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On some complex explicit formulae connected with the Möbius function. I

by

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1. Let $M(x)$ denote the summatory function of the Möbius μ-function,

$$M(x) = \sum_{n \leqslant x} \mu(n).$$

Then $M(x)$ is the difference between the number of squarefree positive integers $n \leqslant x$ with an even number of prime factors and of those with an odd number of prime factors.

An explicit formula which expresses $M(x)$ in terms of zeros of the Riemann zeta function under the assumption of the Riemann hypothesis (RH) is well known (see e.g. [4], p. 318). Assuming the RH and the simplicity of complex zeros ϱ of the Riemann zeta function $\zeta(s)$, Titchmarsh showed that there is a sequence τ_n, $n \leqslant \tau_n \leqslant n+1$ such that, denoting

$$M_0(x) = \big(M(x+0)+M(x-0)\big)/2,$$

we have

$$M_0(x) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ |\mathrm{Im}\,\varrho| < \tau_n}} \frac{x^{\varrho}}{\varrho\,\zeta'(\varrho)} - 2 - \sum_{n=1}^{\infty} \frac{(-1)^n (2\pi/x)^{2n}}{(2n)!\, n\, \zeta(2n+1)}. \tag{1.1}$$

Following some ideas of J. Kaczorowski's paper (see [2]) we will investigate expressions similar to the series over Riemann zeta zeros ϱ in (1.1), considering this series as a function of a complex variable x. More precisely, we will describe the analytic character of some functions $m(z)$ and $\mathcal{M}(z)$ defined in the case where there are no multiple zeros ϱ of the Riemann zeta function, for $\mathrm{Im}\, z > 0$ as follows:

$$m(z) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0 < \mathrm{Im}\,\varrho < T_n}} \frac{e^{\varrho z}}{\zeta'(\varrho)} \tag{1.2}$$

and

$$\mathcal{M}(z) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{e^{\varrho z}}{\varrho\,\zeta'(\varrho)} \tag{1.3}$$

where the summation is over all non-trivial zeros ϱ of $\zeta(s)$. The sequence T_n yields a certain grouping of the zeros.

If $\zeta(s)$ has a multiple zero at $s = \varrho$, the corresponding term in (1.1), (1.2) and (1.3) must be replaced by an appropriate residue. In the following we will consider this general case.

First we show that $m(z)$ is a holomorphic function for $\operatorname{Im} z > 0$. Next we continue $m(z)$ analytically to a meromorphic function on the whole complex plane, which satisfies a certain functional equation. The functional equation for $m(z)$ connects the values of the function m at the points z and $\bar{z}$. Hence from the behaviour of $m(z)$ in the half-plane $\operatorname{Im} z > 0$ it permits one to deduce its behaviour for $\operatorname{Im} z < 0$. Finally, we describe all singularities of $m(z)$.

As an application of analytic properties of the m-function, in the next note we will obtain the classical formula (1.1) without any hypothesis.

In 1985 A. Odlyzko and H. te Riele in their joint paper [3] showed that $\limsup_{x\to\infty} |M(x)|\, x^{-1/2} > 1.06$, which yields a disproof of the Mertens conjecture. What is generally expected is that the true value of this limes superior is ∞. The method we use in this paper may be used to improve on the 1.06 constant.

2. For any complex number $z = x+iy$ from the upper half-plane $H = \{z \in C\colon \operatorname{Im} z > 0\}$ let us consider the integral

$$\int \frac{e^{sz}}{\zeta(s)}\, ds$$

taken round the rectangle $(-1/2, 3/2, 3/2+iT_n, -1/2+iT_n)$ where the T_n $(n \leqslant T_n \leqslant n+1)$ are chosen so that

$$\left|\frac{1}{\zeta(\sigma+iT_n)}\right| < T_n^{c_1} \tag{2.1}$$

for $-1 \leqslant \sigma \leqslant 2$ and c_1 is a numerical constant (see [4], Th. 9.7). Then the integral along the upper side of the contour tends to 0 as $n\to\infty$, and by Cauchy's theorem of residues

$$\int_{-1/2+i\infty}^{-1/2} \frac{e^{sz}}{\zeta(s)}\, ds + \int_{-1/2}^{3/2} \frac{e^{sz}}{\zeta(s)}\, ds + \int_{3/2}^{3/2+i\infty} \frac{e^{sz}}{\zeta(s)}\, ds = 2\pi i m(z) \tag{2.2}$$

where for $\operatorname{Im} z > 0$

$$m(z) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho-1)!} \frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}} \left[(s-\varrho)^{k_\varrho} \frac{e^{sz}}{\zeta(s)}\right]_{s=\varrho} \tag{2.3}$$

and k_ϱ denotes the order of multiplicity of the non-trivial zero ϱ of the zeta function. If there are no multiple zeros of the zeta function then

$$(2.4)\qquad m(z) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{e^{\varrho z}}{\zeta'(\varrho)}.\ (^1)$$

