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On some complex explicit formulae connected with the Möbius function. II

by

K. M. BARTZ (Poznań)

1. The present paper is a sequel to [1] and the notation of that paper is used throughout. We show an application of the theory of the m-function presented in [1].

The explicit formula which expresses the summatory function of the Möbius μ-function, $M(x) = \sum_{n\leqslant x}\mu(n)$, in terms of the zeros of the Riemann zeta function $\zeta(s)$ is well known. According to Titchmarsh ([5], p. 318) if we set

$$M_0(x) = \big(M(x-0)+M(x+0)\big)/2$$

then under the assumption of the Riemann hypothesis and the simplicity of the complex zeros ϱ of $\zeta(s)$ we have

$$M_0(x) = \sum_{\varrho}\frac{x^{\varrho}}{\varrho\,\zeta'(\varrho)} - 2 - \sum_{n=1}^{\infty}\frac{(-1)^n(2\pi/x)^{2n}}{(2n)!\,n\,\zeta(2n+1)} \tag{1.1}$$

where the sum over ϱ converges after grouping the terms into suitable blocks.

The main purpose of the present paper is to obtain the classical formula (1.1) without assuming the Riemann hypothesis, as an application of the analytic properties of the m-function presented in [1].

More precisely, we prove the following theorem.

THEOREM. *There is a sequence T_n, $2^{n-1}T_0 \leqslant T_n < 2^n T_0$ $(n \geqslant 1)$, where T_0 is an absolute positive constant, such that*

$$M_0(x) = \lim_{n\to\infty}\sum_{\substack{\varrho\\ |\mathrm{Im}\,\varrho|<T_n}}\frac{1}{(k_\varrho-1)!}\frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}}\left[(s-\varrho)^{k_\varrho}\frac{x^s}{s\,\zeta(s)}\right]_{s=\varrho} - 2 - \sum_{n=1}^{\infty}\frac{(-1)^n(2\pi/x)^{2n}}{(2n!)\,n\,\zeta(2n+1)}, \tag{1.2}$$

where k_ϱ denotes the order of multiplicity of the non-trivial zero ϱ of the Riemann zeta function.

2. In the proof of the theorem we shall need the following lemmas.

LEMMA 1. (H. Montgomery, cf. [2], Th. 9.4.) *For any given* $\varepsilon > 0$ *there exists a* $T_0 = T_0(\varepsilon)$ *such that for* $T \geqslant T_0$ *the following holds: Between* T *and* $2T$ *there exists a* t *for which*

$$|\zeta(\sigma \pm it)|^{-1} < c_1 t^{\varepsilon} \quad \textit{for} \ -1 \leqslant \sigma \leqslant 2,$$

with an absolute constant $c_1 > 0$.

Remark on notation. Let T_n, where

(2.1) $$2^{n-1} T_0 \leqslant T_n < 2^n T_0, \quad n = 1, 2, \ldots$$

be such that by Lemma 1

(2.2) $$|\zeta(\sigma + iT_n)|^{-1} < c_1 T_n^{1/2} \quad \text{for} \ -1 \leqslant \sigma \leqslant 2.$$

The function $\zeta(s)$ has no zeros on the line $t = T_n$ of course. This yields the grouping of the non-trivial zeros ϱ of $\zeta(s)$ which will be used in the present paper.

The next lemma recalls some basic facts on the m-function proved in [1].

LEMMA 2 (see [1], Ths. 1, 2, 3). *For* $z = x+iy$, $y > 0$ *we define the* m*-function by the formula*

(2.3) $$m(z) = \lim_{n \to \infty} \sum_{\substack{\varrho \\ 0 < \operatorname{Im}\varrho < T_n}} \frac{1}{(k_\varrho - 1)!} \frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}} \left[(s-\varrho)^{k_\varrho} \frac{e^{sz}}{\zeta(s)} \right]_{s=\varrho},$$

where k_ϱ *denotes the order of multiplicity of the non-trivial zero* ϱ *of* $\zeta(s)$. *The function* $m(z)$ *is holomorphic on the upper half-plane and can be continued analytically to a meromorphic function on the whole complex plane, which satisfies the functional equation*

(2.4) $$m(z) + \overline{m(\bar{z})} = A(z),$$

where

$$A(z) = -2 \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cos\left(\frac{2\pi}{n} e^{-z}\right)$$

is an entire function of order 1 *of the variable* $z_1 = e^{-z}$. *The only singularities of* $m(z)$ *are simple poles at the points* $z = \log n$ *on the real axis, where* n *is a product of different primes or* n *is equal to* 1 *with residues*

$$\operatorname*{res}_{z = \log n} m(z) = -\mu(n)/2\pi i.$$

Remark. In the previous note [1] we have chosen T_n $(n \leqslant T_n \leqslant n+1)$ in such a way that

$$|\zeta(\sigma + iT_n)|^{-1} < c_2 T_n^{c_3} \quad \text{for} \ -1 \leqslant \sigma \leqslant 2$$

where c_2 and c_3 are absolute constants. It is easy to see that all theorems on the m-function are also true if we group the zeros of $\zeta(s)$ using Lemma 1.

