ACTA ARITHMETICA
LVII (1991)

The k-functions in multiplicative number theory, IV
On a method of A. E. Ingham

by

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1. Introduction and statement of the Theorem. In 1936 and 1942 A. E. Ingham proved the following, closely related theorems.

THEOREM A ([6]). *Let θ denote the upper bound of the real parts of the zeros of the Riemann zeta function $\zeta(s) = \zeta(\sigma+it)$. If θ is attained, i.e. if $\zeta(s)$ has a zero on the line $\sigma = \theta$, then there exists an absolute constant $c_0 > 1$ such that, for all $x > 1$, the interval $(x, c_0 x)$ contains integers n and n' satisfying*

$$\pi(n) < \operatorname{li} n, \qquad \pi(n') > \operatorname{li} n'.$$

THEOREM B ([7]). *Let*

$$F(s) = \int_0^\infty A(u)e^{-su}\,du,$$

where $A(u)$ is absolutely integrable over every finite interval $0 \leqslant u \leqslant U$, and the integral is convergent in some half-plane $\sigma > \sigma_1 \geqslant 0$.

Let $A^*(u)$ be a real trigonometric polynomial

$$A^*(u) = \sum_{n=-N}^{N} a_n e^{i\gamma_n u} \qquad (\gamma_n \text{ real},\ \gamma_{-n} = -\gamma_n,\ a_{-n} = \bar{a}_n)$$

and let

$$F^*(s) = \int_0^\infty A^*(u)e^{-su}\,du = \sum_{n=-N}^{N} a_n/(s-i\gamma_n) \qquad (\sigma > 0).$$

Suppose that $F(s)-F^*(s)$ (suitably defined outside the half-plane $\sigma > \sigma_1$) is regular in the region $\sigma \geqslant 0$, $-T \leqslant t \leqslant T$, for some $T > 0$ (or, more generally, continuous in this region and regular in the interior).

Then, as $u \to \infty$ (with T fixed)

$$\liminf A(u) \leqslant \liminf A_T^*(u) \leqslant \limsup A_T^*(u) \leqslant \limsup A(u), \tag{1.1}$$

where

$$(1.2)\qquad A_T^*(u) = \sum_{|\gamma_n|<T}\Big(1-\frac{|\gamma_n|}{T}\Big)a_n e^{i\gamma_n u} = a_0 + 2\,\mathrm{Re}\sum_{0<\gamma_n<T}\Big(1-\frac{\gamma_n}{T}\Big)a_n e^{i\gamma_n u}.$$

For generalizations and refinements of Theorem B see [1], [3], [4], [11].

The most characteristic feature of Theorems A and B is the use of some kind of almost periodicity. W. J. Jurkat [8] specified this concept introducing almost periodic functions in a distributional sense. The aim of this note is to prove a general result, which in the most important cases is a substitute for both Theorems A and B, and to show how the k-functions defined in part I of this paper [9] fit to the problems discussed. These functions permit the interpretation of some interesting arithmetic functions (such as $\psi(x)-x = \sum_{n\leqslant x}\Lambda(n)-x$, in usual notation) as boundary values of suitable harmonic functions which are almost periodic, in the sense of Bohr, on an open half-plane. It turns out that the classical Bohr theory is sufficient for the problems under consideration and any extension of it is superfluous; the methods and results of this note are, however, closely related to those of Jurkat [8].

For the sake of brevity, let $\mathfrak{A}$ denote the set of all functions

$$(1.3)\qquad F(z) = \sum_{n=1}^{\infty} a_n e^{iw_n z},\qquad z = x+iy,\ y>0,$$

satisfying the following conditions:

1. $0\leqslant w_1 < w_2 < \ldots$ are real numbers.
2. $a_n\in \boldsymbol{C}$, $n = 1, 2, 3, \ldots$
3. There exists a non-negative integer B such that

$$\sum_{n=2}^{\infty}|a_n| w_n^{-B} < \infty.$$

4. There exists a non-negative number L_0 such that for every x, $|x|\geqslant L_0$, the limit

$$P(x) = \lim_{y\to 0^+}\mathrm{Re}\,F(x+iy)$$

exists and represents a locally bounded function of $x\in\boldsymbol{R}\setminus[-L_0, L_0]$.

Moreover, let

$$\alpha(F) = \inf_{\substack{y>0\\ x\in\boldsymbol{R}}}\mathrm{Re}\,F(x+iy),\qquad \beta(F) = \sup_{\substack{y>0\\ x\in\boldsymbol{R}}}\mathrm{Re}\,F(x+iy).$$

THEOREM. *Let $F\in\mathfrak{A}$. Then for every real number a satisfying $\alpha(F) < a < \beta(F)$ there exists a positive real number $l = l(a, F)$ such that*

$$\inf_{x\in I} P(x) < a < \sup_{x\in I} P(x)$$

for every interval $I\subset\boldsymbol{R}\setminus[-L_0, L_0]$ of length $|I|\geqslant l$.

