} = 1$, and $D_n^{(2m+1)} = D_{n-2m}^{(1)}$ for $n \geqslant 2m$. Then we have

$$\begin{aligned} P_n &= P_{2m-1}\,P_{n-2m}+P_{2m-2}\,Q_{n-2m}, \\ Q_n &= Q_{2m-1}\,P_{n-2m}+Q_{2m-2}\,Q_{n-2m}. \end{aligned} \tag{4.22}$$

In particular, we have

$$P_{2m} = P_{2m-1}+P_{2m-2}, \quad Q_{2m} = Q_{2m-1}+Q_{2m-2}. \tag{4.23}$$

If m is odd, then q^2 is also a primitive mth root of unity. Using (3.21), (4.11), (4.12), and (4.15), we shall study P_{2m}, P_{2m-1}, Q_{2m} and Q_{2m-1} in the following four cases.

(1) If $m \equiv 1 \pmod 8$, we find that

$$\begin{aligned} P_{2m-1} &= 2+q^{3m^2/4+m/4} = 3, \\ P_{2m} &= 2+q^{(m^2-5m+2)/4}+q^{3m^2/4+m/4} = 3+q^{(m+1)/2}, \\ Q_{2m-1} &= q^{3m^2/4-m/4-1/2} = q^{(m-1)/2}, \\ Q_{2m} &= 2+q^{3m^2/4-m/4-1/2}-q^{3m^2/4+5m/4} = 1+q^{(m+1)/2}. \end{aligned}$$

(2) If $m \equiv 3 \pmod 8$, then

$$\begin{aligned} P_{2m-1} &= 2-q^{(3m^2-m)/4} = 1, \\ P_{2m} &= 2-q^{(3m^2-m)/4}-q^{(3m^2+5m+2)/4} = 1-q^{(m+1)/2}, \\ Q_{2m-1} &= -q^{(3m^2+m-2)/4} = -q^{(m-1)/2}, \end{aligned}$$

$$Q_{2m} = 2-q^{(3m^2+m-2)/4}+q^{(3m^2-5m)/4} = 3-q^{(m-1)/2}.$$

(3) If $m \equiv 5 \pmod 8$, we find that

$$P_{2m-1} = 2-q^{3m^2/4+m/4} = 1,$$
$$P_{2m} = 2-q^{(3m^2-5m+2)/4}-q^{3m^2/4+m/4} = 1-q^{(m+1)/2},$$
$$Q_{2m-1} = -q^{(3m^2-m-2)/4} = -q^{(m-1)/2},$$
$$Q_{2m} = 2-q^{(3m^2-m-2)/4}+q^{3m^2/4+5m/4} = 3-q^{(m-1)/2}.$$

(4) If $m \equiv 7 \pmod 8$, we see that

$$P_{2m-1} = 2+q^{(3m^2-m)/4} = 3,$$
$$P_{2m} = 2+q^{(3m^2-m)/4}+q^{(3m^2+5m+2)/4} = 3+q^{(m+1)/2},$$
$$Q_{2m-1} = q^{(3m^2+m-2)/4} = q^{(m-1)/2},$$
$$Q_{2m} = 2-q^{(3m^2-5m)/4}+q^{(3m^2+m-2)/4} = 1+q^{(m-1)/2}.$$

Therefore by (4.23), we have the following tables:

Table 3

	P_{2m}	P_{2m-1}	P_{2m-2}
$m \equiv 1 \pmod 8$	$3+q^{(m+1)/2}$	3	$q^{(m+1)/2}$
$m \equiv 3 \pmod 8$	$1-q^{(m+1)/2}$	1	$-q^{(m+1)/2}$
$m \equiv 5 \pmod 8$	$1-q^{(m+1)/2}$	1	$-q^{(m+1)/2}$
$m \equiv 7 \pmod 8$	$3+q^{(m+1)/2}$	3	$q^{(m+1)/2}$

Table 4

	Q_{2m}	Q_{2m-1}	Q_{2m-2}
$m \equiv 1 \pmod 8$	$1+q^{(m-1)/2}$	$q^{(m-1)/2}$	1
$m \equiv 3 \pmod 8$	$3-q^{(m-1)/2}$	$-q^{(m-1)/2}$	3
$m \equiv 5 \pmod 8$	$3-q^{(m-1)/2}$	$-q^{(m-1)/2}$	3
$m \equiv 7 \pmod 8$	$1+q^{(m-1)/2}$	$q^{(m-1)/2}$	1

From (4.22) and Tables 3 and 4, if $m \equiv \pm 1 \pmod 8$ and $a = q^{(m+1)/2}$, then

$$\begin{aligned} P_{2mt+r} &= 3\,P_{2m(t-1)}+a\,Q_{2m(t-1)+r}, \\ Q_{2mt+r} &= a^{-1}\,P_{2m(t-1)+r}+Q_{2m(t-1)+r}, \end{aligned} \tag{4.24}$$

where $t \geqslant 1$, $0 \leqslant r < 2m$. Similarly, as in (4.19), let

$$P_{2mt+r} = A_t P_r + a B_t Q_r, \qquad Q_{2mt+r} = a^{-1} C_t P_r + D_t Q_r. \tag{4.25}$$

Then by (4.24),

$$A_t = 3A_{t-1} + C_{t-1}, \quad B_t = 3B_{t-1} + D_{t-1}, \qquad (4.26)$$
$$C_t = A_{t-1} + C_{t-1}, \quad D_t = B_{t-1} + D_{t-1}.$$

It is not difficult to prove that

$$B_t = C_t. \qquad (4.27)$$

In fact we shall prove (4.27) by induction on t. For $t = 1$, (4.27) is true obviously. Assuming (4.27) is true for all positive integers less than t, by (4.26), we find that

$$\begin{aligned} B_t &= 3B_{t-1} + D_{t-1} = 3C_{t-1} + (B_{t-1} - 2B_{t-2}) = C_{t-1} + 3C_{t-1} - 2C_{t-2} \\ &= C_{t-1} + 3(A_{t-2} + C_{t-2}) - 2C_{t-2} = C_{t-1} + 3A_{t-2} + C_{t-2} \\ &= C_{t-1} + A_{t-1} = C_t. \end{aligned}$$

Then we have

$$\frac{A_t}{B_t} = \frac{A_t}{C_t} = 3 - \frac{2C_{t-1}}{A_{t-1} + C_{t-1}} = 3 - \frac{2}{1 + \dfrac{A_{t-1}}{B_{t-1}}},$$

$$\frac{B_t}{D_t} = \frac{3B_{t-1} + D_{t-1}}{B_{t-1} + D_{t-1}} = 3 - \frac{2D_{t-1}}{B_{t-1} + D_{t-1}} = 3 - \frac{2}{1 + \dfrac{B_{t-1}}{D_{t-1}}}.$$

Therefore

$$\lim_{t \to 1} \frac{A_t}{B_t} = 3 - \frac{2}{4-}\frac{2}{4-} \cdots = 1 + \sqrt{2},$$

$$\lim_{t \to 1} \frac{B_t}{D_t} = 3 - \frac{2}{4-}\frac{2}{4-} \cdots = 1 + \sqrt{2}.$$

Consequently,

$$\lim_{t \to 1} \frac{P_{2mt+r}}{Q_{2mt+r}} = \lim_{t \to 1} \frac{A_t P_r + aB_t Q_r}{a^{-1} C_t P_r + D_t Q_r} = \lim_{t \to 1} \frac{\dfrac{A_t}{B_t} P_r + aQ_r}{a^{-1} P_r + \left(\dfrac{B_t}{D_t}\right)^{-1} Q_r}$$

$$= \frac{(1+\sqrt{2}) P_r + aQ_r}{a^{-1} P_r + (1+\sqrt{2})^{-1} Q_r} = (1+\sqrt{2})\, a.$$

This limit is independent of r, and here $a = q^{(m+1)/2}$.