LEMMA 3. *The series $m(z)$, $z = x+iy$, is uniformly convergent for $y \geqslant \delta > 0$, almost uniformly with respect to x.*

Proof. We have

$$m(z) = \sum_{n=1}^{\infty}\Bigg(\sum_{\substack{\varrho \\ T_n < \operatorname{Im}\varrho < T_{n+1}}} \frac{1}{(k_\varrho - 1)!}\frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}}\left[(s-\varrho)^{k_\varrho}\frac{e^{sz}}{\zeta(s)}\right]_{s=\varrho}\Bigg)$$

$$+ \sum_{\substack{\varrho \\ 14 < \operatorname{Im}\varrho < T_1}} \frac{1}{(k_\varrho - 1)!}\frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}}\left[(s-\varrho)^{k_\varrho}\frac{e^{sz}}{\zeta(s)}\right]_{s=\varrho} = \sum_{n=0}^{\infty} m_n(z)$$

where by Cauchy's integral formula, for $n \geqslant 1$

$$m_n(z) = \frac{1}{2\pi i}\Big(\int_{-1/2+iT_n}^{3/2+iT_n} + \int_{3/2+iT_n}^{3/2+iT_{n+1}} + \int_{3/2+iT_{n+1}}^{-1/2+iT_{n+1}} + \int_{-1/2+iT_{n+1}}^{-1/2+iT_n}\Big)\frac{e^{sz}}{\zeta(s)}\,ds$$

$$= m_{n1}(z)+m_{n2}(z)+m_{n3}(z)+m_{n4}(z).$$

For $y \geqslant \delta > 0$ we have by (2.2)

$$|m_{n1}(z)| = \frac{1}{2\pi}\left|\int_{-1/2+iT_n}^{3/2+iT_n}\frac{e^{sz}}{\zeta(s)}\,ds\right| < \int_{-1/2}^{3/2}\frac{e^{\sigma x - T_n y}}{|\zeta(\sigma+iT_n)|}\,d\sigma \ll \frac{e^{3|x|/2}\,T_n^{1/2}}{e^{T_n\delta}},$$

$$|m_{n2}(z)| = \frac{1}{2\pi}\left|\int_{3/2+iT_n}^{3/2+iT_{n+1}}\frac{e^{sz}}{\zeta(s)}\,ds\right| \ll e^{3|x|/2}\int_{T_n}^{T_{n+1}}\frac{dt}{e^{ty}} \ll \frac{e^{3|x|/2}}{\delta}\left(e^{-T_n\delta}-e^{-T_{n+1}\delta}\right),$$

$$|m_{n3}(z)| < \int_{-1/2}^{3/2}\frac{e^{\sigma x - T_{n+1}y}}{|\zeta(\sigma+iT_{n+1})|}\,d\sigma \ll \frac{e^{3|x|/2}\,T_{n+1}^{1/2}}{e^{T_{n+1}\delta}},$$

$$|m_{n4}(z)| \ll e^{|x|/2}\int_{T_n}^{T_{n+1}}\frac{dt}{e^{ty}} \ll \frac{e^{|x|/2}}{\delta}\left(e^{-T_n\delta}-e^{-T_{n+1}\delta}\right).$$

Similarly we estimate $m_0(z)$ since

$$m_0(z) = \frac{1}{2\pi i}\Big(\int_{-1/2+14i}^{3/2+14i} + \int_{3/2+14i}^{3/2+T_1 i} + \int_{3/2+T_1 i}^{-1/2+T_1 i} + \int_{-1/2+T_1 i}^{-1/2+14i}\Big)\frac{e^{sz}}{\zeta(s)}\,ds$$

and $|\zeta(\sigma+14i)|^{-1} < c_4$ for $-1 \leqslant \sigma \leqslant 2$.

Finally, we observe that $\sum_{n=0}^{\infty}|m_{n1}(z)|$, $\sum_{n=0}^{\infty}|m_{n2}(z)|$, $\sum_{n=0}^{\infty}|m_{n3}(z)|$ and $\sum_{n=0}^{\infty}|m_{n4}(z)|$ are convergent for $\delta > 0$ and by the Weierstrass criterion the series $m(z)$ is uniformly convergent for $y \geqslant \delta > 0$ almost uniformly for all x.