The analytic character of the m-function is described by the following theorems:

THEOREM 1. *The function $m(z)$ is holomorphic on the upper half-plane H and for $z \in H$ we have*

$$(2.5)\qquad 2\pi i m(z) = m_1(z) + m_2(z) - e^{3z/2} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{3/2}(z-\log n)}$$

where the last term on the right is a meromorphic function on the whole complex plane with poles at $z = \log n$ if n is a product of different primes or n is equal to 1,

$$(2.6)\qquad m_1(z) = \int_{-1/2+i\infty}^{-1/2} \frac{e^{sz}}{\zeta(s)} ds$$

is analytic on H and

$$(2.7)\qquad m_2(z) = \int_{-1/2}^{3/2} \frac{e^{sz}}{\zeta(s)} ds$$

is regular on the whole complex plane.

THEOREM 2. *The function $m(z)$ can be continued analytically to a meromorphic function on the whole complex plane which satisfies the functional equation*

$$(2.8)\qquad m(z) + \overline{m(\bar{z})} = -2 \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cos\left(\frac{2\pi}{n} e^{-z}\right)$$

where the function on the right-hand side has the period πi and is an entire function of order 1 *and type 2π as a function of the variable $z_1 = e^{-z}$.*

· *More explicitly, we have for* $\operatorname{Im} z > 0$

$$m(z) = \frac{1}{2\pi i} \int_{-1/2+i\infty}^{-1/2} \frac{e^{sz}}{\zeta(s)} ds + \frac{1}{2\pi i} \int_{-1/2}^{3/2} \frac{e^{sz}}{\zeta(s)} ds - \frac{e^{3z/2}}{2\pi i} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{3/2}(z-\log n)},$$

for $\operatorname{Im} z < 0$

$$m(z) = -\overline{m(\bar{z})} - 2 \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cos\left(\frac{2\pi}{n} e^{-z}\right)$$

and for $|\operatorname{Im} z| < \pi$

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(1) Let us remark that this definition does not depend on the particular form of the sequence (T_n) satisfying (2.1). (We make use of this comment in part II.)

$$m(z) = \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2} e^{s(z-\log 2\pi - i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)} ds$$

$$-\frac{1}{2\pi i} \int_{-1/2}^{-1/2+i\infty} e^{s(z-\log 2\pi + i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)} ds - \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{e^{-z}(-2\pi i)/n}$$

$$+\frac{1}{2\pi i} \int_{-1/2}^{3/2} \frac{e^{sz}}{\zeta(s)} ds - \frac{e^{3z/2}}{2\pi i} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{3/2}(z-\log n)}.$$

THEOREM 3. *The only singularities of $m(z)$ meromorphic on* **C** *are simple poles at the points $z = \log n$ on the real axis, where n is a product of different primes or n is equal to* 1 *with residue*

$$\operatorname*{res}_{z=\log n} m(z) = -\mu(n)/2\pi i. \tag{2.9}$$

3. Proof of Theorem 1. We have by (2.2) for $z \in H$

$$2\pi i m(z) = m_1(z) + m_2(z) + m_3(z)$$

where the last integral

$$m_3(z) = \int_{3/2}^{3/2+i\infty} \frac{e^{sz}}{\zeta(s)} ds,$$

since $\operatorname{Re} s = 3/2 > 1$ and $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)/n^s$, is equal to

$$m_3(z) = \sum_{n=1}^{\infty} \mu(n) \int_{3/2}^{3/2+i\infty} e^{sz - s\log n} ds = -e^{3z/2} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{3/2}(z-\log n)}.$$

The inversion of the order of integration and summation is justified for $z \in H$ by the uniform convergence of the integral and the series.