Acknowledgement. This paper has been written during the author's Research Fellowship from the Alexander von Humboldt Foundation. The author wishes to thank Prof. dr. W. Schaal for the invitation to Germany and to the Foundation for providing the excellent work environment.

2. We now show how our result implies Theorem A. For brevity let us consider the case $\theta = 1/2$. This is the very case where Theorem A is really deep. We consider the function

$$F(z) = e^{-z/2} K(z, \chi_0) = \sum_{\gamma > 0} \frac{1}{\varrho} e^{i\gamma z},$$

where χ_0 is the primitive principal character and the summation is taken over all non-trivial zeros $\varrho = 1/2 + i\gamma$ of the Riemann zeta function with positive imaginary parts. F belongs to the class $\mathfrak{A}$; conditiom 4 is satisfied with $L_0 = 0$ (cf. [9]). Moreover, by (3.3), (4.3), and (8.11) of [9], we have for $z = x+iy$, $y > 0$, $|x| \leqslant 1/2$

$$K(z, \chi_0) = K(i, \chi_0) + \int_i^z \left(\frac{1}{2\pi i} \frac{e^s}{e^s - 1} \log s + \frac{\alpha}{s} + N(s) \right) ds$$

$$= \frac{1}{4\pi i} \log^2 z + \alpha \log z + O(1),$$

where $\alpha = (\log(2\pi) + C)/(2\pi i) - 1/4$ (C is the Euler constant) and N denotes a function which is holomorphic and bounded for $|x| \leqslant 1/2$, $y \geqslant 0$. Hence

$$\operatorname{Re} K(re^{i\varphi}, \chi_0) = \frac{1}{2\pi} \left(\varphi - \frac{\pi}{2} \right) \log r + O(1), \qquad 0 < r < 1, 0 < \varphi < \pi,$$

and consequently $\alpha(F) = -\infty$, $\beta(F) = \infty$. Moreover, by Theorem 4.1 of [9], we have

$$P(x) = -\tfrac{1}{2} e^{-x/2} (\psi(e^x) - e^x) + O(xe^{-x/2})$$

as $x \to \infty$. Therefore we have

COROLLARY 1 (Assume the Riemann Hypothesis). *For every positive constant A there exists a number $c_1 = c_1(A) > 1$ such that, for all $x > 1$, the interval $(x, c_1 x)$ contains integers n and n' satisfying*

$$\psi(n) - n > A\sqrt{n}, \qquad \psi(n') - n' < -A\sqrt{n'}. \tag{2.1}$$

Observe that the Theorem implies that the numbers n and n' in (2.1) are real. But, in view of $\psi(x) - x = \psi([x]) - [x] + O(1)$, this supports our somewhat stronger formulation. Moreover, by partial summation we have

$$\pi(x) - \operatorname{li} x = \frac{\psi(x) - x}{\log x} + O\left(\frac{\sqrt{x}}{\log x} \right).$$

Hence any solution (n, n') of (2.1) satisfies the inequalities

$$\frac{\pi(n)-\operatorname{li} n}{\sqrt{n}/\log n} > A+O(1), \quad \frac{\pi(n')-\operatorname{li} n'}{\sqrt{n'}/\log n'} < -A+O(1),$$

which proves Theorem A.

The case $\theta > 1/2$ of Theorem A can be established much more simply by appealing directly to the almost periodicity of the (absolutely convergent) series

$$\operatorname{Re} \sum_{\substack{\operatorname{Re}\varrho=\theta \\ \gamma>0}} \frac{1}{\varrho} e^{i\gamma x} \sim \lim_{y\to 0^+} \operatorname{Re} e^{-\theta z} K(z, \chi_0) \sim -\tfrac{1}{2}(\psi(e^x)-e^x)e^{-\theta x}$$

$(z = x+iy,\ x > 0,\ y > 0)$.

3. Relations between the Theorem and Theorem B of Ingham are not so direct since neither of them follows from the other. In spite of that both lead to similar conclusions when applied to specific arithmetic problems. Let us now formulate an obvious consequence of the Theorem.