The next lemma is a generalization of the classical theorem of M. Riesz [4].

LEMMA 4 (see [3], Th. 4.2). *Let $w_n = a_n + ib_n$ ($n = 1, 2, \ldots$) be complex numbers such that $|a_n| \leqslant A$, $b_1 \leqslant b_2 \leqslant \ldots$, $\lim_{n\to\infty} b_n = \infty$ and let c_5 be an absolute positive constant. Moreover, let the series*

$$f(z) = \sum_{n=1}^{\infty} A_n e^{w_n z}, \qquad A_n \in C,$$

converge for $y = \operatorname{Im} z > 0$ *and satisfy the conditions*

$$(2.5) \qquad \Big|\sum_{n=N+1}^{\infty} A_n e^{(w_n - w_N)z}\Big| = o(|y|^{-c_5}), \qquad N \to \infty,$$

for $y \to 0^+$ *almost uniformly with respect to* $x = \operatorname{Re} z$, *and*

$$(2.6) \qquad \Big|\sum_{n=1}^{N} A_n e^{(w_n - w_N)z}\Big| = o(|y|^{-c_5}), \qquad N \to \infty,$$

for $y \to 0^-$ *also almost uniformly with respect to* $x = \operatorname{Re} z$.

If f *is holomorphic at the boundary point* $x_0 \in \boldsymbol{R}$ *then the series* $\sum_{n=1}^{\infty} A_n \exp(w_n x_0)$ *is convergent to* $f(x_0)$. *Moreover, the convergence is uniform on every compact real interval consisting of regular points of* f *only.*

Remark. It is easy to verify that this lemma is also true if the series converges after grouping the terms into suitable blocks. In [3] this lemma is formulated with $c_5 = 1$, but it is easy to see that we can take $c_5 > 1$ as well. In the following we take $c_5 = 2$.

LEMMA 5 (see [3], Th. 4.3). *Let* f *be as in Lemma* 4. *Suppose that for some* $x_0 \in \boldsymbol{R}$, $g \in \boldsymbol{C}$ *and* $r_0 > 0$ *we have*

$$f(z) = g \log(z - x_0) + h(z)$$

for $|z - x_0| < r_0$, $\operatorname{Im} z > 0$, *where h is holomorphic in the whole disc* $|z - x_0| < r_0$. *Then for* T *tending to infinity*

$$\sum_{\operatorname{Im} w_n < T} A_n e^{w_n x_0} = -g \log T - gC + h(x_0) + g\frac{\pi i}{2} + o(1)$$

where C *is the Euler constant.*

3. In this section we consider a function analogous to the series over the non-trivial zeros of the zeta function in (1.1), but as a function of the complex variable x.

For $\operatorname{Im} z > 0$ we define $\mathcal{M}(z)$ by

$$(3.1) \qquad \mathcal{M}(z) = \int_{z+i\infty}^{z} m(s)\, ds,$$

the path of integration being the half-line $s = z + iy$, $0 \leqslant y \leqslant \infty$. For any a on this half-line

$$\mathcal{M}(z) = \mathcal{M}(a) + \int_a^z m(s)\, ds.$$

The function $\mathcal{M}$ can be continued analytically along any path on the complex plane not passing through the poles of the m-function. $\mathcal{M}$ has logarithmic branch points at $z = \log n$, where n is equal to 1 or n is a product of different primes (see Lemma 2). For those n there exists $\eta = \eta(n) > 0$ such that for $|z - \log n| < \eta$

$$\mathcal{M}(z) = -\frac{\mu(n)}{2\pi i}\log(z - \log n) + g(z) \tag{3.2}$$

where g is holomorphic in the disc $|z - \log n| < \eta$ and depends on the choice of the particular branch of $\mathcal{M}$.

For a real x let us write

$$\mathcal{M}_0(x) = \lim_{y\to 0^+} \operatorname{Re}\mathcal{M}(x+iy). \tag{3.3}$$

From the above analysis it follows that this limit exists for every x which is a regular point of $\mathcal{M}$, and also for $x = \log n$, where n is equal to 1 or n is a product of different primes, by (3.2), since $\lim_{y\to 0^+} \operatorname{Arg}(iy) = \pi/2$. Furthermore, since for any $x_0 > 0$, $\lim_{y\to 0^+} \operatorname{Arg}(x_0+iy) = 0$, $\lim_{y\to 0^+} \operatorname{Arg}(-x_0+iy) = \pi$ and $\lim_{y\to 0^+} \operatorname{Arg}(iy) = \pi/2$, we have

$$\mathcal{M}_0(x) = \big(\mathcal{M}_0(x+0) + \mathcal{M}_0(x-0)\big)/2 \tag{3.4}$$

for every real x.

