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On the distribution of natural numbers with divisors from an arithmetic progression

by

P. D. VARBANEC (Odessa)

1. Introduction. Let us consider two sequences of natural numbers $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ and let the function f be defined by

$$f(n) = \sum_{\substack{k_1,k_2 \\ n=a_{k_1}\cdot b_{k_2}}} 1.$$

Let S_f be the summatory function of f,

$$S_f(x) = \sum_{n \leqslant x} f(n).$$

For some "good" sequences $\{b_k\}$ it is possible to obtain nontrivial formulae for the sum

$$S_f(x; a, q) = \sum_{n \leqslant x} f(n; a, q),$$

where $a, q \in N$ and

$$f(n; a, q) = \sum_{\substack{k_1,k_2 \\ n=a_{k_1}\cdot b_{k_2} \\ b_{k_2} \equiv a (\operatorname{mod} q)}} 1.$$

The purpose of this paper is to prove the asymptotic formula for the sum $S_f(x; a, q)$ for sufficiently large class of sequences $\{a_k\}$, $\{b_k\}$. In special cases, such problems have been considered in [2], [5], [6], [7].

2. Notations. For the sequences $\{a_k\}$, $\{b_k\}$ let

$$A(n) = \sum_{\substack{k \\ a_k = n}} 1, \quad B(n) = \sum_{\substack{k \\ b_k = n}} 1.$$

It is obvious that

$$f(n; a, q) = \sum_{\substack{md=n \\ d \equiv a (\operatorname{mod} q)}} A(m)B(d).$$

If we put

$$\text{(1)} \qquad F_1(s) = \sum_{n=1}^{\infty} A(n)/n^s, \qquad F_2(s) = \sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} B(n)/n^s,$$

then

$$\text{(2)} \qquad F(s) = \sum_{n=1}^{\infty} f(n;\, a,\, q)/n^s = F_1(s)F_2(s).$$

We shall consider the sequences $\{a_k\}$, $\{b_k\}$ such that $A(n),\ B(n) = O(n^{\varepsilon})$. Therefore, the Dirichlet series (1) and (2) are convergent for $\operatorname{Re} s > 1$.

3. Statement of the main result.

THEOREM 1. *Let $F_1(s)$, $F_2(s)$, $F(s)$ be analytic in the domain*

$$\operatorname{Re} s > 1 - \frac{c_1}{(\log(|t|+3))^{\gamma}}$$

except at the point $s = 1$, with $s = \sigma + it$, $0 < \gamma < 1$, $c_1 > 0$. If for any $T > 3$ in the domain

$$\operatorname{Re} s > 1 - \frac{c_1}{(\log(T+3))^{\gamma}}, \qquad |\operatorname{Im} s| \leqslant T, \qquad |s-1| > \frac{c_1}{(\log T)^{\gamma}}$$

the following estimate holds:

$$\text{(3)} \qquad F(s) - \frac{B(a)}{a^s} F_1(s) = O\big((1+(1+T)^{\gamma_1(1-\sigma)})q^{-\sigma} \log^{c_2} T\big),$$

with the constants $0 \leqslant \gamma_1 \leqslant 1$, $c_2 > 0$, then for $0 < a \leqslant q$ and $x \to \infty$

$$\text{(4)} \qquad \sum_{n \leqslant x} f(n;\, a,\, q) = \frac{1}{2\pi i} \int_{C_\varrho} \left[F(s) - \frac{B(a)}{a^s} F_1(s) \right] \frac{x^s}{s}\, ds$$
$$+ B(a) \sum_{n \leqslant x/a} A(n) + O\left(\frac{x}{q} e^{-c_0(\log x)^{1-\gamma-\varepsilon}} \right),$$

where $c_0 > 0$ and C_ϱ is the positively oriented circle of radius ϱ centred at $s = 1$ with $s = 1 - \varrho$ removed. The circle lies in the domain of analyticity of $F_1(s)$ and $F(s)$. Now and later on the O-constants can depend only on ε.