Similarly, if $m \equiv \pm 3 \pmod 8$, let $b = -q^{(m+1)/2}$. Then we have

$$P_{2mt+r} = P_{2m(t-1)+r} + b\, Q_{2m(t-1)+r} = A_t P_r + b B_t Q_r,$$

$$Q_{2mt+r} = b^{-1} P_{2m(t-1)+r} + 3\, Q_{2m(t-1)+r} = b^{-1} C_t P_r + D_t Q_r.$$

By using a similar argument in the cases $m \equiv \pm 1 \pmod 8$, we find that

(4.28) $$B_t = C_t,$$

and

(4.29) $$\begin{aligned} A_t &= A_{t-1} + C_{t-1}, & B_t &= B_{t-1} + D_{t-1}, \\ C_t &= A_{t-1} + 3\, C_{t-1}, & D_t &= B_{t-1} + 3\, D_{t-1}. \end{aligned}$$

Then

$$\lim_{t \to 1} \frac{B_t}{A_t} = 1 + \sqrt{2} \quad \text{and} \quad \lim_{t \to 1} \frac{D_t}{B_t} = 1 + \sqrt{2}.$$

Therefore

$$\lim_{t \to 1} \frac{P_{2mt+r}}{Q_{2mt+r}} = \lim_{t \to 1} \frac{A_t P_r + b B_t Q_r}{b^{-1} C_t P_r + D_t Q_r} = \lim_{t \to 1} \frac{\left(\frac{B_t}{A_t}\right)^{-1} P_r + bQ_r}{b^{-1} P_r + \frac{D_t}{B_t} Q_r}$$

$$= \frac{(1+\sqrt{2})^{-1} P_r + bQ_r}{b^{-1} P_r + (1+\sqrt{2}) Q_r} = \frac{b}{1+\sqrt{2}} = (\sqrt{2}-1)\, b.$$

If m is even, then

(4.30) $$D_n^{(m+1)} = D_{n-m}^{(1)} \quad \text{for } n \geqslant m,$$

and

(4.31) $$P_n = P_{m-1} P_{n-m} + P_{m-2} Q_{n-m}, \quad Q_n = Q_{m-1} P_{n-m} + Q_{m-2} Q_{n-m}.$$

In particular,

(4.32) $$P_m = P_{m-1} + P_{m-2}, \quad Q_m = Q_{m-1} + Q_{m-2}.$$

From (3.21), we have

(4.33) $$\begin{aligned} P_{m-1} &= \sum_{\lambda=0}^{\infty} \delta(\lambda)\, q^{f(\lambda)} \begin{bmatrix} m \\ m/2 - \lambda - u(\lambda) \end{bmatrix}_{q^2}, \\ P_m &= \sum_{\lambda=0}^{\infty} \delta(\lambda)\, q^{f(\lambda)} \begin{bmatrix} m+1 \\ m/2 - \lambda \end{bmatrix}_{q^2}, \\ Q_{m-1} &= \sum_{\lambda=0}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)} \begin{bmatrix} m \\ m/2 - 1 - \lambda + u(\lambda) \end{bmatrix}_{q^2}, \\ Q_m &= \sum_{\lambda=0}^{\infty} \varepsilon(\lambda)\, q^{g(\lambda)} \begin{bmatrix} m+1 \\ m/2 - \lambda \end{bmatrix}_{q^2}. \end{aligned}$$

We shall determine P_m, P_{m-1}, Q_m and Q_{m-1} by using (4.13), (4.14), and (3.16)–(3.20).

(5) If $m \equiv 0 \pmod 8$, we find that

$$P_{m-1} = 2+\delta\left(\frac{m}{2}\right)q^{(3m^2+2m)/16}+\delta\left(\frac{m}{2}-1\right)q^{(3m^2-2m)/16}$$

$$= 2+(q^{m/2})^{m/8}\,q^{(3m^2+2m)/16}+(q^{m/2})^{m/8}\,q^{(3m^2-2m)/16} = 2+q^{m/8}+q^{-m/8},$$

$$P_m = P_{m-1}, \qquad Q_{m-1} = 0,$$

$$Q_m = 2+\varepsilon\left(\frac{m}{2}\right)q^{(3m^2+10m)/16}+\varepsilon\left(\frac{m}{2}-1\right)q^{(3m^2-10m)/16}$$

$$= 2+(q^{m/2})^{m/8}\,q^{(3m^2+10m)/16}+(q^{m/2})^{m/8}\,q^{(3m^2-10m)/16}$$

$$= 2+q^{5m/8}+q^{-5m/8} = 2-q^{m/8}-q^{-m/8}.$$

(6) If $m \equiv 2 \pmod 8$, we find that

$$P_{m-1} = 2+\delta\left(\frac{m}{2}\right)q^{(3m^2+2m)/16} = 2+(q^{m/2})^{(m-2)/8}\,q^{(3m^2+2m)/16} = 1,$$

$$P_m = 2+\delta\left(\frac{m}{2}\right)q^{(3m^2+2m)/16}+\delta\left(\frac{m}{2}-1\right)q^{(3m^2-10m+8)/16}$$

$$= 1+(q^{m/2})^{(m-2)/8}\,q^{(3m^2-10m+8)/16} = 1+q^{(-m+2)/4},$$

$$Q_{m-1} = \varepsilon\left(\frac{m}{2}-1\right)q^{(3m^2-2m-8)/16} = (q^{m/2})^{(m-2)/8}\,q^{(3m^2-2m-8)/16} = q^{(m-2)/4},$$

$$Q_m = 2+\varepsilon\left(\frac{m}{2}\right)q^{(3m^2+10m)/16}+\varepsilon\left(\frac{m}{2}-1\right)q^{(3m^2-2m-8)/16}$$

$$= 2+(q^{m/2})^{(m+6)/8}\,q^{(3m^2+10m)/16}+(q^{m/2})^{(m-2)/8}\,q^{(3m^2-2m-8)/16}$$

$$= 1+q^{(m-2)/4}.$$

(7) If $m \equiv 4 \pmod 8$, we find that

$$P_{m-1} = 2,$$

$$P_m = 2+\delta\left(\frac{m}{2}\right)q^{(3m^2+10m+8)/16}+\delta\left(\frac{m}{2}-1\right)q^{(3m^2-10m+8)/16}$$

$$= 2+(q^{m/2})^{(m+4)/8}\,q^{(3m^2+10m+8)/16}+(q^{m/2})^{(m-4)/8}\,q^{(3m^2-10m+8)/16}$$

$$= 2+q^{(-m+4)/8}+q^{(m+4)/8},$$

$$Q_{m-1} = \varepsilon\left(\frac{m}{2}\right) q^{(3m^2+2m-8)/16} + \varepsilon\left(\frac{m}{2}-1\right) q^{(3m^2-2m-8)/16}$$
$$= (q^{m/2})^{(m-4)/8} q^{(3m^2+2m-8)/16} + (q^{m/2})^{(m+4)/8} q^{(3m^2-2m-8)/16}$$
$$= q^{(-m-4)/8} + q^{(m-4)/8},$$
$$Q_m = 2 + Q_{m-1}.$$

(8) If $m \equiv 6 \pmod 8$, we can see that

$$P_{m-1} = 2+\delta\left(\frac{m}{2}-1\right) q^{(3m^2-2m)/16} = 2+(q^{m/2})^{(m+2)/8} q^{(3m^2+2m)/16} = 1,$$
$$P_m = 2+\delta\left(\frac{m}{2}-1\right) q^{(3m^2-2m)/16} + \delta\left(\frac{m}{2}\right) q^{(3m^2+10m+8)/16}$$
$$= 1+(q^{m/2})^{(m+2)/8} q^{(3m^2+10m+8)/16} = 1+q^{(m+2)/4},$$
$$Q_{m-1} = \varepsilon\left(\frac{m}{2}\right) q^{(3m^2+2m-8)/16} = (q^{m/2})^{(m+2)/8} q^{(3m^2+2m-8)/16} = q^{-(m+2)/4},$$
$$Q_m = 2+\varepsilon\left(\frac{m}{2}\right) q^{(3m^2+2m-8)/16} + \varepsilon\left(\frac{m}{2}-1\right) q^{(3m^2-10m)/16}$$
$$= 2+q^{-(m+2)/4} + (q^{m/2})^{(m-6)/8} q^{(3m^2-10m)/16} = 1+q^{-(m+2)/4}.$$

Using (4.32), we obtain the following tables:

Table 5

	P_m	P_{m-1}	P_{m-2}
$m \equiv 0 \pmod 8$	$2+q^{m/8}+q^{-m/8}$	$2+q^{m/8}+q^{-m/8}$	0
$m \equiv 2 \pmod 8$	$1+q^{(-m+2)/4}$	1	$q^{(-m+2)/4}$
$m \equiv 4 \pmod 8$	$2+q^{(-m+4)/8}+q^{(m+4)/8}$	2	$q^{(-m+4)/8}+q^{(m+4)/8}$
$m \equiv 6 \pmod 8$	$1+q^{(m+2)/4}$	1	$q^{(m+2)/4}$

Table 6

	Q_m	Q_{m-1}	Q_{m-2}
$m \equiv 0 \pmod 8$	$2-q^{m/8}-q^{-m/8}$	0	$2-q^{m/8}-q^{-m/8}$
$m \equiv 2 \pmod 8$	$1+q^{(m-2)/4}$	$q^{(m-2)/4}$	1
$m \equiv 4 \pmod 8$	$2+q^{(m-4)/8}+q^{-(m+4)/8}$	$q^{(m-4)/8}+q^{-(m+4)/8}$	2
$m \equiv 6 \pmod 8$	$1+q^{-(m+2)/4}$	$q^{-(m+2)/4}$	1

By (4.31), if $m \equiv 0 \pmod 8$, we find that

$$P_{mt+r} = (2+q^{m/8}+q^{-m/8})^t P_r, \qquad Q_{mt+r} = (2-q^{m/8}-q^{-m/8})^t Q_r$$

for $t \geqslant 1$, $0 \leqslant r \leqslant m-1$. But $Q_{m-1} = 0$ and $P_{m-1} \neq 0$. Therefore $S_2(q)$ diverges.