Let us remark that since the series $m(z)$, $z = x+iy$, is uniformly convergent for $y \geqslant \delta > 0$ almost uniformly with respect to x (see Lemma 3), we can invert summation and integration in (3.1), and we get for $\operatorname{Im} z > 0$

$$\mathcal{M}(z) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho - 1)!}\frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}}\left[(s-\varrho)^{k_\varrho}\frac{e^{sz}}{s\zeta(s)}\right]_{s=\varrho} \tag{3.5}$$

and further for $\operatorname{Im} z > 0$

$$\begin{aligned}\mathcal{M}(z) &= \Big(\sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_1}} + \sum_{n=1}^{\infty}\sum_{\substack{\varrho \\ T_n<\operatorname{Im}\varrho<T_{n+1}}}\Big)\frac{1}{(k_\varrho - 1)!}\frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}}\left[(s-\varrho)^{k_\varrho}\frac{e^{sz}}{s\zeta(s)}\right]_{s=\varrho} \\ &= \sum_{n=0}^{\infty}\mathcal{M}_n(z).\end{aligned} \tag{3.6}$$

What we need is the uniform convergence of the series $\sum_{n=1}^{\infty}\mathcal{M}_n(z)$ for $\operatorname{Im} z \geqslant 0$. We have

LEMMA 6. *For $y = \operatorname{Im} z \geqslant \eta > 0$ the series $\sum_{n=0}^{\infty}\mathcal{M}_n(z)$ converges to $\mathcal{M}(z)$ uniformly with respect to y, almost uniformly with respect to $x = \operatorname{Re} z$. If $y = 0$ and x is not equal to* $\log n$, *where n is* 1 *or n is a product of different primes, then the series $\sum_{n=0}^{\infty}\mathcal{M}_n(x)$ is also convergent to $\mathcal{M}(x)$ and the convergence is uniform in every closed interval containing no points of the form* $\log n$.

Proof. The Cauchy integral formula states that for $n \geqslant 1$

$$\mathcal{M}_n(z) = \frac{1}{2\pi i}\Big(\int\limits_{-1/2+iT_n}^{3/2+iT_n} + \int\limits_{3/2+iT_n}^{3/2+iT_{n+1}} + \int\limits_{3/2+iT_{n+1}}^{-1/2+iT_{n+1}} + \int\limits_{-1/2+iT_{n+1}}^{-1/2+iT_n}\Big)\frac{e^{sz}}{s\zeta(s)}\,ds$$
$$= \mathcal{M}_{n1}(z)+\mathcal{M}_{n2}(z)+\mathcal{M}_{n3}(z)+\mathcal{M}_{n4}(z)$$

where for $y \geqslant \delta > 0$ we have by (2.2)

$$(3.7)\qquad |\mathcal{M}_{n1}(z)| = \frac{1}{2\pi}\left|\int\limits_{-1/2+iT_n}^{3/2+iT_n} \frac{e^{sz}}{s\zeta(s)}\,ds\right| < \int\limits_{-1/2}^{3/2} \frac{e^{\sigma x - T_n y}}{|\sigma+iT_n|\,|\zeta(\sigma+iT_n)|}\,d\sigma$$
$$\ll \frac{e^{3|x|/2}}{T_n^{1/2}\,e^{T_n\delta}},$$

$$(3.8)\qquad |\mathcal{M}_{n2}(z)| = \frac{1}{2\pi}\left|\int\limits_{3/2+iT_n}^{3/2+iT_{n+1}} \frac{e^{sz}}{s\zeta(s)}\,ds\right| \ll e^{3|x|/2}\,T_n^{-1}\int\limits_{T_n}^{T_{n+1}}\frac{dt}{e^{ty}}$$
$$\ll \frac{e^{3|x|/2}}{\delta T_n}\left(e^{-T_n\delta} - e^{-T_{n+1}\delta}\right),$$

$$(3.9)\qquad |\mathcal{M}_{n3}(z)| < \int\limits_{-1/2}^{3/2} \frac{e^{\sigma x - T_{n+1}y}}{|\sigma+iT_{n+1}|\,|\zeta(\sigma+iT_{n+1})|}\,d\sigma \ll \frac{e^{3|x|/2}}{T_{n+1}^{1/2}\,e^{T_{n+1}\delta}},$$

$$(3.10)\qquad |\mathcal{M}_{n4}(z)| \ll e^{|x|/2}\,T_n^{-1}\int\limits_{T_n}^{T_{n+1}}\frac{dt}{e^{ty}} \ll \frac{e^{|x|/2}}{\delta T_n}\left(e^{-T_n\delta} - e^{-T_{n+1}\delta}\right)$$

and similarly we estimate $|\mathcal{M}_0(z)|$, since $|\zeta(\sigma+i14)|^{-1} < c_4$ for $-1 \leqslant \sigma \leqslant 2$.