4. Auxiliary results.

LEMMA 1. *If $x \in \boldsymbol{R}$, $a, q \in \boldsymbol{N}$ with $0 < a \leqslant q$, $x^{1/2} \leqslant q < x$, then for any $n_0 \leqslant x$ of the form $n_0 = (a + m_0 q) r_0$, $m_0, r_0 \in \boldsymbol{N}$, in the interval $(n_0 - q, n_0 + q)$ there exist $O\big(x^{\varepsilon}(1 + (ax)^{1/2}/q)\big)$ numbers of the form $n = (a + mq) r$ with $m, r \in \boldsymbol{N}$.*

Proof. If $(ax)^{1/2} \leqslant q$, then for any $n \leqslant x$, $n = (a+mq)r$, we have

$$r \leqslant x/q \leqslant (x/a)^{1/2}$$

and

$$a|r-r_0| \leqslant a(x/a)^{1/2} \leqslant (ax)^{1/2} \leqslant q.$$

It follows from

$$q > |n-n_0| = |a(r-r_0)+q(mr-m_0r_0)| \geqslant q|mr-m_0r_0|-a|r-r_0|$$

that the inequality $|n-n_0| < q$ is valid if $mr-m_0r_0 = 0, \pm 1$. But, as $m_0r_0 \leqslant x$, the last equalities have $d(m_0r_0+\delta) = O(x^\varepsilon)$ solutions, where $\delta = 0, \pm 1$ and $d(n)$ denotes the number of natural divisors of the number n.

Now, let $x^{1/2} \leqslant q \leqslant (ax)^{1/2}$. We shall prove that for any $n_1 = (a+m_1q)r_1 \leqslant x$, $m_1, r_1 \in N$, there are $O\left(x^\varepsilon \frac{(ax)^{1/2}}{q}\right)$ numbers $n_2 = (a+m_2)r_2$, $m_2, r_2 \in N$, such that $0 < n_2-n_1 < q$. We have

$$m_1r_1q \leqslant n_1 < (1+m_1)r_1q, \quad m_2r_2q \leqslant n_2 < (1+m_2)r_2q.$$

If $a = q$, then it follows from

$$n_2-n_1 = \big(m_2(r_2+1)-m_1(r_1+1)\big)q$$

that the inequality $0 < n_2-n_1 < q$ is impossible. Therefore, we shall assume that $0 < a < q$.

There are five cases to consider:

1° $m_1r_1q < (1+m_1)r_1q \leqslant m_2r_2q < (1+m_2)r_2q$,
2° $m_1r_1q \leqslant m_2r_2q \leqslant (1+m_1)r_1q \leqslant (1+m_2)r_2q$,
3° $m_2r_2q \leqslant m_1r_1q \leqslant (1+m_1)r_1q \leqslant (1+m_2)r_2q$,
4° $m_1r_1q \leqslant m_2r_2q \leqslant (1+m_2)r_2q \leqslant (1+m_1)r_1q$,
5° $m_2r_2q \leqslant m_1r_1q \leqslant (1+m_2)r_2q \leqslant (1+m_1)r_1q$.

In the first case we have

$$n_2-n_1 \geqslant [m_2r_2-(1+m_1)r_1]q.$$

Thus, $n_2-n_1 < q$ if $m_2r_2-(1+m_1)r_1 = 0$. But for fixed m_1, r_1 the last equation has no more than $d\big((1+m_1)r_1\big) = O(x^\varepsilon)$ solutions m_2, r_2.

In case 2°, $r_2 > r_1$ or $r_2 < r_1$, because the inequalities $0 < n_2-n_1 < q$ for $r_1 = r_2$ do not hold.

Let $r_2 > r_1$. If we put $r_2 = r_1+l$, $m_2 = m_1-k$ with $l, k \in N$, then

$$0 < n_2-n_1 = l(a+m_1q)-kq(r_1+l).$$

It follows from $n_2-n_1 < q$ that

$$0 < m_1l-r_1k-kl+al/q < 1. \tag{5}$$

But from $m_1r_1q \leqslant m_2r_2q \leqslant (1+m_1)r_1q$ we get

$$0 \leqslant m_1l-r_1k-kl. \tag{6}$$

Therefore, from (5) and (6) we obtain

$$m_1 l - r_1 k - kl = 0.$$

As $m_1 l - r_1 k - kl = m_2 r_2 - m_1 r_1$, so the equation $m_2 r_2 - m_1 r_1 = 0$ has $O(x^\varepsilon)$ solutions.