If $m \equiv 4 \pmod 8$, by (4.31) and Table 5 we have

(4.34) $$P_{mt+r} = 2P_{m(t-1)+r} + cQ_{m(t-1)+r}, \quad Q_{mt+r} = \bar{c}P_{m(t-1)+r} + 2Q_{m(t-1)+r},$$

where $c = q^{(-m+4)/8} + q^{(m+4)/8}$.

Similarly, let

(4.35) $$P_{mt+r} = A_t P_r + cB_t Q_r, \quad Q_{mt+r} = \bar{c}C_t P_r + D_t Q_r.$$

From (4.31), it is easy to see that for any positive integer t,

(4.36) $$\begin{aligned} A_t &= 2A_{t-1} + |c|^2 C_{t-1}, & B_t &= 2B_{t-1} + D_{t-1}, \\ C_t &= A_{t-1} + 2C_{t-1}, & D_t &= |c|^2 B_{t-1} + 2D_{t-1}, \end{aligned}$$

and

(4.37) $$A_t = D_t, \quad B_t = C_t.$$

Therefore

$$\frac{A_t}{B_t} = 2 + \cfrac{|c|^2 - 4}{2 + \cfrac{A_{t-1}}{B_{t-1}}}$$

and

(4.38) $$\lim_{t \to 1} \frac{A_t}{B_t} = 2 + \frac{|c|^2 - 4}{4} + \frac{|c|^2 - 4}{4} + \ldots.$$

The right-hand side of (4.38) is a periodic continued fraction with period $k = 1$. Since $|c|^2 - 4 \neq 0$, by a well-known theorem for convergence of periodic continued fractions (cf. [8], Theorems 3.1 and 3.2), we find that (4.38) converges and the limit is $|c|$. Consequently, by (4.35), (4.37) and (4.38),

$$\lim_{t \to 1} \frac{P_{mt+r}}{Q_{mt+r}} = \lim_{t \to 1} \frac{A_t P_r + cB_t Q_r}{\bar{c}C_t P_r + D_t Q_r} = \lim_{t \to 1} \frac{\frac{A_t}{B_t} P_r + cQ_r}{\bar{c}P_r + \frac{D_t}{B_t} Q_r}$$

$$= \frac{|c| P_r + cQ_r}{\bar{c}P_r + |c| Q_r} = \frac{c}{|c|}.$$

This limit does not depend on r. Therefore $S_2(q)$ converges.

If $m \equiv \pm 2 \pmod 8$, let

$$d = \begin{cases} q^{(-m+2)/4}, & \text{if } m \equiv 2 \pmod 8, \\ q^{(m+2)/4}, & \text{if } m \equiv -2 \pmod 8). \end{cases}$$

Then by (4.31),

$$P_{mt+r} = P_{m(t-1)+r} + dQ_{m(t-1)+r}, \quad Q_{mt+r} = d^{-1} P_{m(t-1)+r} + Q_{m(t-1)+r}.$$

Consequently, $P_{mt+r}/Q_{mt+r} = d$, for $t \geqslant 1, 0 \leqslant r \leqslant m-1$. We conclude that $S_2(q)$ converges to d in this case. Therefore this theorem has been proved completely.

THEOREM 9. *Let*

$$S_3(q) = 1 + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \dots,$$

where q is a primitive m-th root of unity. Then $S_3(q)$ diverges if and only if $m \equiv 0 \pmod 3$. Furthermore,

$$S_3(q) = \begin{cases} 2\,q^{(2m+1)/3}, & \textit{if } m \equiv 1 \pmod 6, \\ 2\,q^{(m+1)/3}, & \textit{if } m \equiv -1 \pmod 6, \\ q^{(-m+2)/6}, & \textit{if } m \equiv 2 \pmod 6, \\ q^{(m+2)/6}, & \textit{if } m \equiv -2 \pmod 6. \end{cases}$$

Proof. It is obvious that in this case $a_m = 2$ and $D_n^{(m+1)} = D_{n-m}^{(1)}$. By (4.7), we have for $n \geqslant m$

(4.39) $$P_n = P_{m-1}\,P_{n-m} + 2\,P_{m-2}\,Q_{n-m}, \qquad Q_n = Q_{m-1}\,P_{n-m} + 2\,Q_{m-2}\,Q_{n-m}.$$

In particular,

(4.40) $$P_m = P_{m-1} + 2\,P_{m-2}, \qquad Q_m = Q_{m-1} + 2\,Q_{m-2}.$$

We shall determine P_{m-1}, P_m, Q_{m-1}, and Q_m by using Theorem 6.

For m even, by (3.32),

(4.41) $$\begin{aligned} P_{m-1} &= \sum_{\lambda=0}^{\infty} q^{f(\lambda)} \begin{bmatrix} m \\ m/2 - a(\lambda) \end{bmatrix}_{q^2}, \\ P_m &= \sum_{\lambda=0}^{\infty} q^{f(\lambda)} \begin{bmatrix} m+1 \\ m/2 - b(\lambda) \end{bmatrix}_{q^2}, \\ Q_{m-1} &= \sum_{\lambda=0}^{\infty} q^{g(\lambda)} \begin{bmatrix} m \\ m/2 - c(\lambda) \end{bmatrix}_{q^2}, \\ Q_m &= \sum_{\lambda=0}^{\infty} q^{g(\lambda)} \begin{bmatrix} m+1 \\ m/2 - d(\lambda) \end{bmatrix}_{q^2}. \end{aligned}$$

By using (4.41), (4.13) and (4.14), we study P_{m-1}, P_m, Q_{m-1}, and Q_m in the following 3 cases.

If $m \equiv 0 \pmod 6$, then

$$\begin{aligned} P_{m-1} &= 2 + q^{f(2m/3)} + q^{f((2m-3)/3)} = 2 + q^{m/6} + q^{-m/6}, \\ P_m &= 2 + q^{f(2m/3)} + q^{f((2m-3)/3)} = 2 + q^{m/6} + q^{-m/6}, \\ Q_{m-1} &= 0, \\ Q_m &= 2 + q^{g(m/3)} + q^{g((m-3)/3)} = 2 + q^{m/2} + q^{m/2} = 0. \end{aligned}$$

If $m \equiv 2 \pmod 6$, then

$$P_{m-1} = 2+q^{f((2m-1)/3)} = 2+q^{m(m+1)/6} = 1,$$

$$P_m = 2+q^{f((2m+2)/3)}+q^{f((2m-1)/3)}+q^{f((2m-4)/3)} = 1+2\,q^{(2-m)/6},$$

$$Q_{m-1} = q^{g((m-2)/3)} = q^{(m-2)/6},$$

$$Q_m = 2+q^{g((m-2)/3)} = 2+q^{(m-2)/6}.$$

If $m \equiv 4 \pmod 6$, then

$$P_{m-1} = 2+q^{f((2m-2)/3)} = 2+q^{m(m-1)/6} = 1,$$

$$P_m = 2+q^{f((2m+1)/3)}+q^{f((2m-5)/3)} = 1+2\,q^{(m+2)/6},$$

$$Q_{m-1} = 2+q^{g((m-1)/3)} = q^{-(m+2)/6},$$

$$Q_m = 2+q^{g((m-1)/3)} = 2+q^{-(m+2)/6}.$$

For m odd, by (3.32),

$$P_{m-1} = \sum_{\lambda=0}^{\infty} q^{f(\lambda)} \begin{bmatrix} m \\ (m-1)/2-b(\lambda) \end{bmatrix}_{q^2},$$

$$P_m = \sum_{\lambda=0}^{\infty} q^{f(\lambda)} \begin{bmatrix} m+1 \\ (m+1)/2-a(\lambda) \end{bmatrix}_{q^2},$$

$$Q_{m-1} = \sum_{\lambda=0}^{\infty} q^{g(\lambda)} \begin{bmatrix} m \\ (m-1)/2-d(\lambda) \end{bmatrix}_{q^2},$$

$$Q_m = \sum_{\lambda=0}^{\infty} q^{g(\lambda)} \begin{bmatrix} m+1 \\ (m+1)/2-c(\lambda) \end{bmatrix}_{q^2}.$$

Using (4.9), (4.10) and the formulas above, and noting q^2 is also an mth primitive root of unity, we study P_{m-1}, P_m, Q_{m-1} and Q_m in the following 3 cases.