Finally, we see that $\sum_{n=0}^{\infty}|\mathcal{M}_{n2}(z)|$ and $\sum_{n=0}^{\infty}|\mathcal{M}_{n4}(z)|$ are convergent for $\delta > 0$ and $\sum_{n=0}^{\infty}|\mathcal{M}_{n1}(z)|$ and $\sum_{n=0}^{\infty}|\mathcal{M}_{n3}(z)|$ for $\delta \geqslant 0$. Hence by the Weierstrass criterion the series $\mathcal{M}(z)$ is uniformly convergent for $y \geqslant \delta > 0$, almost uniformly for all x.

Next we prove the convergence of the series $\sum_{n=0}^{\infty}\mathcal{M}_n(z)$ on the line $y = 0$ at regular points of $m(z)$. Let $x \neq \log n$, where n is a product of different primes or $n = 1$. Let us number the complex zeros of $\zeta(s)$ lying on the upper half-plane according to increasing imaginary parts: $\varrho_1, \varrho_2, \ldots$, so that $\operatorname{Im}\varrho_n \leqslant \operatorname{Im}\varrho_{n+1}$. Let $\varrho_k = \sigma_k + it_k$ be the last zero before the line $t = T_N$, where

$$|\zeta(\sigma+iT_N)|^{-1} \ll T_N^{1/2} \quad \text{for } -1 \leqslant \sigma \leqslant 2$$

and where there are no zeros of $\zeta(s)$. We have by (2.1)

$$2^{N-2}T_0 \leqslant T_{N-1} < t_k < T_N < 2^N T_0 \quad \text{for } N \text{ sufficiently large.}$$

First we verify the condition (2.5) of Lemma 4. By Cauchy's integral formula, since

$$\sum_{\substack{\varrho \\ T_{n-1} < \operatorname{Im}\varrho < T_n}} \frac{1}{(k_\varrho - 1)!}\frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}}\left[(s-\varrho)^{k_\varrho}\frac{e^{(s-\varrho_k)z}}{s\zeta(s)}\right]_{s=\varrho} = e^{-\varrho_k z}\mathcal{M}_{n-1}(z),$$

using the estimates (3.7)–(3.10), we get for $y > 0$

$$B_1(z) = \left| \sum_{n=N+1}^{\infty} \sum_{\substack{\varrho \\ T_{n-1} < \operatorname{Im}\varrho < T_n}} \frac{1}{(k_\varrho - 1)!} \frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}} \left[(s-\varrho)^{k_\varrho} \frac{e^{(s-\varrho_k)z}}{s\,\zeta(s)} \right]_{s=\varrho} \right|$$

$$\ll \frac{e^{5|x|/2}}{y} \sum_{n=N+1}^{\infty} \frac{1}{T_{n-1}^{1/2}\, e^{y(T_{n-1}-t_k)}}$$

$$\ll \frac{e^{5|x|/2}}{y} \left(T_N^{-1/2} + T_{N+1}^{-1/2} + \sum_{n=N+3}^{\infty} \frac{1}{T_{n-1}^{1/2}\,(T_{n-1}-t_k)\,y} \right).$$

Further, since for $n \geqslant N+3$ we have $T_{n-1} - t_k > T_{n-1}/2$ $(T_{n-1} \geqslant 2^{n-2}\, T_0 \geqslant 2^{N+1}\, T_0 > 2\, T_N > 2\, t_k)$, we obtain

$$B_1(z) \ll \frac{e^{5|x|/2}}{y} \left(2^{-N/2} + \frac{1}{y} \sum_{n=N+3}^{\infty} 2^{-3n/2} \right) \underset{N\to\infty}{=} o(y^{-2})$$

for $y \to 0^+$ almost uniformly with respect to $x = \operatorname{Re} z$.