Since $|\Gamma(-1/2+it)| \ll \exp(-\pi t/2)$, the functional equation for $\zeta(s)$ implies

$$\left|\frac{1}{\zeta(-1/2+it)}\right| = \frac{2\sqrt{2\pi}\,|\sin(3/2-it)(\pi/2)|\,|\Gamma(-1/2+it)|}{|\zeta(3/2-it)|}$$

$$\ll |e^{i(3/2-it)\pi/2} - e^{-i(3/2-it)\pi/2}|\,|\Gamma(-1/2+it)| \ll 1.$$

Thus we have

$$|m_1(z)| = \left|\int_{-1/2+\infty}^{-1/2} \frac{e^{sz}}{\zeta(s)} ds\right| \ll e^{-x/2} \int_0^{\infty} e^{-ty} dt = \frac{e^{-x/2}}{y}$$

and $m_1(z)$ is absolutely convergent for $y = \operatorname{Im} z > 0$.

4. We shall first prove that the function $m(z)$ analytic for $y > 0$ can be continued to a meromorphic function for $y > -\pi$.

Let us consider the integral

$$m_1(z) = -\int_{-1/2}^{-1/2+i\infty} \frac{e^{sz}}{\zeta(s)}\, ds$$

convergent for $y > 0$. By the functional equation for $\zeta(s)$ we get

$$(4.1)\qquad \begin{aligned} m_1(z) &= -\int_{-1/2}^{-1/2+i\infty} e^{s(z-\log 2\pi - i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)}\, ds \\ &\quad -\int_{-1/2}^{-1/2+i\infty} e^{s(z-\log 2\pi + i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)}\, ds \\ &= m_{11}(z)+m_{12}(z). \end{aligned}$$

Since $|\Gamma(-1/2+it)/\zeta(3/2-it)| \ll e^{-\pi t/2}$, $m_{11}(z)$ is regular for $y > 0$ and $m_{12}(z)$ for $y > -\pi$.

We have

$$(4.2)\qquad \begin{aligned} m_{11}(z) &= -\int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi - i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)}\, ds \\ &\quad +\int_{-1/2-i\infty}^{-1/2} e^{s(z-\log 2\pi - i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)}\, ds \\ &= I_1(z)+I_2(z). \end{aligned}$$

It is easy to verify that the integral $I_2(z)$ is convergent for $y < \pi$. Since $m_{11}(z)$ is regular for $y > 0$, the integral $I_1(z)$ is convergent for $0 < y < \pi$. Thus we can reduce $I_1(z)$ to a case of Mellin's inversion formula as follows. We have formally

$$(4.3)\qquad I_1(z) = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi - i\pi/2+\log n)}\, \Gamma(s)\, ds.$$

To justify the inversion of the order of summation and integration for $0 \le y < \pi$ we will see that the integral and the series converge uniformly.

First, by Cauchy's theorem of residues

$$(4.4)\qquad \begin{aligned} &-\int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi - i\pi/2+\log n)}\, \Gamma(s)\, ds \\ &= -\int_{1-i\infty}^{-1+i\infty} e^{s(z-\log 2\pi - i\pi/2+\log n)}\, \Gamma(s)\, ds + 2\pi i \operatorname*{res}_{s=0} e^{s(z-\log 2\pi - i\pi/2+\log n)}\, \Gamma(s) \\ &= -\int_{1-i\infty}^{1+i\infty} e^{s(z-\log 2\pi - i\pi/2+\log n)}\, \Gamma(s)\, ds + 2\pi i. \end{aligned}$$

Since $\operatorname{Re} e^{-(z-\log 2\pi - i\pi/2+\log n)} = (2\pi/ne^{x})\sin y > 0$ for $0 < y < \pi$, using Mellin's inversion formula we get

$$\int_{1-i\infty}^{1+i\infty} e^{s(z-\log 2\pi - i\pi/2+\log n)}\,\Gamma(s)\,ds = 2\pi i e^{-e^{-z}2\pi i/n} \tag{4.5}$$

and by (4.3), (4.4) and (4.5)

$$I_1(z) = -2\pi i \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-e^{-z}2\pi i/n}. \tag{4.6}$$

Since $\sum_{n=1}^{\infty} \mu(n)/n = 0$, we have

$$\begin{aligned}\left|\sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-2\pi i/ne^z}\right| &= \left|\sum_{n=1}^{\infty} \frac{\mu(n)}{n}(e^{-2\pi i/ne^z}-1)\right| \\ &\leqslant \sum_{n=1}^{\infty} \frac{1}{n}(e^{|-2\pi i/ne^z|}-1) = \sum_{n=1}^{\infty}\frac{1}{n}(e^{2\pi/ne^x}-1) \\ &\leqslant e^{2\pi/e^x} \sum_{n\leqslant[2\pi/e^x]} \frac{1}{n} + \frac{2\pi(e-1)}{e^x} \sum_{n\geqslant[2\pi/e^x]+1} \frac{1}{n^2} \ll c_2(x)\end{aligned}$$

and the series on the right of (4.6) is absolutely convergent for all y.