COROLLARY 2. *We have*

$$\limsup_{x\to\pm\infty} P(x) = \beta(F), \quad \liminf_{x\to\pm\infty} P(x) = \alpha(F).$$

What makes Theorem B very useful is the fact that it estimates the lower and upper limits of $A(u)$ as $u\to\infty$ in terms of a finite almost periodic polynomial. To relate our Theorem with such polynomials we only have to convolve F in (1.3) with an appropriate kernel $\hat{r}\in L^1(-\infty, \infty)$. Then for every positive y we have

$$\int_{-\infty}^{\infty} F(u+iy)\hat{r}(u-x)du = \sum_n a_n r\left(\frac{w_n}{2\pi}\right)e^{iw_n z} = F^*(z), \tag{3.1}$$

say, where r is the inverse Fourier transform of $\hat{r}$:

$$r(u) = \int_{-\infty}^{\infty} \hat{r}(v)e(vu)\,dv \quad (e(\theta) = e^{2\pi i\theta}).$$

In case $\hat{r} \geqslant 0$ and $r(0) = 1$ from (3.1) it is apparent that

$$\alpha(F) \leqslant \alpha(F^*) \leqslant \beta(F^*) \leqslant \beta(F).$$

So we get the following

COROLLARY 3. *The Theorem is true for any a satisfying*

$$\alpha(F^*) < a < \beta(F^*).$$

Moreover, we have

$$\limsup_{x\to\pm\infty} P(x) \geqslant \beta(F^*), \quad \liminf_{x\to\pm\infty} P(x) \leqslant \alpha(F^*).$$

From this we get Ingham's relations (1.1), (1.2) by putting

$$r(u) = \begin{cases} 1-|u|/T & \text{for } |u| \leqslant T, \\ 0 & \text{for } |u| \geqslant T, \end{cases}$$

$$\hat{r}(v) = T\left(\frac{\sin(\pi T v)}{\pi T v}\right)^2$$

Ingham writes [6] that his choice of weight function was suggested in part by the systematic use of the Fejér kernel by N. Wiener in his work on Tauberian theorems. It is, however, clear that his method of proof leads to a statement more general than Theorem B and very similar to Corollary 3.

4. Estimates of type (1.1), (1.2) have proved to be very useful in numerical computations related to some conjectures such as the Pólya conjecture (concerning the sums $L(x) = \sum_{n \leqslant x}(-1)^{\Omega(n)}$, cf. [5]) or the Mertens conjecture (cf. [10])

$$M(x) = \sum_{n \leqslant x} \mu(n) \leqslant \sqrt{x}, \qquad x \geqslant x_0. \tag{4.1}$$

Using Theorem B and the Kronecker Approximation Theorem Ingham was able to prove that if the Riemann Hypothesis is true and the imaginary parts of the non-trivial zeros of the Riemann zeta function are linearly independent over Q, then

$$\liminf_{x\to\infty} M(x)/\sqrt{x} = -\infty, \tag{4.2}$$

$$\limsup_{x\to\infty} M(x)/\sqrt{x} = \infty. \tag{4.3}$$

Similar equalities hold when $M(x)$ is replaced by $L(x)$. Moreover,

$$\liminf_{x\to\infty} \frac{\pi(x, q, a)-(1/\varphi(q))\operatorname{li} x}{\sqrt{x}/\log x} = -\infty, \tag{4.4}$$

$$\limsup_{x\to\infty} \frac{\pi(x, q, a)-(1/\varphi(q))\operatorname{li} x}{\sqrt{x}/\log x} = \infty, \tag{4.5}$$

$$\liminf_{x\to\infty} \frac{\pi(x, q, a)-\pi(x, q, b)}{\sqrt{x}/\log x} = -\infty, \tag{4.6}$$

$$\limsup_{x\to\infty} \frac{\pi(x, q, a)-\pi(x, q, b)}{\sqrt{x}/\log x} = \infty, \tag{4.7}$$

for each reduced residue class $a \pmod q$ and $b \pmod q$, $a \not\equiv b \pmod q$, upon assuming the Generalized Riemann Hypothesis for Dirichlet's L-functions $(\operatorname{mod} q)$ and the linear independence over Q of the imaginary parts of their non-trivial zeros lying in the upper half-plane. In the proof of (4.2) and (4.3) Ingham made use of the fact that

$$\sum_{\varrho} \frac{1}{|\varrho\zeta'(\varrho)|} = \infty \tag{4.8}$$

(we assume, as we may in this case, that all zeros are simple). He needed Kronecker's theorem to solve the following system of unhomogeneous diophantine inequalities:

$$|\gamma x - \psi_\gamma - 2\pi m_\gamma| < \varepsilon, \quad 0 < \gamma < T, \quad m_\gamma \in \boldsymbol{Z}, \quad \psi_\gamma = \operatorname{Arg}(\varrho\zeta'(\varrho)). \tag{4.9}$$

For any x satisfying (4.9) we have

$$A_T^*(x) = \sum_{|\gamma| \leqslant T} \frac{1}{|\varrho\zeta'(\varrho)|}\left(1 - \frac{|\gamma|}{T}\right)(1 + O(\varepsilon)),$$

which is large when T is large enough according to (4.8). Hence $\beta(A_T^*) \to \infty$ as $T \to \infty$, which proves (4.3). Similar arguments with $\psi'_\gamma = \psi_\gamma + \pi$ in place of ψ_γ lead to (4.2). The relations (4.4)–(4.7) follow in the same manner from the divergence of $\sum 1/|\varrho|$.