Next, applying (3.7)–(3.10), we get

$$B_2'(z) = \left| \sum_{n=2}^{N} \sum_{\substack{\varrho \\ T_{n-1} < \operatorname{Im}\varrho < T_n}} \frac{1}{(k_\varrho - 1)!} \frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}} \left[(s-\varrho)^{k_\varrho} \frac{e^{(s-\varrho_k)z}}{s\,\zeta(s)} \right]_{s=\varrho} \right|$$

$$\ll \frac{e^{3|x|/2}}{|y|} \sum_{n=2}^{N} \frac{1}{T_{n-1}^{1/2}\, e^{y(T_n - t_k)}}$$

$$\ll \frac{e^{3|x|/2}}{|y|} \left(\sum_{n=2}^{N-1} \frac{1}{\sqrt{2^{n-2}}\, e^{-y(t_k - T_n)}} + \frac{e^{-y(T_N - t_k)}}{T_{N-1}^{1/2}} \right)$$

and further, since for $x > 0$ we have $e^x > x$,

$$B_2'(z) \ll \frac{e^{3|x|/2}}{|y|^2} \left(\sum_{n=2}^{N-3} \frac{1}{\sqrt{2^{n-2}}\,(t_k - 2^n\, T_0)} + \frac{1}{\sqrt{2^{N-4}}\,(t_k - T_{N-2})} + \frac{1}{\sqrt{2^{N-3}}\,(t_k - T_{N-1})} \right) + \frac{e^{3|x|/2}}{|y|} \frac{e^{-y(T_N - t_k)}}{T_{N-1}^{1/2}}.$$

Finally, since for $4 \leqslant n \leqslant N-3$ we have $2^{(n-2)/2}\,(t_k - 2^n\, T_0) \geqslant t_k - T_0$,

$$(3.11) \qquad B_2'(z) \ll \frac{e^{3|x|/2}}{|y|^2} \left(\frac{1}{t_k - 4\, T_0} + \frac{1}{\sqrt{2}\,(t_k - 8\, T_0)} + \frac{N-6}{t_k - T_0} + \frac{1}{\sqrt{2^{N-4}}\,(t_k - T_{N-2})} + \frac{1}{\sqrt{2^{N-3}}\,(t_k - T_{N-1})} \right) + \frac{e^{3|x|/2}}{|y|} \frac{e^{-y(T_N - t_k)}}{T_{N-1}^{1/2}} \underset{N\to\infty}{=} o(|y|^{-2})$$

because it is easy to see that all factors are convergent to 0 as $N \to \infty$, since $2^{N-2}\, T_0 < t_k \leqslant 2^N\, T_0$ and $N/(t_k - T_0) \underset{N\to\infty}{\to} 0$. The last factor converges to 0 because $T_N - t_k < 8$ for N sufficiently large and since $y \to 0^-$, we can take $-y < c_6$ where c_6 is a positive constant. So, we have verified (2.6) – the second condition of Lemma 4:

$$(3.12)\qquad B_2(z) = \left| \Big(\sum_{14 < \operatorname{Im}\varrho < T_1} + \sum_{n=2}^{N} \sum_{\substack{\varrho \\ T_{n-1} < \operatorname{Im}\varrho < T_n}} \Big) \frac{1}{(k_\varrho - 1)!} \frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}} \times \left[(s-\varrho)^{k_\varrho} \frac{e^{(s-\varrho_k)z}}{s\,\zeta(s)} \right]_{s=\varrho} \right| \underset{N\to\infty}{=} o(|y|^{-2})$$

for $y \to 0^-$ almost uniformly with respect to $x = \operatorname{Re} z$.

Hence by Lemma 4 according to (3.11) and (3.12) the proof is complete.

4. Proof of the Theorem. Suppose first that $x > 0$ and x is not equal to a product of different primes and x is not equal to 1. Then by Lemma 2

$$M_0(x) = M(x) = \sum_{n \leqslant x} \mu(n) = -2\pi i \sum_{n \leqslant x} \operatorname*{res}_{z=\log n} m(z)$$

and by Cauchy's integral formula, since $m(z)$ is regular at $z = \log x$ we have for any $a > 0$

$$(4.1)\qquad M(x) = \int_{l(-a,\log x)} m(z)\,dz - \int_{\overline{l(-a,\log x)}} m(z)\,dz$$

where l is a simple and smooth curve $\tau\colon [0,1] \to \boldsymbol{C}$ such that $-\tau(0) = a$, $\tau(1) = \log x$ and $\operatorname{Im}\tau(t) > 0$ for $t \in (0,1)$. By the functional equation for $m(z)$ (see (2.4)) we obtain

$$\int_{\overline{l(-a,\log x)}} m(z)\,dz = \int_{l(-a,\log x)} m(\bar z)\,d\bar z = \int_{l(-a,\log x)} \big(-\overline{m(z)} + \overline{A(z)}\big)\,d\bar z$$
$$= -\overline{\int_{l(-a,\log x)} m(z)\,dz} + \overline{\int_{l(-a,\log x)} A(z)\,dz}$$

and by (4.1)

$$(4.2)\qquad M(x) = 2\operatorname{Re} \int_{l(-a,\log x)} m(z)\,dz - \overline{\int_{l(-a,\log x)} A(z)\,dz}$$

where

$$(4.3)\qquad \int_{l(-a,\log x)} A(z)\,dz = -2 \int_{-a}^{\log x} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cos\frac{2\pi}{ne^z}\,dz = -\int_{-a}^{\log x} \sum_{n=1}^{\infty} \frac{dz}{e^{2nz}\,\zeta'(-2n)}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n \big((2\pi)^{2n} x^{-2n} - (2\pi e^a)^{2n}\big)}{(2n)!\, n\, \zeta(2n+1)}.$$