If $m \equiv 1 \pmod 6$, then

$$P_{m-1} = q^{f((2m-2)/3)} = 1,$$

$$P_m = q^{f((2m+1)/3)}+q^{f((2m-2)/3)}+q^{f((2m-5)/3)} = 1+2\,q^{(2m+1)/3},$$

$$Q_{m-1} = q^{g((m-1)/3)} = q^{(m-1)/3},$$

$$Q_m = q^{g((m-1)/3)} = q^{(m-1)/3}.$$

If $m \equiv 3 \pmod 6$, then

$$P_{m-1} = q^{f(2m/3)}+q^{f((2m-3)/3)} = q^{-m/3}+q^{m/3},$$

$$P_m = q^{f(2m/3)}+q^{f((2m-3)/3)} = q^{-m/3}+q^{m/3},$$

$$Q_{m-1} = 0,$$

$$Q_m = q^{g(m/3)}+q^{g((m-3)/3)} = 2.$$

If $m \equiv 5 \pmod 6$, then

$$P_{m-1} = q^{f((2m-1)/3)} = 1,$$
$$P_m = q^{f((2m+2)/3)} + q^{f((2m-1)/3)} + q^{f((2m-4)/3)} = 1 + 2q^{(m+1)/3},$$
$$Q_{m-1} = q^{g((m-2)/3)} = q^{-(m+1)/3},$$
$$Q_m = q^{g((m-2)/3)} = q^{-(m+1)/3}.$$

Therefore, by (4.40) we have the following tables:

Table 7

	P_m	P_{m-1}	P_{m-2}
$m \equiv 0 \pmod 6$	$2+q^{m/6}+q^{-m/6}$	$2+q^{m/6}+q^{-m/6}$	0
$m \equiv 1 \pmod 6$	$1+2q^{(2m+1)/3}$	1	$q^{(2m+1)/3}$
$m \equiv 2 \pmod 6$	$1+2q^{(2-m)/6}$	1	$q^{(2-m)/6}$
$m \equiv 3 \pmod 6$	$q^{m/3}+q^{-m/3}$	$q^{m/3}+q^{-m/3}$	0
$m \equiv 4 \pmod 6$	$1+2q^{(m+2)/6}$	1	$q^{(m+2)/6}$
$m \equiv 5 \pmod 6$	$1+2q^{(m+1)/3}$	1	$q^{(m+1)/3}$

Table 8

	Q_m	Q_{m-1}	Q_{m-2}
$m \equiv 0 \pmod 6$	0	0	0
$m \equiv 1 \pmod 6$	$q^{(m-1)/3}$	$q^{(m-1)/3}$	0
$m \equiv 2 \pmod 6$	$2+q^{(m-2)/6}$	$q^{(m-2)/6}$	1
$m \equiv 3 \pmod 6$	2	0	1
$m \equiv 4 \pmod 6$	$2+q^{-(m+2)/6}$	$q^{-(m+2)/6}$	1
$m \equiv 5 \pmod 6$	$q^{-(m+1)/3}$	$q^{-(m+1)/3}$	0

If $m \equiv 0 \pmod 6$, by (4.39), we find that

$$P_{mt+r} = (2+q^{m/6}+q^{-m/6})^t P_r \quad \text{and} \quad Q_{mt+r} = 0,$$

for $t \geqslant 1$, $0 \leqslant r < m$. Thus it is obvious that $S_3(q)$ diverges.

If $m \equiv 3 \pmod 6$, from (4.39), we find that, for $0 \leqslant r < m$,

$$P_{mt+r} = (q^{m/3}+q^{-m/3})^t P_r \quad \text{and} \quad Q_{mt+r} = 2^t Q_r.$$

It is obvious that

$$\lim_{t\to\infty} \left(\frac{q^{m/3}+q^{-m/3}}{2}\right)^t = 0.$$

Thus, for $Q_r \neq 0$,

$$\lim_{t\to\infty} \frac{P_{mt+r}}{Q_{mt+r}} = 0.$$

But $Q_{m-1} = 0$, and $P_{m-1} \neq 0$, so $S_3(q)$ diverges.

If $m \equiv \pm 1 \pmod 6$, and

$$a = \begin{cases} q^{(2m+1)/3}, & \text{if } m \equiv 1 \pmod 6, \\ q^{(m+1)/3}, & \text{if } m \equiv -1 \pmod 6, \end{cases}$$

then, by (4.39), we find that for a positive integer t

$$P_{mt+r} = P_{m(t-1)+r} + 2aQ_{m(t-1)+r} \quad \text{and} \quad Q_{mt+r} = a^{-1}P_{m(t-1)+r}.$$

Therefore

$$P_{mt+r} = P_{m(t-1)+r} + 2P_{m(t-2)+r}.$$

Consequently,

$$\frac{P_{mt+r}}{Q_{mt+r}} = a\frac{P_{mt+r}}{P_{m(t-1)+r}} = a\left[1 + \cfrac{2}{\cfrac{P_{m(t-1)+r}}{P_{m(t-2)+r}}}\right].$$

Then

$$\lim_{t\to\infty}\frac{P_{mt+r}}{Q_{mt+r}} = a\left(1 + \frac{2}{1+}\frac{2}{1+}\ \ldots\right) = 2a.$$

If $m \equiv \pm 2 \pmod 6$, let

$$b = \begin{cases} q^{(2-m)/6}, & \text{if } m \equiv 2 \pmod 6, \\ q^{(m+2)/6}, & \text{if } m \equiv -2 \pmod 6. \end{cases}$$

Then, by (4.39) and Tables 7, 8, we have

$$P_{mt+r} = P_{m(t-1)+r} + 2bQ_{m(t-1)+r} \quad \text{and} \quad Q_{mt+r} = b^{-1}P_{m(t-1)+r} + 2Q_{m(t-1)+r},$$

for $t \geqslant 1$, $0 \leqslant r < m$. Thus $P_{mt+r}/Q_{mt+r} = b$ and $S_3(q) = b$.

Thus Theorem 9 has been proved.

5. Continued fractions II: $|q| > 1$. In this section, we shall discuss the Ramanujan–Selberg continued fractions $S_1(q)$, $S_2(q)$ and $S_3(q)$ when $|q| > 1$.

Let $x = q^{-1}$, then $0 < |x| < 1$. It is easy to see that

$$\text{(5.1)} \qquad \begin{bmatrix} A \\ B \end{bmatrix}_{x^{-1}} = x^{-B(A-B)}\begin{bmatrix} A \\ B \end{bmatrix}_x,$$

$$\text{(5.2)} \qquad \lim_{n\to\infty}\begin{bmatrix} 2n \\ n-l \end{bmatrix}_x = \lim_{n\to\infty}\frac{(1-x)\ldots(1-x^{2n})}{(1-x)\ldots(1-x^{n-l})(1-x)\ldots(1-x^{n+l})}$$

$$= (x;x)_\infty^{-1}, \quad \text{for any integer } l.$$

We first discuss $S_1(q)$.