Despite the fact that no way of proving the linear independence of zeta zeros has been found so far, and that we cannot provide a solution of (4.9) for each ε and T, we still can find a real number x satisfying (4.9) for certain values of ε and T by means of numerical computations. Applying Theorem B, this leads to inequalities of type

$$\liminf_{x \to \infty} M(x)/\sqrt{x} \leqslant -c_2, \tag{4.10}$$

$$\limsup_{x \to \infty} M(x)/\sqrt{x} \geqslant c_3. \tag{4.11}$$

This approach culminated in the important paper by A. Odlyzko and H. J. J. te Riele [10], who deduced that $c_2 = 1.009$, $c_3 = 1.06$, disproving in this way the Mertens conjecture (4.1) (cf. [10] for detailed history of the problem).

5. Using k-functions (or their modifications) we can prove conditional relations of types (4.2)–(4.7) using another principle. To establish the proper level of generality let us introduce the following notation. Let $\Omega = T^\infty$ denote the infinite-dimensional torus, i.e. the topological product of infinitely many copies of $T^1 = \{z \in \boldsymbol{C}\colon |z| = 1\}$. We have the continuous homomorphism $\lambda\colon \boldsymbol{R} \to \Omega$ defined by

$$\lambda(t) = \left(e(w_n t)\right)_{n=1}^{\infty},$$

where $w_1, w_2, \ldots$ denote the exponents in the definition of F (see (1.3)). Let $\Gamma = \Gamma(w_1, w_2, \ldots)$ denote the group $\overline{\lambda(\boldsymbol{R})}$, where the bar denotes the closure in the Tikhonov topology of Ω. Ω has a natural structure of T^1-set given by

$$T^1 \times \Omega \ni (u, \alpha) \mapsto u\alpha \in \Omega,$$

where $u\alpha = (u, u, \ldots)\alpha$. We consider the stabilizer S_0 of Γ, i.e. $S_0 = \{u \in T^1: u\Gamma = \Gamma\}$. Since evidently S_0 is a closed subgroup of T^1 we have only two possibilities: $S_0 = T^1$ or S_0 is a finite, cyclic group generated by a root of unity: $S_0 = \{1, \zeta, \zeta^2, \ldots, \zeta^{q-1}\}$, $\zeta^q = 1$. For instance, if the numbers $w_1, w_2, \ldots$ are linearly independent over $\boldsymbol{Q}$, then $S_0 = T^1$, by Kronecker's Approximation Theorem. However, it is not difficult to find examples of linearly dependent (over $\boldsymbol{Q}$) sequences $w_1, w_2, \ldots$ with non-trivial stabilizer S_0.

COROLLARY 4. *Let* $F \in \mathfrak{A}$ *and* $\zeta \in S_0$, $\zeta \neq \pm 1$. *Then* $\beta(F) \geqslant cm(F)$, $\alpha(F) \leqslant -cm(F)$, *where*

$$m(F) = \sup_{\substack{y>0 \\ x \in \boldsymbol{R}}} |F(x+iy)|$$

and $c = (\sqrt{3}-1)/2$. *In particular, the Theorem is true for every* $a \in (-cm(F), cm(F))$.

Let us apply Corollary 4 to the functions (cf. (9.1) below)

$$F_1(z) = e^{-z/2} \frac{1}{\varphi(q)} \sum_{\chi (\bmod q)} \overline{\chi(a)}\, K(z, \chi'), \qquad (a, q) = 1, \tag{5.1}$$

$$F_2(z) = e^{-z/2} \frac{1}{\varphi(q)} \sum_{\chi (\bmod q)} \big(\overline{\chi(a)} - \overline{\chi(b)}\big) K(z, \chi'), \tag{5.2}$$

$$(a, q) = (b, q) = 1, \qquad a \not\equiv b \,(\bmod\, q),$$

where χ' denotes the primitive Dirichlet character induced by χ (we assume the Generalized Riemann Hypothesis). They are unbounded on the upper half-plane. In fact, $|F_j(z)| \to \infty$, $j = 1, 2$, as z approaches a logarithmic singularity on the real axis (F_1 and F_2 have such singularities at every point of the form $x = \log p$, p prime, $p \nmid q$, $p \equiv a (\bmod\, q)$). Hence $m(F_j) = \infty$, $j = 1, 2$. Similar comments apply to

$$F_3(z) = \sum_{\gamma > 0} \frac{1}{\varrho \zeta'(\varrho)} e^{i\gamma z}, \qquad \zeta(\varrho) = 0,\ 0 < \operatorname{Re} \varrho < 1,\ \gamma = \operatorname{Im} \varrho > 0 \tag{5.3}$$

(we assume the Riemann Hypothesis and simplicity of zeros). We have also $m(F_3) = \infty$ (compare the forthcoming paper by K. M. Bartz [2]).