In particular, by Cauchy's integral theorem and (3.3)

$$(4.4)\qquad \operatorname{Re} \int_{l(-a,\log x)} m(z)\,dz = \lim_{y\to 0^+} \operatorname{Re}\Big(\int_{\log x + i\infty}^{\log x + iy} - \int_{-a+i\infty}^{-a+iy} \Big) m(z)\,dz$$

$$= \lim_{y\to 0+} \operatorname{Re}\big(\mathcal{M}(\log x+iy)-\mathcal{M}(-a+iy)\big) = \mathcal{M}_0(\log x)-\mathcal{M}_0(-a)$$

because $\int_{-a+i\infty}^{\log x+i\infty} m(z)\,dz = 0$ by the uniform convergence of $m(z)$ for $\operatorname{Im} z > 0$ (Lemma 3).

Moreover, by Lemma 6 (the uniform convergence of the series $\sum_{n=0}^{\infty} \mathcal{M}_n(z)$ for $\operatorname{Im} z = 0$) we get

$$(4.5)\qquad \mathcal{M}_0(\log x) = \operatorname{Re} \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho-1)!}\frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}}\left[(s-\varrho)^{k_\varrho}\frac{x^s}{s\,\zeta(s)}\right]_{s=\varrho}$$

and

$$(4.6)\qquad \mathcal{M}_0(-a) = \operatorname{Re} \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho-1)!}\frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}}\left[(s-\varrho)^{k_\varrho}\frac{(e^{-a})^s}{s\,\zeta(s)}\right]_{s=\varrho}.$$

Therefore, by (4.2)–(4.6) we obtain

$$(4.7)\qquad M(x) = 2\operatorname{Re} \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho-1)!}\frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}}\left[(s-\varrho)^{k_\varrho}\frac{x^s-e^{-as}}{s\,\zeta(s)}\right]_{s=\varrho}$$
$$-\sum_{n=1}^{\infty} \frac{(-1)^n\left((2\pi)^{2n}x^{-2n}-(2\pi e^{a})^{2n}\right)}{(2n!)\,n\,\zeta(2n+1)}$$

for $x \neq 1$ and x not equal to a product of different primes.

It is easy to verify that

$$(4.8)\qquad 2\operatorname{Re} \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho-1)!}\frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}}\left[(s-\varrho)^{k_\varrho}\frac{x^s}{s\,\zeta(s)}\right]_{s=\varrho}$$
$$= \lim_{n\to\infty} \sum_{\substack{\varrho \\ |\operatorname{Im}\varrho|<T_n}} \frac{1}{(k_\varrho-1)!}\frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}}\left[(s-\varrho)^{k_\varrho}\frac{x^s}{s\,\zeta(s)}\right]_{s=\varrho}$$

since

$$\overline{\sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho-1)!}\frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}}\left[(s-\varrho)^{k_\varrho}\frac{x^s}{s\,\zeta(s)}\right]_{s=\varrho}}$$
$$= \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho-1)!}\frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}}\left[(s-\varrho)^{k_\varrho}\frac{x^s}{s\,\zeta(s)}\right]_{s=\varrho}$$

and setting $s = \sigma + i\tau$, we have further

$$= \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho-1)!}\frac{\partial^{k_\varrho-1}}{\partial\sigma^{k_\varrho-1}}\left[(s-\varrho)^{k_\varrho}\frac{x^s}{s\,\zeta(s)}\right]_{s=\varrho}$$

$$= \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho-1)!} \frac{\partial^{k_\varrho-1}}{\partial\sigma^{k_\varrho-1}} \left[(\bar{s}-\bar{\varrho})^{k_\varrho} \frac{x^{\bar{s}}}{\bar{s}\,\zeta(\bar{s})} \right]_{s=\varrho}$$

$$= \sum_{\substack{\varrho \\ -T_n<\operatorname{Im}\varrho<0}} \frac{1}{(k_\varrho-1)!} \frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}} \left[(s-\varrho)^{k_\varrho} \frac{x^{s}}{s\,\zeta(s)} \right]_{s=\varrho}.$$