Let $r_2 < r_1$. We put $r_2 = r_1 - l$, $m_2 = m_1 + k$, $l, k \in N$. Then

$$0 < n_2 - n_1 = -al + r_1 kq - m_1 lq - klq < q.$$

Hence

(7) $$al/q < kr_1 - lm_1 - kl < 1 + al/q.$$

From $m_2 r_2 q \leqslant (1+m_1) r_1 q \leqslant (1+m_2) r_2 q$ we have

(8) $$kr_1 - m_1 l - kl \geqslant l.$$

By (7) and (8) we get

$$l < 1 + al/q \Leftrightarrow l < a/(q-a).$$

This means that if $a \leqslant q - q^{1-\varepsilon} = q(1-q^{-\varepsilon})$, then $l < q^\varepsilon$ and for each l there exists no more than one k such that

$$n_2 = [a + (m_1 + k) q](r_1 - l), \quad n_2 - n_1 < q.$$

Let us assume that $q(1-q^{-\varepsilon}) < a < q$. We put

$$a = q(1 - \vartheta q^{-\varepsilon}), \quad \text{where } 0 < \vartheta < 1.$$

We have

$$n_2 - n_1 = [kr_1 - m_1 l - kl - l] q + \vartheta l q^{1-\varepsilon}.$$

Let us notice that for such a, $kr_1 - m_1 l - kl = l$, because if $kr_1 - m_1 l - kl \geqslant l + 1$, then $n_2 - n_1 \geqslant q + \vartheta l q^{1-\varepsilon} > q$. Thus $(1+m_1) r_1 = (1+m_2) r_2$ and for fixed m_1, r_1 there are $O(x^\varepsilon)$ solutions of the last equation. This completes the proof in case 2°.

In case 3° we have

(9) $$0 < n_2 - n_1 = (r_2 - r_1) a + (m_2 r_2 - m_1 r_1) q < q$$

and $r_2 > r_1$.

Hence, by (9)

$$m_2 r_2 - m_1 r_1 \leqslant 0.$$

There are $O(x^\varepsilon)$ pairs m_2, r_2 satisfying the condition

$$m_2 r_2 - m_1 r_1 = 0.$$

Let

(10) $$m_2 r_2 - m_1 r_1 < 0.$$

For the pair m_2, r_2 satisfying (10) and $m_2 > a$ we have

$$(a+m_2 q)r_2 \leqslant x \leqslant q^2 \Rightarrow r_2 \leqslant q/m_2 \leqslant q/a.$$

It follows from $(r_2-r_1)a < q$ that there are no solutions of (9) ($(m_2 r_2 - m_1 r_1)q \leqslant -q$). Thus, we can assume that $1 \leqslant m_2 \leqslant a$.

Let us take δ, $0 < \delta < 1$ (the exact value of δ will be defined later). There are $O(a^\delta)$ pairs m_2, r_2 satisfying (9) with $m_2 \leqslant a^\delta$.

If $a^\delta < m_2 \leqslant a$, then among r_2 satisfying $(r_2-r_1)a > q$ we choose the least number and we denote it by $r_2^{(1)}$. For this $r_2^{(1)}$ there are $O(x^\varepsilon)$ values of r_2 such that $(r_2-r_1) > q/a$ and $0 < (r_2-r_2^{(1)})a \leqslant q$.

In fact, from

$$|n_2-n_2^{(1)}| < q \Rightarrow |(r_2-r_2^{(1)})a-(m_2 r_2 - m_2^{(1)} r_2^{(1)})q| \leqslant q$$

we get

$$m_2 r_2 - m_2^{(1)} r_2^{(1)} = 0,\ 1.$$

Let $r_2^{(2)}$ be the least number satisfying $r_2 > r_2^{(1)} + q/a$. Analogously we define $r_2^{(3)}, \ldots, r_2^{(k)}$. It follows from

$$r_2 \leqslant \frac{x}{m_2 q} \leqslant \frac{x}{a^\delta q}$$

that

$$k \leqslant \frac{x/(a^\delta q)}{q/a} = \frac{a^{1-\delta}x}{q^2}.$$

Thus, beside $O(x^\varepsilon a^\delta)$ pairs m_2, r_2 satisfying $m_2 \leqslant a^\delta$, there are $O(a^{1-\delta}x^{1+\varepsilon}/q^2)$ pairs with $a^\delta < m_2 \leqslant a$.