COROLLARY 5. *Assume the Generalized Riemann Hypothesis for Dirichlet's L-functions* (mod q) *and let the stabilizer of* $\Gamma(\gamma'_1, \gamma'_2, \ldots)$, *where* $\gamma'_1, \gamma'_2, \ldots$ *denote* (*positive*) *imaginary parts of zeros of all L-functions* (mod q), *contain a number* $\zeta \neq \pm 1$. *Then the relations* (4.4)–(4.7) *hold. Similarly, assume that* *the stabilizer of* $\Gamma(\gamma_1, \gamma_2, \ldots)$, *where* $\gamma_1, \gamma_2, \ldots$ *denote (positive) imaginary parts of zeros of the Riemann zeta function, contains a number* $\zeta \neq \pm 1$. *Then the relations* (4.2) *and* (4.3) *hold.*

Although, as in the case of linear independence, we cannot check whether the stabilizers under consideration contain a number $\zeta \neq \pm 1$ or not, we can use the underlying ideas to the proof of results of type (4.10) and (4.11). This time we have to solve the following diophantine problem (written in general symbols referring to F defined by (1.3)):

$$(5.4) \qquad |w_n x - \alpha - 2\pi m_n| < \varepsilon, \quad m_n \in \mathbf{Z}, \quad n = 1, 2, \ldots, N,$$
$$\alpha \text{ fixed}, \ 0 < \alpha < 2\pi, \ \alpha \neq \pi.$$

For any solution x of (5.4) and any $z = x + iy$, $y \geqslant 0$, we have

$$(5.5) \qquad \begin{aligned} \beta(F) &\geqslant |\sin\alpha|\,|\mathrm{Im}\, F^*(z)| + (\cos\alpha)\mathrm{Re}\, F^*(z) - \varepsilon \|F^*\|, \\ \alpha(F) &\leqslant -|\sin\alpha|\,|\mathrm{Im}\, F^*(z)| + (\cos\alpha)\mathrm{Re}\, F^*(z) + \varepsilon \|F^*\| \end{aligned}$$

where F^* is defined by (3.1) with the help of a weight function $\hat{r} \in L^1(-\infty, \infty)$, $\hat{r} \geqslant 0$, $r(0) = 1$, such that $r(u) = 0$ for $u > w_N/(2\pi)$, and

$$\|F^*\| = \sum_{n=1}^{N} |a_n| \hat{r}(w_n/(2\pi)).$$

If now $\mathrm{Im}\, F$ is unbounded on the upper half-plane, as it happens in the case of $F = F_j$, $j = 1, 2, 3$, defined by (5.1)–(5.3), it is quite natural to expect that $\mathrm{Im}(F^*)$ is "large". For instance, $\mathrm{Im}(F_1^*)$ and $\mathrm{Im}(F_2^*)$ should be "large" at $x = k \log p$, $p \nmid q$, p prime, $p \equiv a \pmod q$ or $p \equiv b \pmod q$, $k \in \mathbf{Z}$, since F_j^*, $j = 1, 2$, approximates F_j in some sense. Similarly $\mathrm{Im}(F_3^*)$ should be "large" at $x = \log n$, $\mu(n) \neq 0$. Moreover, the diophantine problem (5.4) seems to be easier than (4.9). Hence one may hope at least to improve the numerical values of c_2 and c_3 in (4.10) and (4.11) using the above-described method. It would be very desirable to test whether numerical computations confirm such hope.

6. We have the following simple consequence of Corollary 2.

COROLLARY 6. *We have*

$$\liminf_{x \to -\infty} P(x) = \liminf_{x \to \infty} P(x),$$

$$\limsup_{x \to -\infty} P(x) = \limsup_{x \to \infty} P(x).$$

This result is of importance to number theory when both functions $P|_{(-\infty,0)}$ and $P|_{(0,\infty)}$ have arithmetic interpretations. For instance, applying this principle to F_1 defined by (5.1) we obtain the following result, which is interesting as long as neither of (4.4) and (4.5) is proved.