THEOREM 10. *Let*

$$S_1(q) = 1 + \frac{q}{1+}\,\frac{q+q^2}{1}\,\frac{q^3}{+1+}\,\frac{q^2+q^4}{1}\,+\ldots,$$

where $|q| > 1$. *Set* $x = q^{-1}$. *Then the odd indexed convergents tend to*

$$\frac{(-x^3;x^{10})_\infty(-x^5;x^{10})_\infty(-x^7;x^{10})_\infty}{x(-x^2;x^{10})_\infty(-x^8;x^{10})_\infty(-x^{10};x^{10})_\infty}, \tag{5.3}$$

while the even indexed convergents tend to

$$\frac{(-x;x^{10})_\infty(-x^5;x^{10})_\infty(-x^9;x^{10})_\infty}{(-x^4;x^{10})_\infty(-x^6;x^{10})_\infty(-x^{10};x^{10})_\infty}. \tag{5.4}$$

Proof. From (3.10) and (5.1), we find that

$$\begin{aligned}
P_{2n-1}(q) &= P_{2n-1}(x^{-1})\\
&= \sum_{\lambda=0}^{\infty}(-1)^{[\lambda/2]}\left(x^{-f(\lambda)}\begin{bmatrix}2n\\ n-2\lambda\end{bmatrix}_{x^{-1}} - x^{-h(\lambda)}\begin{bmatrix}2n\\ n-2\lambda-2\end{bmatrix}_{x^{-1}}\right)\\
&= \sum_{\lambda=0}^{\infty}(-1)^{[\lambda/2]}\left(x^{-n^2+4\lambda^2-f(\lambda)}\begin{bmatrix}2n\\ n-2\lambda\end{bmatrix}_x\right.\\
&\qquad\left. - x^{-n^2+4(\lambda+1)^2h(\lambda)}\begin{bmatrix}2n\\ n-2\lambda-2\end{bmatrix}_x\right)\\
&= x^{-n^2}\sum_{\lambda=0}^{\infty}(-1)^{[\lambda/2]}\left(x^{5\lambda^2/2+(-1)^{\lambda+1}\lambda/2}\begin{bmatrix}2n\\ n-2\lambda\end{bmatrix}_x\right.\\
&\qquad\left. - x^{5\lambda^2/2+5\lambda+5/2+(-1)^{\lambda+1}(\lambda+1)/2}\begin{bmatrix}2n\\ n-2\lambda-2\end{bmatrix}_x\right),
\end{aligned}$$

$$\begin{aligned}
P_{2n}(q) &= P_{2n}(x^{-1})\\
&= \sum_{\lambda=0}^{\infty}(-1)^{[\lambda/2]}\left(x^{-f(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda+v(\lambda)\end{bmatrix}_{x^{-1}} - x^{-h(\lambda)}\begin{bmatrix}2n+1\\ n-2\lambda-2+v(\lambda)\end{bmatrix}_{x^{-1}}\right)\\
&= x^{-n^2-n}\sum_{\lambda=0}^{n}(-1)^{[\lambda/2]}\left(x^{5\lambda^2/2+(-1)^{\lambda}3/2}\begin{bmatrix}2n+1\\ n-2\lambda+v(\lambda)\end{bmatrix}_x\right.\\
&\qquad\left. - x^{5\lambda^2/2+5\lambda/2+3/4+(-1)^{\lambda+1}(3\lambda/2+3/4)}\begin{bmatrix}2n+1\\ n-2\lambda-2+v(\lambda)\end{bmatrix}_x\right),
\end{aligned}$$

$$\begin{aligned}
Q_{2n-1}(q) &= Q_{2n-1}(x^{-1})\\
&= \sum_{\lambda=0}^{\infty}(-1)^{[(\lambda+1)/2]}x^{-g(\lambda)}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_{x^{-1}}\\
&= \sum_{\lambda=0}^{n}(-1)^{[(\lambda+1)/2]}x^{-n^2+(2\lambda+1)^2-g(\lambda)}\begin{bmatrix}2n\\ n-2\lambda-1\end{bmatrix}_x
\end{aligned}$$

$$= x^{-n^2} \sum_{\lambda=0}^{n} (-1)^{[(\lambda+1)/2]} x^{5\lambda^2/2+5\lambda/2+3/4+(-1)^\lambda(2\lambda+1)/4} \begin{bmatrix} 2n \\ n-2\lambda-1 \end{bmatrix}_x,$$

$$Q_{2n}(q) = Q_{2n}(x^{-1})$$

$$= \sum_{\lambda=0}^{\infty} (-1)^{[(\lambda+1)/2]} x^{-g(\lambda)} \begin{bmatrix} 2n+1 \\ n-2\lambda-v(\lambda) \end{bmatrix}_{x^{-1}}$$

$$= x^{-n^2-n} \sum_{\lambda=0}^{\infty} (-1)^{[(\lambda+1)/2]} x^{5\lambda^2/2+5\lambda/2+3/4+(-1)^{\lambda+1}(3\lambda/2+3/4)} \begin{bmatrix} 2n+1 \\ n-2\lambda-v(\lambda) \end{bmatrix}_x.$$

Let

$$C_0(q) = \lim_{n\to\infty} \frac{P_{2n-1}(q)}{Q_{2n-1}(q)} \quad \text{and} \quad C_e(q) = \lim_{n\to\infty} \frac{P_{2n}(q)}{Q_{2n}(q)}.$$

Then, by (5.2),

$$C_0(q) = C_0(x^{-1}) = \frac{\sum_{\lambda=0}^{\infty} (-1)^{[\lambda/2]} (x^{S(\lambda)} - x^{T(\lambda)})}{\sum_{\lambda=0}^{\infty} (-1)^{[(\lambda+1)/2]} x^{R(\lambda)}}, \tag{5.5}$$

$$C_e(q) = C_e(x^{-1}) = \frac{\sum_{\lambda=0}^{\infty} (-1)^{[\lambda/2]} (x^{S_1(\lambda)} - x^{T_1(\lambda)})}{\sum_{\lambda=0}^{\infty} (-1)^{[(\lambda+1)/2]} x^{R_1(\lambda)}}, \tag{5.6}$$

where

$$\begin{aligned}
S(\lambda) &= \frac{5\lambda^2}{2} + (-1)^{\lambda+1}\frac{\lambda}{2}, \\
T(\lambda) &= \frac{5\lambda^2}{2} + 5\lambda + \frac{5}{2} + (-1)^{\lambda+1}\frac{\lambda+2}{2}, \\
R(\lambda) &= \frac{5\lambda^2}{2} + \frac{5\lambda}{2} + \frac{3}{4} + (-1)^{\lambda}\frac{2\lambda+2}{4}, \\
S_1(\lambda) &= \frac{5\lambda^2}{2} + (-1)^{\lambda}\frac{3\lambda}{2}, \\
T_1(\lambda) &= \frac{5\lambda^2}{2} + 5\lambda + \frac{5}{2} + (-1)^{\lambda}\left(\frac{3\lambda}{2} + \frac{3}{2}\right), \\
R_1(\lambda) &= \frac{5\lambda^2}{2} + \frac{5\lambda}{2} + \frac{3}{4} + (-1)^{\lambda+1}\left(\frac{3\lambda}{2} + \frac{3}{4}\right).
\end{aligned} \tag{5.7}$$

The Ramanujan theta function $f(a,b)$ (cf. [1]) is defined by

(5.8) $$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2},$$

where $|ab| < 1$. The following two results will be used in our proof.

ENTRY 19 ([1], p. 30). *We have*

(5.9) $$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty.$$

ENTRY 31 ([1], p. 46). *Let*

$$u_n = a^{n(n+1)/2} b^{n(n-1)/2} \quad \text{and} \quad v_n = a^{n(n-1)/2} b^{n(n+1)/2}$$

for a positive integer n. Then

(5.10) $$f(u_1, v_1) = \sum_{r=0}^{n-1} u_r f\left(\frac{u_{n+r}}{u_r}, \frac{v_{n-r}}{u_r}\right).$$

From (5.5), we have

(5.11) $$C_0(x^{-1}) = \frac{\sum\limits_{\substack{\lambda=0 \\ \lambda \equiv 0 \pmod 2}}^{\infty} (-1)^{[\lambda/2]}(x^{S(\lambda)} - x^{T(\lambda)}) + \sum\limits_{\substack{\lambda=1 \\ \lambda \equiv 1 \pmod 2}}^{\infty} (-1)^{[\lambda/2]}(x^{S(\lambda)} - x^{T(\lambda)})}{\sum\limits_{\substack{\lambda=0 \\ \lambda \equiv 0 \pmod 2}}^{\infty} (-1)^{[(\lambda+1)/2]} x^{R(\lambda)} + \sum\limits_{\substack{\lambda=1 \\ \lambda \equiv 1 \pmod 2}}^{\infty} (-1)^{[(\lambda+1)/2]} x^{R(\lambda)}}.$$