Let k be a positive integral number and let $C_{k,n}$ denote the rectangle $(-2k+1-iT_n,\ 3/2-iT_n,\ 3/2+iT_n,\ -2k+1+iT_n)$. Then, since

$$\int_{3/2-i\infty}^{3/2+i\infty} \frac{y^s}{s}\, ds = 0 \quad \text{for } 0<y<1,$$

we have

$$\lim_{\substack{k\to\infty \\ n\to\infty}} \int_{C_{k,n}} \frac{x^s}{s\,\zeta(s)}\, ds = 0$$

for $0<x<1$ and by the calculus of residues since $\operatorname{res}_{s=0}\left(x^s/s\,\zeta(s)\right) = -2$ we deduce that for any $a>0$

$$\text{(4.9)}\quad \lim_{n\to\infty} \sum_{\substack{\varrho \\ |\operatorname{Im}\varrho|<T_n}} \frac{1}{(k_\varrho-1)!} \frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}} \left[(s-\varrho)^{k_\varrho} \frac{(e^{-a})^s}{s\,\zeta(s)} \right]_{s=\varrho} + \sum_{n=1}^{\infty} \frac{(e^a)^{2n}}{(-2n)\,\zeta'(-2n)} = 2.$$

Finally, by (4.7)–(4.9) for $x>0$, $x \neq 1$ and x not equal to a product of different primes we obtain

$$\text{(4.10)}\quad \sum_{n\le x} \mu(n) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ |\operatorname{Im}\varrho|<T_n}} \frac{1}{(k_\varrho-1)!} \frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}} \left[(s-\varrho)^{k_\varrho} \frac{x^s}{s\,\zeta(s)} \right]_{s=\varrho} - 2 - \sum_{n=1}^{\infty} \frac{(-1)^n (2\pi)^{2n} x^{-2n}}{(2n)!\, n\,\zeta(2n+1)}.$$

Now, let x be a product of different primes or $x=1$. Then $\log x$ is not a regular point of $m(s)$. We find by (4.5), (4.8) and (4.10) that

$$\mathcal{M}_0(\log x+0) = \frac{1}{2} \sum_{n\le x} \mu(n) + 1 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n (2\pi)^{2n} x^{-2n}}{(2\,n!)\, n\,\zeta(2\,n+1)},$$

$$\mathcal{M}_0(\log x-0) = \frac{1}{2} \sum_{n< x} \mu(n) + 1 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n (2\pi)^{2n} x^{-2n}}{(2\,n!)\, n\,\zeta(2\,n+1)},$$

and by (3.4)

$$\text{(4.11)}\quad 2\,\mathcal{M}_0(\log x) = M(x) - \frac{1}{2}\mu(x) + 2 + \sum_{n=1}^{\infty} \frac{(-1)^n (2\pi)^{2n} x^{-2n}}{(2\,n!)\, n\,\zeta(2\,n+1)}.$$

Finally, using (3.2) we get

$$\mathcal{M}_0(\log x) = \lim_{y\to 0^+} \operatorname{Re}(\log x + iy) = \lim_{y\to 0^+}\left(-\frac{\mu(x)}{2\pi}\operatorname{Arg}(iy) + \operatorname{Re} g(\log x + iy)\right)$$
$$= -\mu(x)/4 + \operatorname{Re} g(\log x)$$

and combining this with Lemma 5 and (4.8) we have

$$(4.12)\qquad \mathcal{M}_0(\log x) = \frac{1}{2}\lim_{n\to\infty} \sum_{\substack{\varrho \\ |\operatorname{Im}\varrho| < T_n}} \frac{1}{(k_\varrho - 1)!}\frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}}\left[(s-\varrho)^{k_\varrho}\frac{x^s}{s\,\zeta(s)}\right]_{s=\varrho}.$$

(4.11) and (4.12) complete the proof of our Theorem.

Let us remark that if we choose T_n in the way (2.1)–(2.2) in Titchmarsh's proof of formula (1.1) (see [5], p. 318), then we can omit the RH assumption, too. But if we choose T_n in the same way as Titchmarsh in our Theorem, then we can obtain the explicit formula (1.1) without RH, but instead of $\sum_\varrho x^\varrho/(\varrho\,\zeta'(\varrho))$ we will have $\lim_{y\to 0^+}\sum_\varrho e^{(\log x + iy)\varrho}/(\varrho\,\zeta'(\varrho))$, which exists as we have proved. This means that our method permits us to omit RH in a more natural way. Only in the proof of Lemma 6 (the almost uniform convergence of the series $\sum_{n=0}^{\infty}\mathcal{M}_n(x)$ to $\mathcal{M}(x)$ for real x) we need the estimate (2.2) with $\varepsilon < 1$.

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INSTITUTE OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY
Matejki 48/49, 60-769 Poznań, Poland

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