COROLLARY 7. *Assume the Generalized Riemann Hypothesis. Then*

$$\liminf_{x\to\infty}\frac{\psi(x,q,a)-\frac{1}{\varphi(q)}x}{\sqrt{x}} = -\limsup_{x\to\infty}\sqrt{x}\left(\tilde{\psi}(x,q,\bar{a})-\frac{1}{\varphi(q)}\log x+b(q,\bar{a})\right),$$

$$\limsup_{x\to\infty}\frac{\psi(x,q,a)-\frac{1}{\varphi(q)}x}{\sqrt{x}} = -\liminf_{x\to\infty}\sqrt{x}\left(\tilde{\psi}(x,q,\bar{a})-\frac{1}{\varphi(q)}\log x+b(q,\bar{a})\right),$$

where $\bar{a}$ denotes the residue class $(\operatorname{mod} q)$ *such that* $a\bar{a}\equiv 1 \pmod q$, *and*

$$\tilde{\psi}(x,q,\bar{a}) = \sum_{\substack{n\leqslant x\\ n\equiv\bar{a}(\operatorname{mod} q)}} \Lambda(n)/n.$$

Moreover, for every residue class $a(\operatorname{mod} q)$ *we put*

$$b(q,\bar{a}) = \frac{1}{\varphi(q)}\sum_{\chi(\operatorname{mod} q)}\overline{\chi(a)}\,C(\chi')-\sum_{p^k\|q}\left(1-\frac{1}{p^{g_p}}\right)^{-1}\frac{1}{p^{l_0}\varphi(p^k)}\log p,$$

where $C(\chi')$ is defined by (4.6) *and* (4.8) *of* [9] *and the last sum is taken over all prime divisors p of q such that the class $\bar{a}$* $(\operatorname{mod} qp^{-k})$ *belongs to the cyclic group generated by* $p(\operatorname{mod} qp^{-k})$; *$g_p$ denotes the order of* $p(\operatorname{mod} qp^{-k})$ *and $l_0\in N$ is uniquely determined by the conditions*

$$0<l_0\leqslant g_p,\qquad \bar{a}\equiv p^{l_0}(\operatorname{mod} qp^{-k}).$$

This result describes a somewhat unexpected connection between the primes in two (in general) different arithmetic progressions $n\equiv a(\operatorname{mod} q)$ and $n\equiv\bar{a}(\operatorname{mod} q)$, $a\bar{a}\equiv 1 \pmod q$. This connection seems to be of a deeper nature and we shall return to this subject later.

7. Proof of the Theorem (cf. [8]). Let us first prove the Theorem for a function $F\in\mathfrak{A}$ satisfying condition 3 with $B=0$. In this case F is holomorphic for $z=x+iy$, $y>0$, and continuous for $y\geqslant 0$ and $P(z)=\operatorname{Re}F(z)$ is harmonic on the upper half-plane. By the maximum-modulus principle for harmonic functions we get

$$\alpha(F)=\inf_{x\in\mathbf{R}}P(x),\qquad \beta(F)=\sup_{x\in\mathbf{R}}P(x).$$

Hence our assertion follows at once from the almost periodicity of P.

In the general case, consider the subsidiary function

$$F_\delta(z)=\sum_{n=1}^{\infty}a_nS^B(\delta w_n)e^{iw_nz},\qquad z=x+iy,\ y>0,\ \delta>0,$$

where

$$S(u) = \begin{cases} (\sin u)/u, & u \neq 0, \\ 1, & u = 0. \end{cases}$$

Since $S(u) \leqslant \min(1, 1/|u|)$, the sum F_δ is absolutely convergent for $y \geqslant 0$. Moreover, $F_\delta \to F$ as $\delta \to 0$ almost uniformly on the upper half-plane and thus $\alpha(F_\delta) \to \alpha(F)$ and $\beta(F_\delta) \to \beta(F)$ as $\delta \to 0$. Fix a positive δ_0 so small that

$$\alpha(F_{\delta_0}) < a < \beta(F_{\delta_0}).$$

Using the Lebesgue bounded convergence theorem we have for $|x| > L_0 + B\delta_0$

$$\operatorname{Re} F_{\delta_0}(x) = \frac{1}{(2\delta_0)^B} \int_{-\delta_0}^{\delta_0} \ldots \int_{-\delta_0}^{\delta_0} P(x+t_1+\ldots+t_B)\,dt_1 \ldots dt_B.$$

Hence for such x we obtain

$$\min_{|t-x| \leqslant B\delta_0} P(t) \leqslant \operatorname{Re} F_{\delta_0}(x) \leqslant \max_{|t-x| \leqslant B\delta_0} P(x).$$

Since our Theorem is true for F_{δ_0} this concludes the proof: it suffices to take

$$l(a, F) = l(a, F_{\delta_0}) + 2B\delta_0.$$

8. Proof of Corollary 4 and relation (5.5). Suppose that $\zeta \in S_0$, $\zeta = e + if$, $f > 0$. Then $\bar{\zeta} \in S_0$ and there exist two sequences (x_n), (y_n) of real numbers such that

$$\lim_{n \to \infty} F(z + x_n) = \zeta F(z), \qquad \lim_{n \to \infty} F(z + y_n) = \bar{\zeta} F(z)$$

almost uniformly for $z = x + iy$, $y > 0$. Hence

$$\beta(F) \geqslant f\,|\operatorname{Im} F(z)| + e \operatorname{Re} F(z) \tag{8.1}$$

for every z from the upper half-plane.