Finally, by (4.1), (4.2) and (4.6), we obtain the following analytic continuation of $m_1(z)$ to $y > -\pi$. For $|y| < \pi$

$$\begin{aligned} m_1(z) = -2\pi i \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-e^{-z}2\pi i/n} &+ \int_{-1/2-i\infty}^{-1/2} e^{s(z-\log 2\pi - i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)}\,ds \\ &- \int_{-1/2}^{-1/2+i\infty} e^{s(z-\log 2\pi + i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)}\,ds \end{aligned} \tag{4.7}$$

where the first term is holomorphic for all y, the second for $y < \pi$ and the third for $y > -\pi$.

In accordance with Theorem 1, (4.7) completes the continuation of $m(z)$ to the region $y > -\pi$.

5. Let us consider the function

$$m^-(z) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ -T_n < \operatorname{Im}\varrho < 0}} \frac{1}{(k_\varrho - 1)!} \frac{d^{k_\varrho - 1}}{ds^{k_\varrho - 1}} \left[(s-\varrho)^{k_\varrho} \frac{e^{sz}}{\zeta(s)} \right]_{s=\varrho} \tag{5.1}$$

where k_ϱ denotes the order of multiplicity of the non-trivial zero ϱ of $\zeta(s)$, defined for z belonging to

$$H^- = \{z \in C\colon \operatorname{Im} z < 0\}. \tag{5.2}$$

Since $\zeta(\bar{s}) = \overline{\zeta(s)}$ we have $|\zeta(\bar{s})| = |\zeta(s)|$ and by (2.1) we choose T_n ($n \leqslant T_n \leqslant n+1$) such that

(5.3) $$\left|\frac{1}{\zeta(\sigma - iT_n)}\right| < T_n^{c_1} \quad \text{for } -1 \leqslant \sigma \leqslant 2.$$

If $\zeta(s)$ has only simple zeros, then

(5.4) $$m^-(z) = \lim_{n\to\infty} \sum_{\substack{\varrho \\ -T_n < \operatorname{Im}\varrho < 0}} \frac{e^{z\varrho}}{\zeta'(\varrho)}.$$

Now taking the integral

$$\int \frac{e^{sz}}{\zeta(s)}\, ds$$

round the rectangle $(-1/2, 3/2, 3/2 - iT_n, -1/2 - iT_n)$ with $n \to \infty$, we have by Cauchy's residue theorem

(5.5) $$2\pi i m^-(z) = m_1^-(z) + m_2^-(z) + m_3^-(z)$$

where

(5.6) $$m_1^-(z) = -\int_{-1/2-i\infty}^{-1/2} \frac{e^{sz}}{\zeta(s)}\, ds$$

is regular for $y < 0$ (the proof similar to that for $m_1(z)$),

(5.7) $$m_2^-(z) = \int_{3/2}^{-1/2} \frac{e^{sz}}{\zeta(s)}\, ds$$

is analytic on the whole complex plane and

(5.8) $$m_3^-(z) = \int_{3/2-i\infty}^{3/2} \frac{e^{sz}}{\zeta(s)}\, ds = \sum_{n=1}^{\infty} \mu(n) \int_{3/2-i\infty}^{3/2} e^{s(z-\log n)}\, ds$$
$$= e^{3z/2} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{3/2}(z - \log n)}.$$

Thus $m_3^-(z)$ is meromorphic on the whole complex plane. The inversion of the order of integration and summation is justified for $z \in H^-$ by the uniform convergence of the integral and the series.

Now $m_1^-(z)$ analytic for $y < 0$ we have to continue to $y < \pi$ just as $m_1(z)$ in Section 4. We have by the functional equation for $\zeta(s)$

(5.9) $$m_1^-(z) = m_{11}^-(z) + m_{12}^-(z)$$

where

(5.10) $$m_{11}^-(z) = -\int_{-1/2-i\infty}^{-1/2} e^{s(z-\log 2\pi - i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)}\, ds$$

is absolutely convergent for $y < \pi$ and

$$(5.11)\qquad m_{12}^{-}(z) = -\int_{-1/2-i\infty}^{-1/2} e^{s(z-\log 2\pi+i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)}\,ds$$

is absolutely convergent for $y < 0$.