Finally, if we take δ such that $a^\delta = a^{1-\delta}x/q^2$, then there are $O\left(x^\varepsilon (aq)^{1/2}/q\right)$ numbers n_2 satisfying $0 < n_2 - n_1 < q$. This completes the proof in case 3°.

Case 4° can be proved in the same way as 3°. We should only notice that $r_1 > r_2$.

For case 5°, let us notice that from $m_2 r_2 \leqslant m_1 r_1$ and

$$0 < n_2 - n_1 = (r_2-r_1)a + (m_2 r_2 - m_1 r_1)q < q$$

we get $r_2 > r_1$.

By

$$(1+m_1)r_1 \geqslant (1+m_2)r_2 \Rightarrow m_2 r_2 - m_1 r_1 \leqslant -(r_2-r_1)$$

we have

$$n_2 - n_1 \leqslant (r_2-r_1)a - (r_2-r_1)q = (r_2-r_1)(a-q) < 0.$$

This contradiction ends the proof in the fifth case. Thus, there are no pairs m_2, r_2 satisfying 5°.

This completes the proof of Lemma 1.

LEMMA 2. *Let* $\{a_k\}$, $\{b_k\}$ *be sequences of natural numbers such that* $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = \infty$. *Let* $A(n)$, $B(n)$, $f(n)$, $f(n; a, q)$, $F(s)$ *be the functions defined in Section* 2. *If* $A(n) = B(n) = O(n^{\varepsilon})$, *then for any* $C > 1$, $T > 1$

$$(11)\quad \sum_{n\leqslant x} f(n; a, q) = \frac{1}{2\pi i}\int_{C-iT}^{C+iT}\left[F(s)-\frac{B(a)}{a^s}\sum_{n=1}^{\infty}\frac{A(n)}{n^s}\right]\frac{x^s}{s}ds + B(a)\sum_{n\leqslant x/a} A(n) + R(x, a, q, T),$$

where

$$R(x, a, q, T) = \begin{cases} O\left(\dfrac{x^{C+\varepsilon}}{T}\right) & \text{if } q < x^{1/2}, \\ O\left(\dfrac{x^{C+\varepsilon}}{qT}\left(1+\dfrac{(ax)^{1/2}}{q}\right)\right) & \text{if } x^{1/2} \leqslant q < x. \end{cases}$$

Proof. The case $q < x^{1/2}$ is trivial, so we can assume that $x^{1/2} \leqslant q < x$. If we put

$$f^*(n; a, q) = \begin{cases} f(n; a, q) & \text{if } a\nmid n, \\ f(n; a, q) - B(a) & \text{if } a\mid n, \end{cases}$$

then

$$\sum_{n\leqslant x} f(n; a, q) = \sum_{n\leqslant x} f^*(n; a, q) + B(a)\sum_{n\leqslant x/a} A(n).$$

For $\operatorname{Re} s > 1$ we have

$$\sum_{n=1}^{\infty}\frac{f^*(n; a, q)}{n^s} = F(s) - \frac{B(a)}{a^s}\sum_{n=1}^{\infty}\frac{A(n)}{n^s}.$$

We shall use the well-known relation

$$\frac{1}{2\pi i}\int_{C-iT}^{C+iT}\frac{y^s}{s}ds = \begin{cases} 1 + O\big(y^C/(T\log y)\big) & \text{if } y > 1, \\ O\big(y^C/(T\,|\log y|)\big) & \text{if } 0 < y < 1 \end{cases}$$

with $C > 1$ and $T > 1$.