Now let $z = x + iy$, $y > 0$, be fixed and consider two cases.

Case I. $|\operatorname{Re} F(z)| \leqslant \beta(F)$. It is easy to see that if S_0 contains an element $\zeta \neq \pm 1$, then it contains a root of unity $\zeta^* = e(1/m)$ with $m \geqslant 3$ as well. Hence there exists a natural number l such that

$$|\operatorname{Arg} \zeta^{*l} - \pi/2| \leqslant \pi/m \leqslant \pi/3.$$

When $\zeta^{*l} = e^* + if^*$, we have $|e^*| \leqslant 1/2$, $f^* \geqslant \sqrt{3}/2$. Hence, by (8.1) we obtain

$$\beta(F) \geqslant f^* |\operatorname{Im} F(z)| - |e^*| \beta(F) \geqslant f^* |F(z)| - (f^* + |e^*|) \beta(F)$$

and consequently

$$\beta(F) \geqslant \frac{f^*}{1 + f^* + |e^*|} |F(z)| \geqslant \frac{\sqrt{3}-1}{2} |F(z)|. \tag{8.2}$$

Case II. $|\mathrm{Re}\, F(z)| > \beta(F)$. Then we have $\mathrm{Re}\, F(z) < 0$ and if $\zeta^* = e(1/m) \in S_0$, then $2 \nmid m$. Indeed, otherwise $-1 = (\zeta^*)^{m/2} \in S_0$ and it follows from (8.1) that $|\mathrm{Re}\, F(z)| \leqslant \beta(F)$, a contradiction.

If $m = 3$, then by (8.1), we have

$$\beta(F) \geqslant \frac{\sqrt{3}}{2}|\mathrm{Im}\, F(z)| + \tfrac{1}{2}|\mathrm{Re}\, F(z)| \geqslant \tfrac{1}{2}|F(z)| \geqslant \frac{\sqrt{3}-1}{2}|F(z)|. \tag{8.3}$$

Assume that $m \geqslant 5$. Then

$$\min_{l \in N} \mathrm{Re}\, \zeta^{*l} = -\cos(\pi/m) < -1 + (\pi^2/(2m^2)).$$

Hence, using (8.1) we have

$$\beta(F) \geqslant \left(1 - \frac{\pi^2}{2m^2}\right)|\mathrm{Re}\, F(z)|. \tag{8.4}$$

Take an arbitrary element $\zeta = e + if \in S_0$, $f \neq 0$, $e < 0$. By (8.1) we have

$$\beta(F) \geqslant |e|\,|\mathrm{Re}\, F(z)| + f\,|\mathrm{Im}\, F(z)| \geqslant |e|\beta(F) + f\,|\mathrm{Im}\, F(z)|.$$

Hence

$$\beta(F) \geqslant \frac{1 - \pi^2/(2m^2)}{2 - \pi^2/(2m^2)}|F(z)| \geqslant \frac{\sqrt{3}-1}{2}|F(z)| \tag{8.5}$$

because $m \geqslant 5$. Combining (8.2), (8.3) and (8.6) we obtain

$$\beta(F) \geqslant \frac{\sqrt{3}-1}{2} m(F).$$

Applying the above arguments to the function $-F$ we get the corresponding inequality for $\alpha(F)$. This concludes the proof.

For the proof of (5.5) observe that for x satisfying (5.4) we have

$$|w_n(-x) + \alpha + 2\pi m_n| < \varepsilon, \qquad n = 1, 2, \ldots, N.$$

Hence for every $z = x + iy$, $y > 0$, we obtain

$$F^*(z \pm iy) = e^{\pm i\alpha} F^*(z) + \theta\varepsilon \|F^*\|, \qquad |\theta| \leqslant 1.$$

Taking real parts and choosing the appropriate sign in the exponent we obtain (5.5).

9. Proof of Corollary 7. All we have to prove is that for $x > 0$

$$\lim_{y \to 0^+} \mathrm{Re}\, F_1(x + iy) = -\tfrac{1}{2} e^{-x/2}\left(\psi_0(e^x, q, a) - \frac{1}{\varphi(q)} e^x\right) + c_4 + g_1(x) \tag{9.1}$$

and for $x<0$

$$\lim_{y\to 0^+} \operatorname{Re} F_1(x+iy) \tag{9.2}$$
$$= \tfrac{1}{2} e^{|x|/2}\left(\tilde{\psi}_0(e^{|x|}, q, \bar{a}) - \frac{1}{\varphi(q)} x + b(q, \bar{a})\right) + c_4 + g_2(x),$$

where $c_4 = c_4(a, q)$ denotes a constant and g_1, g_2 are functions tending to zero as x tends to ∞ or $-\infty$ respectively.