Replacing λ by $2k$ in the first summations and λ by $2k-1$ in the second summations of the numerator and denominator of (5.11), we find that

$$C_0(x^{-1}) = \frac{\sum\limits_{k=0}^{\infty} (-1)^k (x^{10k-k} - x^{10k^2+9k+2}) + \sum\limits_{k=1}^{\infty} (-1)^{k-1} (x^{10k^2-9k+2} - x^{10k^2+k})}{\sum\limits_{k=0}^{\infty} (-1)^k x^{10k^2+6k+1} + \sum\limits_{k=1}^{\infty} (-1)^k x^{10k^2-6k+1}}$$

$$= \frac{\sum\limits_{k=-\infty}^{\infty} (-1)^{k^2} x^{10k^2-k} - x^2 \sum\limits_{k=-\infty}^{\infty} (-1)^{k^2} x^{10k^2+9k}}{x \sum\limits_{k=-\infty}^{\infty} (-1)^{k^2} x^{10k^2+6k}}$$

$$= \frac{f(-x^9, -x^{11}) - x^2 f(-x^{19}, -x)}{x f(-x^{16}, -x^4)}.$$

Letting $n = 2$, $a = -x^2$, and $b = x^3$ in (5.10), we see that

$$f(-x^2, x^3) = f(-x^9, -x^{11}) - x^2 f(-x^{19}, -x).$$

By (5.9), we find that

$$C_0(x^{-1}) = \frac{f(-x^2, x^3)}{xf(-x^{16}, -x^4)} = \frac{(x^2; -x^5)_\infty (-x^3; -x^5)_\infty (-x^5; -x^5)_\infty}{x(x^{16}; x^{20})_\infty (x^4; x^{20})_\infty (x^{20}; x^{20})_\infty}$$

$$= \frac{(x^2; x^{10})_\infty (-x^7; x^{10})_\infty (-x^3; x^{10})_\infty (x^8; x^{10})_\infty (-x^5; x^{10})_\infty (-x^{10}; x^{10})_\infty}{x(x^{16}; x^{20})_\infty (x^4; x^{20})_\infty (x^{20}; x^{20})_\infty}$$

$$= \frac{(-x^3; x^{10})_\infty (-x^5; x^{10})_\infty (-x^7; x^{10})_\infty}{x(-x^2; x^{10})_\infty (-x^8; x^{10})_\infty (-x^{10}; x^{10})_\infty}.$$

Then (5.3) follows.

Similarly, from (5.6), we find that

$$C_e(x^{-1})$$

$$= \frac{\sum\limits_{\substack{\lambda=0 \\ \lambda\equiv 0 \pmod 2}}^{\infty} (-1)^{[\lambda/2]}(x^{S_1(\lambda)} - x^{T_1(\lambda)}) + \sum\limits_{\substack{\lambda=1 \\ \lambda\equiv 1 \pmod 2}}^{\infty} (-1)^{[\lambda/2]}(x^{S_1(\lambda)} - x^{T_1(\lambda)})}{\sum\limits_{\substack{\lambda=0 \\ \lambda\equiv 0 \pmod 2}}^{\infty} (-1)^{[(\lambda+1)/2]} x^{R_1(\lambda)} + \sum\limits_{\substack{\lambda=1 \\ \lambda\equiv 1 \pmod 2}}^{\infty} (-1)^{[(\lambda+1)/2]} x^{R_1(\lambda)}}$$

$$= \frac{\sum\limits_{k=0}^{\infty} (-1)^k (x^{10k^2+3k} - x^{10k^2+13k+4}) + \sum\limits_{k=1}^{\infty} (-1)^{k-1} (x^{10k^2-13k+4} - x^{10k^2-3k})}{\sum\limits_{k=0}^{\infty} (-1)^k x^{10k^2+2k} + \sum\limits_{k=1}^{\infty} (-1)^k x^{10k^2-2k}}$$

$$= \frac{\sum\limits_{k=-\infty}^{\infty} (-1)^{k^2} x^{10k^2+3k} - x^4 \sum\limits_{k=-\infty}^{\infty} (-1)^{k^2} x^{10k^2+13k}}{\sum\limits_{k=-\infty}^{\infty} (-1)^{k^2} x^{10k^2+2k}}$$

$$= \frac{f(-x^{13}, -x^7) - x^4 f(-x^{23}, -x^{-3})}{f(-x^{12}, -x^8)}.$$

Using (5.10), $n = 2$, $a = -x^4$, and $b = x$, we find that

$$f(-x^4, x) = f(-x^{13}, -x^7) - x^4 f(-x^{23}, -x^{-3}).$$

Therefore, by (5.9),

$$C_e(x^{-1}) = \frac{f(-x^4, x)}{f(-x^{12}, -x^8)} = \frac{(x^4; -x^5)_\infty (-x; -x^5)_\infty (-x^5; -x^5)_\infty}{(x^{12}; x^{20})_\infty (x^8; x^{20})_\infty (x^{20}; x^{20})_\infty}$$

$$= \frac{(x^4; x^{10})_\infty (-x^9; x^{10})_\infty (-x^6; x^{10})_\infty (-x; x^{10})_\infty (-x^5; x^{10})_\infty (x^{10}; x^{10})_\infty}{(x^6; x^{10})_\infty (-x^6; x^{10})_\infty (x^4; x^{10})_\infty (-x^4; x^{10})_\infty (x^{10}; x^{10})_\infty (-x^{10}; x^{10})_\infty}$$

$$= \frac{(-x;x^{10})_\infty (-x^5;x^{10})_\infty (-x^9;x^{10})_\infty}{(-x^4;x^{10})_\infty (-x^6;x^{10})_\infty (-x^{10};x^{10})_\infty}.$$

The theorem has been proved.

THEOREM 11. *Let*

$$S_1(q) = 1 + \frac{q+q^2}{1} \, \frac{q^4}{+1+} \, \frac{q^3+q^6}{1} \, \frac{q^8}{+1+\ldots},$$

where $|q| > 1$. *Set* $x = q^{-1}$. *Then the odd indexed convergents tend to*

$$\frac{(x^{19};x^{40})_\infty (x^{21};x^{40})_\infty + x(x^{29};x^{40})_\infty (x^{11};x^{40})_\infty}{x^2(x^{31};x^{40})_\infty (x^9;x^{40})_\infty - x^6(x^{41};x^{40})_\infty (x^{-1};x^{40})_\infty},$$

while the even indexed convergents tend to

$$\frac{(x^{27};x^{40})_\infty (x^{13};x^{40})_\infty + x^3(x^3;x^{40})_\infty (x^{37};x^{40})_\infty}{(x^{17};x^{40})_\infty (x^{23};x^{40})_\infty - x^2(x^7;x^{40})_\infty (x^{33};x^{40})_\infty}.$$

Proof. From (3.21) and (5.1), we find that

$$P_{2n-1}(q) = P_{2n-1}(x^{-1}) = \sum_{\lambda=0}^{\infty} \delta(\lambda) x^{-f(\lambda)} \begin{bmatrix} 2n \\ n-\lambda-u(\lambda) \end{bmatrix}_{x^{-2}}$$

$$= \sum_{\lambda=0}^{n} \delta(\lambda) x^{-2n^2+2(\lambda+u(\lambda))^2-f(\lambda)} \begin{bmatrix} 2n \\ n-\lambda-u(\lambda) \end{bmatrix}_{x^2}$$

$$= x^{-2n^2} \sum_{\lambda=0}^{n} \delta(\lambda) x^{S(\lambda)} \begin{bmatrix} 2n \\ n-\lambda-u(\lambda) \end{bmatrix}_{x^2},$$

$$P_{2n}(q) = P_{2n}(x^{-1}) = \sum_{\lambda=0}^{\infty} \delta(\lambda) x^{-f(\lambda)} \begin{bmatrix} 2n+1 \\ n-2 \end{bmatrix}_{x^{-2}}$$

$$= \sum_{\lambda=0}^{n} \delta(\lambda) x^{-2n^2+2\lambda^2+2n-f(\lambda)} \begin{bmatrix} 2n+1 \\ n-2 \end{bmatrix}_{x^2}$$

$$= x^{-2n^2-2n} \sum_{\lambda=0}^{n} \delta(\lambda) x^{S_1(\lambda)} \begin{bmatrix} 2n+1 \\ n-2 \end{bmatrix}_{x^2},$$