By this relation we get

$$\sum_{n\leqslant x-q} f^*(n; a, q) = \frac{1}{2\pi i}\int_{C-iT}^{C+iT}\left[F(s)-\frac{B(a)}{a^s}\sum_{n=1}^{\infty}\frac{A(n)}{n^s} - \sum_{x-q<n<x+q}\frac{f^*(n; a, q)}{n^s}\right]\frac{x^s}{s}ds$$
$$+ O\left(\left\{\sum_{n\leqslant x-q} + \sum_{n\geqslant x+q}\right\}\frac{f^*(n; a, q)x^C}{n^C T\,|\log(x/n)|}\right).$$

If $n \leqslant 2x$, then

$$f^*(n; a, q) \ll x^\varepsilon d^*(n; a, q),$$

where $d^*(n; a, q)$ is the number of representations of n in the form $n = (a+qm)r$, $m, r \in N$. Therefore, if $n_0 \leqslant 2x$ satisfies $d^*(n_0; a, q) \neq 0$, then by Lemma 1 there are $O\big(x^\varepsilon(1+(ax)^{1/2}/q)\big)$ numbers in the interval (n_0-q, n_0+q) such that $f^*(n; a, q) \neq 0$. Therefore

$$\sum_{x-q<n\leqslant x} f^*(n; a, q) = O\left(x^{2\varepsilon}\left(1+\frac{(ax)^{1/2}}{q}\right)\right), \tag{12}$$

$$\frac{1}{2\pi i}\int_{C-iT}^{C+iT} \sum_{x-q\leqslant n\leqslant x+q} \frac{f^*(n; a, q)}{n^s}\frac{x^s}{s}\,ds = O\left(x^{2\varepsilon}\left(1+\frac{(ax)^{1/2}}{q}\right)\log T\right). \tag{13}$$

Thus

$$\sum_{n\leqslant x} f^*(n; a, q) = \frac{1}{2\pi i}\int_{C-iT}^{C+iT}\left[F(s)-\frac{B(a)}{a^s}\sum_{n=1}^{\infty}\frac{A(n)}{n^s}\right]\frac{x^s}{s}\,ds \tag{14}$$
$$+O\left(x^{2\varepsilon}\left(1+\frac{(ax)^{1/2}}{q}\right)\log T\right)$$
$$+O\left(\Big\{\sum_{n\leqslant x-q}+\sum_{n\geqslant x+q}\Big\}\frac{x^{C+\varepsilon}d^*(n; a, q)}{n^C T|\log(x/n)|}\right).$$

Analogously as in [6] we split the last sum on the right hand side of (14) into three parts:

$$n \leqslant x/2, \quad x/2 < n < 2x, \quad n \geqslant 2x.$$

We have

$$\sum_{n\leqslant x/2}+\sum_{n\geqslant 2x} = O\left(\frac{x^{C+\varepsilon}}{T}\sum_{n=1}^{\infty}\frac{d^*(n; a, q)}{n^C}\right) \tag{15}$$
$$= O\left(\frac{x^{C+\varepsilon}}{T}\sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(rqm)^C}\right) = O\left(\frac{x^{C+\varepsilon}}{qT(C-1)^2}\right).$$

Let us consider the interval of summation

$$J = \{n\colon\ x/2 < n < 2x,\ n \notin (x-q, x+q),\ d^*(n; a, q) \neq 0\}.$$

Let n_1 be the least number from J. By n_2 we denote the least number $n \in J$ such that $n-n_1 > q$. Analogously we define $n_3, \ldots, n_N$. Obviously $N = O(x/q)$. For each n_j there exist $O\big(x^\varepsilon(1+(ax)^{1/2}/q)\big)$ integers from J. If $n_j \leqslant n \leqslant n_{j+1}$ and $d^*(n; a, q) \neq 0$, then

$$c_1'/|\log(x/n_j)| \leqslant 1/|\log(x/n)| \leqslant c_2'/|\log(x/n_{j+1})|.$$

Therefore

$$\sum_{n\in J} = O\left(\frac{x^{C+\varepsilon}}{Tx^C}\left(1+\frac{(ax)^{1/2}}{q}\right)\sum_{n_j\in J} 1/|\log(x/n_j)|\right) \tag{16}$$

$$= O\left(\frac{x^{\varepsilon}}{T}\left(1+\frac{(ax)^{1/2}}{q}\right)\sum_{j=1}^{N}\frac{x}{jq}\right) = O\left(\frac{x^{1+2\varepsilon}(1+((ax)^{1/2}/q))}{qT}\right)$$

(as usual, we put $n_j = x+\vartheta qj$, where $1/2 \leqslant |\vartheta| \leqslant 2$). The assertion of Lemma 2 follows from (14)–(16).