Next we get

$$(5.12)\qquad \begin{aligned} m_{12}^{-}(z) &= -\int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi+i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)}\,ds \\ &\quad + \int_{-1/2}^{-1/2+i\infty} e^{s(z-\log 2\pi+i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)}\,ds \\ &= I_1^{-}(z)+I_2^{-}(z). \end{aligned}$$

It is easy to verify that the integral $I_2^{-}(z)$ is convergent for $y > -\pi$. Since $m_{12}^{-}(z)$ is regular for $y < 0$, the integral $I_1^{-}(z)$ is convergent for $-\pi < y < 0$ and we can apply Mellin's inversion formula.

We have formally

$$I_1^{-}(z) = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi+i\pi/2+\log n)}\,\Gamma(s)\,ds$$

and by Cauchy's theorem of residues

$$(5.13)\qquad I_1^{-}(z) = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \int_{1-i\infty}^{1+i\infty} e^{s(z-\log 2\pi+i\pi/2+\log n)}\,\Gamma(s)\,ds.$$

To justify the inversion of the order of summation and integration for $-\pi < y < 0$, we will see that the integral and the series converge uniformly.

Since

$$\operatorname{Re} e^{-(z-\log 2\pi+i\pi/2+\log n)} = -\frac{2\pi}{ne^{x}}\sin y > 0$$

for $-\pi < y < 0$, using Mellin's inversion formula we get

$$(5.14)\qquad \int_{1-i\infty}^{1+i\infty} e^{s(z-\log 2\pi+i\pi/2+\log n)}\,\Gamma(s)\,ds = 2\pi i e^{e^{-z}2\pi i/n}$$

and by (5.13)

$$(5.15)\qquad I_1^{-}(z) = -2\pi i \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{e^{-z}2\pi i/n}$$

where the series on the right is absolutely convergent for all y.

Finally, by (5.9), (5.10), (5.12) and (5.15)

$$(5.16)\qquad m_1^{-}(z) = -2\pi i \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{e^{-z}2\pi i/n} - \int_{-1/2-i\infty}^{-1/2} e^{s(z-\log 2\pi-i\pi/2)} \frac{\Gamma(s)}{\zeta(1-s)}\,ds$$

$$+\int_{-1/2}^{-1/2+i\infty} e^{s(z-\log 2\pi+i\pi/2)}\frac{\Gamma(s)}{\zeta(1-s)}\,ds,$$

which completes the continuation of $m^-(z)$ analytic for $y<0$ to the half-plane $y<\pi$.

6. Proof of Theorem 2. By (4.7) and (5.16) for $|y|<\pi$

$$m_1(z)+m_1^-(z) = -4\pi i\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\cos\left(\frac{2\pi}{n}e^{-z}\right). \tag{6.1}$$

It is obvious that

$$m_2(z)+m_2^-(z) = \int_{-1/2}^{3/2}\frac{e^{sz}}{\zeta(s)}\,ds+\int_{3/2}^{-1/2}\frac{e^{sz}}{\zeta(s)}\,ds = 0 \tag{6.2}$$

and by (5.8) and Theorem 1

$$m_3(z)+m_3^-(z) = 0. \tag{6.3}$$

Thus for $|y|<\pi$ we have

$$m(z)+m^-(z) = -2\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\cos\left(\frac{2\pi}{n}e^{-z}\right).$$

Hence according to Theorem 1 for all $y<\pi$

$$m(z) = -m^-(z)-2\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\cos\left(\frac{2\pi}{n}e^{-z}\right)$$

by the principle of analytic continuation and for $y>-\pi$

$$m^-(z) = -m(z)-2\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\cos\left(\frac{2\pi}{n}e^{-z}\right).$$

This implies that $m(z)$ and $m^-(z)$ can be continued analytically over the whole plane as a meromorphic function. And for all z

$$m(z)+m^-(z) = -2\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\cos\left(\frac{2\pi}{n}e^{-z}\right). \tag{6.4}$$