$$Q_{2n-1}(q) = Q_{2n-1}(x^{-1}) = \sum_{\lambda=0}^{\infty} \varepsilon(\lambda) x^{-g(\lambda)} \begin{bmatrix} 2n \\ n-1-\lambda+u(\lambda) \end{bmatrix}_{x^{-2}}$$

$$= \sum_{\lambda=0}^{n} \varepsilon(\lambda) x^{-2n^2+2(1+\lambda-u(\lambda))^2-g(\lambda)} \begin{bmatrix} 2n \\ n-1-\lambda+u(\lambda) \end{bmatrix}_{x^2}$$

$$= x^{-2n^2} \sum_{\lambda=0}^{n} \varepsilon(\lambda) x^{R(\lambda)} \begin{bmatrix} 2n \\ n-1-\lambda+u(\lambda) \end{bmatrix}_{x^2},$$

$$Q_{2n}(q) = Q_{2n}(x^{-1}) = \sum_{\lambda=0}^{\infty} \varepsilon(\lambda)\, x^{-g(\lambda)} \begin{bmatrix} 2n+1 \\ n-\lambda \end{bmatrix}_{x^{-2}}$$

$$= x^{-2n^2-2n} \sum_{\lambda=0}^{n} \varepsilon(\lambda)\, x^{2\lambda^2+2\lambda-g(\lambda)} \begin{bmatrix} 2n+1 \\ n-2 \end{bmatrix}_{x^2}$$

$$= x^{-2n^2-2n} \sum_{\lambda=0}^{n} \varepsilon(\lambda)\, x^{R_1(\lambda)} \begin{bmatrix} 2n+1 \\ n-2 \end{bmatrix}_{x^2},$$

where

$$S(\lambda) = 2(\lambda+u(\lambda))^2 - f(\lambda) = \begin{cases} \frac{5\lambda^2}{4} - \frac{\lambda}{4}, & \text{if } \lambda \equiv 0, 1 \pmod 4, \\ \frac{5\lambda^2}{4} + \frac{11\lambda}{4} + \frac{3}{2}, & \text{if } \lambda \equiv 2, 3 \pmod 4, \end{cases}$$

$$R(\lambda) = 2(1+\lambda-u(\lambda))^2 - g(\lambda) = \begin{cases} \frac{5\lambda^2}{4} + \frac{11\lambda}{4} + 2, & \text{if } \lambda \equiv 0, 1 \pmod 4, \\ \frac{5\lambda^2}{4} + \frac{\lambda}{4} + \frac{1}{2}, & \text{if } \lambda \equiv 2, 3 \pmod 4, \end{cases}$$

$$S_1(\lambda) = 2\lambda^2 + 2\lambda - f(\lambda) = \begin{cases} \frac{5\lambda^2}{4} - \frac{7\lambda}{4}, & \text{if } \lambda \equiv 0, 1 \pmod 4, \\ \frac{5\lambda^2}{4} + \frac{3\lambda}{4} - \frac{1}{2}, & \text{if } \lambda \equiv 2, 3 \pmod 4, \end{cases}$$

$$R_1(\lambda) = 2\lambda^2 + 2\lambda - g(\lambda) = \begin{cases} \frac{5\lambda^2}{4} - \frac{3\lambda}{4}, & \text{if } \lambda \equiv 0, 1 \pmod 4, \\ \frac{5\lambda^2}{4} + \frac{7\lambda}{4} + \frac{1}{2}, & \text{if } \lambda \equiv 2, 3 \pmod 4. \end{cases}$$

Then

$$\sum_{\lambda=0}^{\infty} \delta(\lambda)\, x^{S(\lambda)} = \sum_{\substack{\lambda=0 \\ \lambda \equiv 0,1 \,(\mathrm{mod}\, 4)}}^{\infty} \delta(\lambda)\, x^{S(\lambda)} + \sum_{\substack{\lambda=2 \\ \lambda \equiv 2,3 \,(\mathrm{mod}\, 4)}}^{\infty} \delta(\lambda)\, x^{S(\lambda)}$$

$$= \sum_{k=0}^{\infty} (-1)^k (x^{20k^2-k} + x^{20k^2+9k+1}) + \sum_{k=1}^{\infty} (-1)^k (x^{20k^2-9k+1} + x^{20k^2+k})$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k x^{20k^2-k} + x \sum_{k=-\infty}^{\infty} (-1)^k x^{20k^2+9k}$$

$$= f(-x^{19}, -x^{21}) + x f(-x^{29}, -x^{11})$$

and

$$\sum_{\lambda=0}^{\infty} \varepsilon(\lambda) x^{R(\lambda)} = \sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1 \pmod 4}}^{\infty} \varepsilon(\lambda) x^{R(\lambda)} + \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3 \pmod 4}}^{\infty} \varepsilon(\lambda) x^{R(\lambda)}$$

$$= \sum_{k=0}^{\infty} (-1)^k (x^{20k^2+11k+2} - x^{20k^2+21k+6})$$

$$- \sum_{k=1}^{\infty} (-1)^k (x^{20k^2-21k+6} - x^{20k^2-11k+2})$$

$$= x^2 f(-x^{31}, -x^9) - x^6 f(-x^{41}, -x^{-1}).$$

Therefore

$$C_0(q) = C_0(x^{-1}) = \frac{f(-x^{19}, -x^{21}) + x f(-x^{29}, -x^{11})}{x^2 f(-x^{31}, x^9) - x^6 f(-x^{41}, x^{-1})}$$

$$= \frac{(x^{19}; x^{40})_\infty (x^{21}; x^{40})_\infty + x (x^{29}; x^{40})_\infty (x^{11}; x^{40})_\infty}{x^2 (x^{31}; x^{40})_\infty (x^9; x^{40})_\infty - x^6 (x^{41}; x^{40})_\infty (x^{-1}; x^{40})_\infty}.$$

On the other hand,

$$\sum_{\lambda=0}^{\infty} \delta(\lambda) x^{S_1(\lambda)} = \sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1 \pmod 4}}^{\infty} \delta(\lambda) x^{S_1(\lambda)} + \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3 \pmod 4}}^{\infty} \delta(\lambda) x^{S_1(\lambda)}$$

$$= \sum_{k=0}^{\infty} (-1)^k (x^{20k^2+7k} + x^{20k^2+17k+3})$$

$$+ \sum_{k=1}^{\infty} (-1)^k (x^{20k^2-17k+3} + x^{20k^2-7k})$$

$$= \sum_{k=-\infty}^{\infty} (-1)^{k^2} x^{20k^2+7k} + x^3 \sum_{k=-\infty}^{\infty} (-1)^{k^2} x^{20k^2+17k}$$

$$= f(-x^{27}, -x^{13}) + x^3 f(-x^{37}, -x^3)$$

and

$$\sum_{\lambda=0}^{\infty} \varepsilon(\lambda) x^{R_1(\lambda)} = \sum_{\substack{\lambda=0 \\ \lambda\equiv 0,1 \pmod 4}}^{\infty} \varepsilon(\lambda) x^{R_1(\lambda)} + \sum_{\substack{\lambda=2 \\ \lambda\equiv 2,3 \pmod 4}}^{\infty} \varepsilon(\lambda) x^{R_1(\lambda)}$$

$$= \sum_{k=0}^{\infty} (-1)^k (x^{20k^2+3k} - x^{20k^2+13k+2})$$

$$+ \sum_{k=1}^{\infty} (-1)^k (-x^{20k^2-13k+2} + x^{20k^2-3k})$$

$$= f(-x^{23}, -x^{17}) - x^2 f(-x^{33}, -x^7).$$

Therefore

$$C_e(q) = C_e(x^{-1}) = \frac{(x^{27};x^{40})_\infty (x^{13};x^{40})_\infty + x^3 (x^3;x^{40})_\infty (x^{37};x^{40})_\infty}{(x^{17};x^{40})_\infty (x^{23};x^{40})_\infty - x^2 (x^7;x^{40})_\infty (x^{33};x^{40})_\infty}.$$

The theorem follows immediately.