5. Proof of Theorem 1. We put $\varrho = c_1/(\log(T+3))^{\gamma}$. Let us consider the contour L which is shown in Figure 1.

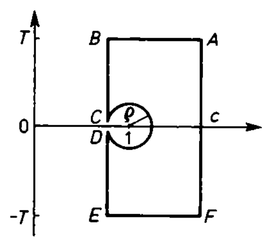

Fig. 1

By (3) with $c = 1+\varepsilon$ we get

$$\frac{1}{2\pi i}\left\{\int_{ABC}+\int_{DEF}\right\}\left(F(s)-\frac{B(a)}{a^s}F_1(s)\right)\frac{x^s}{s}\,ds$$

$$= O\left(\frac{x^c}{qT}\log^{c_2} T\right)+O\left(\left(\frac{x}{q}\right)^{1-\varrho} T^{\gamma_1\varrho}\log^{c_3} T\right),$$

where $c_3 \leqslant c_2+1$.

Hence, by Lemma 2 we obtain

$$\sum_{n\leqslant x} f(n;a,q) \tag{17}$$

$$= \frac{1}{2\pi i}\int_{C_\varrho}\left[F(s)-\frac{B(a)}{a^s}F_1(s)\right]\frac{x^s}{s}\,ds + B(a)\sum_{n\leqslant x/a} A(n) + R(x,a,q,T)$$

$$+O\left(\frac{x^{1+\varepsilon}}{qT}\log^{c_3} T\right)+O\left(\left(\frac{x}{q}\right)^{1-\varrho} T^{\gamma_1\varrho}\log^{c_3} T\right).$$

Therefore, if we put $T = qx^3$, then for $q \leqslant x^{1/2}$ we have

$$R(x,a,q,T)+O\left(\frac{x^{1+\varepsilon}}{qT}\log^{c_3}T\right)+O\left(\left(\frac{x}{q}\right)^{1-\varrho} T^{\gamma_1\varrho}\log^{c_3}T\right)$$

$$= O\left(\frac{x}{q}(x^{-\varepsilon}+x^{-(1-\gamma_1-3\varepsilon)\varrho}\log^{c_3}x)\right) = O\left(\frac{x}{q}e^{-c_4(\log x)^{1-\gamma}}\right).$$

In the case $x^{1/2} \leqslant q < x$ we put $T = (x/q)^{1/\gamma_1} e^{-c_5(\log x)^{\gamma_2}}$, where $\gamma_2 = 1-\varepsilon$. In this case the remainder terms in the formula (17) are $O\left(\frac{x}{q} e^{-c_6(\log x)^{1-\gamma-\varepsilon}}\right)$.

This completes the proof of Theorem 1.

6. An application. We shall use Theorem 1 for studying the distribution of values of the function

$$\tau_B(n; a, q) = \sum_{\substack{n = md \\ \vartheta(m) = 1, d \equiv a (\mathrm{mod}\, q)}} 1,$$

where

$$\vartheta(m) = \begin{cases} 1 & \text{if } m = u^2 + v^2 \quad (u, v \in \boldsymbol{Z}), \\ 0 & \text{otherwise.} \end{cases}$$

If we put

$$\{a_n\}_{n=1}^{\infty} = \{\vartheta(n) n\}_{n=1}^{\infty}, \quad \{b_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty},$$

then $A(n) = \vartheta(n)$, $B(n) = 1$ and for $\mathrm{Re}\, s > 1$

$$F_1(s) = \sum_{n=1}^{\infty} \frac{\vartheta(n)}{n^s} = g_0(s) \sqrt{\zeta(s) L(s, \chi_4)},$$

where $g_0(s)$ is analytic for $\mathrm{Re}\, s > 1/2$, $\zeta(s)$ is the Riemann zeta function and χ_4 in the L-function $L(s, \chi_4)$ denotes the non-principal Dirichlet character modulo 4. Moreover,

$$F_2(s) = \sum_{\substack{n=1 \\ n \equiv a (\mathrm{mod}\, q)}}^{\infty} \frac{1}{n^s} = \frac{1}{q^s} \zeta(s, a/q),$$

where $\zeta(s, a/q)$ is the Hurwitz zeta function.