To prove the functional equation (2.8) observe that if ϱ is a non-trivial zero of $\zeta(s)$ then so is $\bar{\varrho}$. For $z\in H$ we have

$$\overline{m(z)} = \overline{\lim_{n\to\infty}\sum_{\substack{\varrho\\ 0<\operatorname{Im}\varrho<T_n}}\frac{1}{(k_\varrho-1)!}\frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}}\left[(s-\varrho)^{k_\varrho}\frac{e^{sz}}{\zeta(s)}\right]_{s=\varrho}}$$

and further setting $s=\sigma+i\tau$

$$m(z) = \lim_{n\to\infty} \overline{\sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho-1)!}\frac{\delta^{k_\varrho-1}}{\delta\sigma^{k_\varrho-1}}\overline{\left[(s-\varrho)^{k_\varrho}\frac{e^{sz}}{\zeta(s)}\right]_{s=\varrho}}}.$$

Now since $\zeta(\bar{s}) = \overline{\zeta(s)}$ we get

$$m(z) = \lim_{n\to\infty} \overline{\sum_{\substack{\varrho \\ 0<\operatorname{Im}\varrho<T_n}} \frac{1}{(k_\varrho-1)!}\frac{\delta^{k_\varrho-1}}{\delta\sigma^{k_\varrho-1}}\left[(\bar{s}-\bar{\varrho})^{k_\varrho}\frac{e^{\bar{s}\bar{z}}}{\zeta(\bar{s})}\right]_{s=\varrho}}$$

and finally

$$(6.5)\quad m(z) = \lim_{n\to\infty} \overline{\sum_{\substack{\varrho \\ -T_n<\operatorname{Im}\varrho<0}} \frac{1}{(k_\varrho-1)!}\frac{d^{k_\varrho-1}}{ds^{k_\varrho-1}}\left[(s-\varrho)^{k_\varrho}\frac{e^{s\bar{z}}}{\zeta(s)}\right]_{s=\varrho}} = \overline{m^-(\bar{z})}.$$

Next using (6.4) we have for $z \in H$

$$m(z) = \overline{m^-(\bar{z})} = -\overline{m(\bar{z})} - 2\,\overline{\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\cos\left(\frac{2\pi}{n}e^{-\bar{z}}\right)}$$

$$= -\overline{m(\bar{z})} - 2\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\cos\left(\frac{2\pi}{n}e^{-z}\right)$$

and by complex conjugation for $z \in H^-$ and by the principle of analytic continuation for z with $\operatorname{Im} z = 0$. This proves (2.8).

Set

$$A(z) = -2\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\cos\left(\frac{2\pi}{n}e^{-z}\right).$$

Let $z_1 = e^{-z}$. Then

$$A(z_1) = -2\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\cos\left(\frac{2\pi}{n}z_1\right)$$

and since $\sum_{n=1}^{\infty}\mu(n)/n = 0$, we get

$$A(z_1) = -\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(e^{i\frac{2\pi}{n}z_1} + e^{-i\frac{2\pi}{n}z_1} - 2\right)$$

$$= -\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\sum_{k=1}^{\infty}\frac{\left(i\frac{2\pi}{n}z_1\right)^k}{k!} + \sum_{k=1}^{\infty}\frac{\left(-i\frac{2\pi}{n}z_1\right)^k}{k!}\right)$$

$$= 2\sum_{k=1}^{\infty}\frac{(-1)^k(2\pi z_1)^{2k}}{(2k)!\,\zeta(2k+1)}$$

and if $|z_1| = r$, then

$$|A(z_1)| \leqslant 3\sum_{k=1}^{\infty}\frac{(2\pi r)^{2k}}{(2k)!} = \frac{3}{2}(e^{2\pi r}+e^{-2\pi r}-2) < 2e^{2\pi r}. \tag{6.6}$$

Moreover, we have

$$A(ir) = 2\sum_{k=1}^{\infty}\frac{(2\pi r)^{2k}}{(2k)!\,\zeta(2k+1)}$$

and

$$|A(ir)| \geqslant \frac{4}{3}\sum_{k=1}^{\infty}\frac{(2\pi r)^{2k}}{(2k)!} = \frac{2}{3}(e^{2\pi r}+e^{-2\pi r})-\frac{4}{3} \geqslant \frac{1}{3}e^{2\pi r} \tag{6.7}$$

for r sufficiently large. By (6.6) and (6.7) the order of $A(z_1)$ is essentially 1 and the type is 2π.

Theorem 3 is a simple corollary of Theorem 2.

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