THEOREM 12. *Let*

$$S_3(q) = 1 + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \frac{q^4+q^8}{1} + \dots,$$

where $|q| > 1$. *Set* $x = q^{-1}$. *Then the odd indexed convergents tend to*

$$\frac{(-x^{11};x^{24})_\infty (-x^{13};x^{24})_\infty + x(-x^5;x^{24})_\infty (-x^{19};x^{24})_\infty}{x^2(-x^3;x^{24})_\infty (-x^{21};x^{24})_\infty},$$

while the even indexed convergents tend to

$$\frac{(-x^7;x^{24})_\infty (-x^{17};x^{24})_\infty + x^3(-x^{25};x^{24})_\infty (-x^{-1};x^{24})_\infty}{(-x^9;x^{24})_\infty (-x^{15};x^{24})_\infty}.$$

Proof. From (3.32) and (5.1), we find that

$$\begin{aligned}
P_{2n-1}(q) = P_{2n-1}(x^{-1}) &= \sum_{\lambda=0}^{\infty} x^{-f(\lambda)} \begin{bmatrix} 2n \\ n-a(\lambda) \end{bmatrix}_{x^{-2}} \\
&= \sum_{\lambda=0}^{2n} x^{-f(\lambda)} \begin{bmatrix} 2n \\ n-a(\lambda) \end{bmatrix}_{x^{-2}} \\
&= \sum_{\lambda=0}^{2n} x^{-2n^2+2a(\lambda)^2-f(\lambda)} \begin{bmatrix} 2n \\ n-a(\lambda) \end{bmatrix}_{x^2} \\
&= x^{-2n^2} \sum_{\lambda=0}^{2n} \delta(\lambda) x^{S(\lambda)} \begin{bmatrix} 2n \\ n-a(\lambda) \end{bmatrix}_{x^2}, \\
P_{2n}(q) = P_{2n}(x^{-1}) &= \sum_{\lambda=0}^{\infty} x^{-f(\lambda)} \begin{bmatrix} 2n+1 \\ n-b(\lambda) \end{bmatrix}_{x^{-2}} \\
&= \sum_{\lambda=0}^{2n} x^{-2n^2-2n+2b(\lambda)^2+2b(\lambda)-f(\lambda)} \begin{bmatrix} 2n+1 \\ n-b(\lambda) \end{bmatrix}_{x^2} \\
&= x^{-2n^2-2n} \sum_{\lambda=0}^{2n} \delta(\lambda) x^{S_1(\lambda)} \begin{bmatrix} 2n+1 \\ n-2 \end{bmatrix}_{x^2}, \\
Q_{2n-1}(q) = Q_{2n-1}(x^{-1}) &= \sum_{\lambda=0}^{\infty} x^{-g(\lambda)} \begin{bmatrix} 2n \\ n-c(\lambda) \end{bmatrix}_{x^{-2}} \\
&= \sum_{\lambda=0}^{2n} x^{-2n^2+2c(\lambda)^2-g(\lambda)} \begin{bmatrix} 2n \\ n-c(\lambda) \end{bmatrix}_{x^2} \\
&= x^{-2n^2} \sum_{\lambda=0}^{2n} x^{R(\lambda)} \begin{bmatrix} 2n \\ n-c(\lambda) \end{bmatrix}_{x^2},
\end{aligned}$$

$$Q_{2n}(q) = Q_{2n}(x^{-1}) = \sum_{\lambda=0}^{\infty} x^{-g(\lambda)} \begin{bmatrix} 2n+1 \\ n-d(\lambda) \end{bmatrix}_{x^{-2}}$$

$$= \sum_{\lambda=0}^{2n} x^{-2n^2-2n+2d(\lambda)^2+2d(\lambda)-g(\lambda)} \begin{bmatrix} 2n+1 \\ n-d(\lambda) \end{bmatrix}_{x^2}$$

$$= x^{-2n^2-2n} \sum_{\lambda=0}^{2n} x^{R_1(\lambda)} \begin{bmatrix} 2n+1 \\ n-d(\lambda) \end{bmatrix}_{x^2},$$

where

$$S(\lambda) = 2a(\lambda)^2 - f(\lambda) = 2\left(\lambda - \left[\frac{\lambda}{4}\right]\right)^2 - (\tfrac{3}{8}\lambda^2 + \tfrac{3}{8}\lambda + \tfrac{1}{16}) + (-1)^\lambda (\tfrac{1}{8}\lambda + \tfrac{1}{16}),$$

$$S_1(\lambda) = 2b(\lambda)^2 + 2b(\lambda) - f(\lambda)$$

$$= 2\left(\lambda - \left[\frac{\lambda+2}{4}\right]\right)^2 + 2\left(\lambda - \left[\frac{\lambda+2}{4}\right]\right) - (\tfrac{3}{8}\lambda^2 + \tfrac{3}{8}\lambda + \tfrac{1}{16}) + (-1)^\lambda (\tfrac{1}{8}\lambda + \tfrac{1}{16}),$$

$$R(\lambda) = 2C(\lambda)^2 - g(\lambda) = 2\left(\lambda - \left[\frac{\lambda}{2}\right] + 1\right)^2 - \tfrac{3}{2}\lambda(\lambda+1),$$

$$R_1(\lambda) = 2d(\lambda)^2 + 2d(\lambda) - g(\lambda) = 2\left(\lambda + \left[\frac{\lambda+1}{2}\right]\right)^2 + 2\left(\lambda + \left[\frac{\lambda+1}{2}\right]\right) - \tfrac{3}{2}\lambda(\lambda+1).$$

Therefore

$$C_0(q) = \frac{\sum_{\lambda=0}^{\infty} x^{S(\lambda)}}{\sum_{\lambda=0}^{\infty} x^{R(\lambda)}} = \frac{\sum_{\substack{\lambda=0 \\ \lambda \equiv 0,1 \pmod 4}}^{\infty} x^{S(\lambda)} + \sum_{\substack{\lambda=2 \\ \lambda = 2,3 \pmod 4}}^{\infty} x^{S(\lambda)}}{\sum_{\substack{\lambda=0 \\ \lambda \equiv 0 \pmod 2}}^{\infty} x^{R(\lambda)} + \sum_{\substack{\lambda=1 \\ \lambda \equiv 1 \pmod 2}}^{\infty} x^{R(\lambda)}}$$

$$= \frac{\sum_{k=0}^{\infty} (x^{12k^2-k} + x^{12k^2+7k+1}) + \sum_{k=1}^{\infty} (x^{12k^2-7k+1} + x^{12k^2+k})}{\sum_{k=0}^{\infty} x^{12k^2+9k+2} + \sum_{k=1}^{\infty} x^{12k^2-9k+2}}$$

$$= \frac{f(x^{11}, x^{13}) + xf(x^{19}, x^5)}{x^2 f(x^{21}, x^3)}$$

$$= \frac{(-x^{11}; x^{24})_\infty (-x^{13}; x^{24})_\infty + x(-x^5; x^{24})_\infty (-x^{19}; x^{24})_\infty}{x^2 (-x^3; x^{24})_\infty (-x^{21}; x^{24})_\infty},$$

and

$$C_e(q) = C_e(x^{-1}) = \frac{\sum_{\lambda=0}^{\infty} x^{S_1(\lambda)}}{\sum_{\lambda=0}^{\infty} x^{R_1(\lambda)}} = \frac{\sum_{\substack{\lambda=0\\ \lambda\equiv 0,1 \pmod 4}}^{\infty} x^{S_1(\lambda)} + \sum_{\substack{\lambda=2\\ \lambda\equiv 2,3 \pmod 4}}^{\infty} x^{S_1(\lambda)}}{\sum_{\substack{\lambda=0\\ \lambda\equiv 0 \pmod 2}}^{\infty} x^{R_1(\lambda)} + \sum_{\substack{\lambda=1\\ \lambda\equiv 1 \pmod 2}}^{\infty} x^{R_1(\lambda)}}.$$

$$= \frac{\sum_{k=0}^{\infty} (x^{12k^2+5k} + x^{12k^2+13k+3}) + \sum_{k=1}^{\infty} (x^{12k^2-13k+3} + x^{12k^2-5k})}{\sum_{k=0}^{\infty} x^{12k^2+3k} + \sum_{k=1}^{\infty} x^{12k^2-3k}}$$

$$= \frac{f(x^{17}, x^7) + x^3 f(x^{25}, x^{-1})}{f(x^{15}, x^9)}$$

$$= \frac{(-x^7; x^{24})_\infty (-x^{17}; x^{24})_\infty + x^3 (-x^{25}; x^{24})_\infty (-x^{-1}; x^{24})_\infty}{(-x^9; x^{24})_\infty (-x^{15}; x^{24})_\infty}.$$

Thus theorem has been proved.

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