It is clear that the functions $F_1(s)$, $F_2(s)$, $F(s) = F_1(s) F_2(s)$ satisfy the conditions of Theorem 1 with $\gamma = 2/3 + \varepsilon$, $\gamma_1 = 1/3$, $c_2 = 4$.

In fact, in the region

$$\mathrm{Re}\, s \geqslant 1 - \frac{c_1}{(\log(|t|+3))^{2/3+\varepsilon}}$$

$\zeta(s) L(s, \chi_4) \neq 0$, so $F_1(s)$, $F_2(s)$, $F(s)$ are analytic in this region except at $s = 1$.

It follows from Richert's estimate [4]

$$\zeta(s),\ L(s, \chi_4) = O\big((1 + T^{\alpha(1-\sigma)^{3/2}}) \log T\big)$$

with

$$0 \leqslant \mathrm{Re}\, s = \sigma \leqslant 2, \quad |\mathrm{Im}\, s| \leqslant T, \quad |s-1| \geqslant (\log T)^{-1}$$

that

$$\sqrt{\zeta(s)L(s,\chi_4)} = O(\log T)$$

for

$$\operatorname{Re} s \geqslant 1-\frac{c_1}{(\log(T+3))^{2/3+\varepsilon}}, \quad |\operatorname{Im} s| \leqslant T, \quad |s-1| \geqslant (\log T)^{(-2/3)-\varepsilon}.$$

Moreover, if

$$|s-1| \geqslant \log T^{-1}, \quad \operatorname{Re} s \geqslant 1/2, \quad |\operatorname{Im} s| \leqslant T,$$

then

$$\zeta(s, a/q)-\frac{1}{(a/q)^s} = O\big((1+T^{(1-\sigma)/3})\log T\big).$$

So that

$$F(s)-\frac{B(a)}{a^s}F_1(s) = g_0(s)\sqrt{\zeta(s)L(s,\chi_4)}\left[\frac{1}{q^s}\zeta(s, a/q)-\frac{1}{a^s}\right]$$
$$= O\big((1+(1+T)^{(1-\sigma)/3})q^{-\sigma}\log^2 T\big).$$

By Theorem 1 we get

$$(18) \quad \sum_{n\leqslant x} \tau_B(n; a, q)$$
$$= \frac{1}{2\pi i}\int_{C_\varrho} g_0(s)\sqrt{\zeta(s)L(s,\chi_4)}\left(\zeta(s, a/q)-\frac{1}{(a/q)^s}\right)\left(\frac{x}{q}\right)^s\frac{ds}{s}$$
$$+\sum_{n\leqslant x/a}\vartheta(n)+O\left(\frac{x}{q}e^{-c_0(\log x)^{1/3-2\varepsilon}}\right).$$

The integral term of (18) can be computed by using Kaczorowski's approach ([1], the proof of the main lemma). We notice that

$$g_0(s)\sqrt{\zeta(s)L(s,\chi_4)} = \frac{1}{(s-1)^{1/2}}g_1(s),$$

where

$$g_1(s) = \sum_{k=0}^{\infty}\alpha_k(s-1)^k, \quad \alpha_k = O(1),$$

$$\zeta(s, a/q)-\frac{1}{(a/q)^s} = \frac{1}{s-1}+g_2(s)$$

with

$$g_2(s) = \sum_{k=0}^{\infty}\beta_k(s-1)^k, \quad \beta_k = O(1).$$

Therefore

$$g_0(s)\sqrt{\zeta(s)L(s,\chi_4)}\left(\zeta(s,a/q)-\frac{1}{(a/q)^s}\right)=\frac{g_1(s)}{(s-1)^{3/2}}+\frac{g_1(s)g_2(s)}{(s-1)^{1/2}}.$$

In this way we get

$$(19)\quad \frac{1}{2\pi i}\int_{C_Q} g_0(s)\sqrt{\zeta(s)L(s,\chi_4)}\left[\zeta(s,a/q)-\frac{1}{(a/q)^s}\right]\frac{x^s}{q}\frac{ds}{s}$$

$$=\frac{A_0 x(\log(x/q))^{1/2}}{q}+\frac{x}{q}\sum_{k=0}^{K}\frac{B_k}{(\log(x/q))^{k+1/2}}+O\left((c_1K)^K\frac{x}{q(\log(x/q))^{K+1/2}}\right),$$

where

$$A_0=\left(\frac{L(1,\chi_4)}{2}\prod_{\substack{p\\ p\equiv 3(\mathrm{mod}\,4)}}\left(1-\frac{1}{p^2}\right)\right)^{1/2}\frac{1}{\Gamma(3/2)}=\left(2\prod_{p\equiv 3(4)}\left(1-\frac{1}{p^2}\right)\right)^{1/2},$$

$$B_k=(\alpha_{k+1}+\sum_{\nu+\mu=k}\alpha_\nu\beta_\mu)j_k,\qquad k=0,\ldots,K,$$

$$j_k=\frac{1}{2\pi i}\int_{L_0} e^z z^{k-(3/2)}\log z^{-k}dz$$

with the contour L_0 shown in Fig. 2

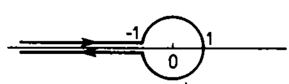

Fig. 2

(K is a positive integer, $K\leqslant c_2\frac{\sqrt{\log x}}{\log\log x}$). The O-constant does not depend on K, x, a, q. Further, for any natural number $M\leqslant c_2\frac{\sqrt{\log(x/a)}}{\log\log(x/a)}$

$$\sum_{n\leqslant x/a}\vartheta(n)=\left(\frac{1}{2}\prod_{p\equiv 3(4)}\left(1-\frac{1}{p^2}\right)\right)^{1/2}\frac{x}{a(\log(x/a))^{1/2}}$$

$$+\frac{x}{a}\sum_{m=1}^{M-1}\frac{l_m}{(\log(x/a))^{m+1/2}}+O(c_1M)^M\left(\frac{x/a}{(\log(x/a))^{M+1/2}}\right)$$

with the computable constants l_m (for the proof of this formula see [3], p. 393; [1], the main lemma).

In this way we have proved the following theorem:

THEOREM 2. *Let* $0 < a \leqslant q < x$, $a, q \in N$, $x \in \mathbf{R}$. *If* $\varepsilon > 0$ *and* $K \leqslant c_2 \frac{\sqrt{\log x}}{\log\log x}$, *then*

$$(20)\quad \sum_{n \leqslant x} \tau_B(n; a, q) = A_0 \frac{x}{q}\left(\log \frac{x}{q}\right)^{1/2} + A_0' \frac{x}{a}\left(\log \frac{x}{a}\right)^{-1/2}$$
$$+ \sum_{k=0}^{K-1}\left(\frac{x}{q} B_k \left(\log \frac{x}{q}\right)^{-k-1/2} + \frac{x}{a} l_k \left(\log \frac{x}{a}\right)^{-k-3/2}\right)$$
$$+ O\left((c_1 K)^K \left(\frac{x}{a}\left(\log \frac{x}{a}\right)^{-K-3/2}\right) + \frac{x}{q}\left(\log \frac{x}{q}\right)^{-K-1/2}\right)$$
$$+ O\left(\frac{x}{q} e^{-c(\log x)^{1/3-\varepsilon}}\right),$$

where the O-constants do not depend on x, a, q, K.

In the same way the sum

$$\sum_{n \leqslant x} \tau_P(n; a, q)$$

can be investigated, where $\tau_P(n; a, q)$ is the number of representations of n in the form $n = pm$, p is a prime number, $m \in N$, $m \equiv a(\operatorname{mod} q)$.

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DEPARTMENT OF MATHEMATICS AND MECHANICS
ODESSA STATE UNIVERSITY
Petra Velikogo 2, 270000 Odessa, U.S.S.R.

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