We have

$$\operatorname{Re} F_1(z) = \operatorname{Re} \frac{e^{-z/2}}{\varphi(q)} \sum_{\chi (\operatorname{mod} q)} \overline{\chi(a)} K(z, \chi')$$
$$= \frac{1}{2} \operatorname{Re} \frac{e^{-z/2}}{\varphi(q)} \sum_{\chi (\operatorname{mod} q)} \overline{\chi(a)} \{K(z, \chi') + \overline{K(z, \bar{\chi}')}\}.$$

Hence by Theorem 4.1 of [9] we obtain (using the notation of [9])

$$\lim_{y\to 0^+} \operatorname{Re} F_1(x+iy) = \frac{1}{2} e^{-x/2} \frac{1}{\varphi(q)} \sum_{\chi (\operatorname{mod} q)} \overline{\chi(a)} F(x, \chi'), \tag{9.3}$$

where for $x>0$

$$F(x, \chi') = -2m(1/2, \chi) e^{x/2} - \psi_0(e^x, \chi') + e(\chi) e^x - e_1(\chi) x - R(x, d) + B(\chi), \tag{9.4}$$

and for $x<0$

$$F(x, \chi') = -2m(1/2, \chi) e^{x/2} + \tilde{\psi}_0(e^x, \bar{\chi}') + e(\chi) e^x + e(\chi) x + R(|x|, 1-d) + C(\chi). \tag{9.5}$$

Here $m(1/2, \chi)$ denotes the multiplicity of the zero of $L(s, \chi)$ at $s = 1/2$ ($m(1/2, \chi) = 0$ when $L(1/2, \chi) \neq 0$).

Inserting (9.4) into (9.3) and taking into account that

$$R(x, d) = O(e^{-x}), \qquad x \to \infty, \tag{9.6}$$

and that by the orthogonality law for characters,

$$\frac{1}{\varphi(q)} \sum_{\chi (\operatorname{mod} q)} \overline{\chi(a)} \psi_0(e^x, \chi') = \sum_{\substack{n \leqslant e^x \\ n \equiv a (\operatorname{mod} q)}} \Lambda(n) + O\Big(\sum_{\substack{p^k \leqslant e^x \\ p|q}} \log p\Big)$$
$$= \psi_0(e^x, q, a) + O(x)$$

we get (9.1) with

$$c_4 = -\frac{1}{\varphi(q)} \sum_{\chi (\operatorname{mod} q)} \overline{\chi(a)} m(1/2, \chi) \quad \text{and} \quad g_1(x) = O(x e^{-x/2}).$$

Considering the case $x<0$ we have to be more careful with constants. As before we insert (9.5) into (9.3). By (9.6), the sum involving $R(|x|, 1-d)$ contributes at most $O(e^{-|x|/2})$. Moreover, we have

$$(9.7)\quad \frac{1}{\varphi(q)}\sum_{\chi(\bmod q)}\overline{\chi(a)}\tilde{\psi}_0(e^{|x|}, \bar{\chi}')$$
$$= \tilde{\psi}_0(e^{|x|}, q, \bar{a}) + \sum_{\substack{p^m \leqslant e^x \\ p|q}} \frac{\log p}{p^m}\left(\frac{1}{\varphi(q)}\sum_{\chi(\bmod q)}\overline{\chi(\bar{a})}\chi'(p^m)\right)$$
$$= \tilde{\psi}_0(e^{|x|}, q, \bar{a}) + R, \quad \text{say.}$$

To estimate R we have to compute the inner sum. Writing $q_p = qp^{-k}$ when $p^k \| q$ we obtain

$$\frac{1}{\varphi(q)}\sum_{\chi(\bmod q)}\overline{\chi(\bar{a})}\chi'(p^m) = \frac{1}{\varphi(q)}\sum_{d|q}\sum_{\chi(\bmod q)}^{*}\overline{\chi(\bar{a})}\chi(p^m)$$
$$= \frac{1}{\varphi(q)}\sum_{d|q_p}\sum_{\chi(\bmod d)}^{*}\overline{\chi(\bar{a})}\chi(p^m) = \frac{1}{\varphi(q)}\sum_{\chi(\bmod q_p)}\overline{\chi(\bar{a})}\chi(p^m)$$
$$= \begin{cases} \dfrac{1}{\varphi(p^k)} & \text{if } p^m \equiv \bar{a} (\bmod q_p), \\ 0 & \text{otherwise.} \end{cases}$$

The star by the sum sign indicates that the summation is taken over the primitive characters only. Therefore we have

$$R = \sum_{p|q}\sum_{\substack{m \\ p^m \equiv \bar{a}(\bmod q_p)}} \frac{\log p}{\varphi(p^k)p^m} + O(e^{-|x|})$$
$$= \sum_{p|q}\frac{\log p}{\varphi(p^k)}\sum_{\substack{m \\ p^m \equiv \bar{a}(\bmod q_p)}} \frac{1}{p^m} + O(e^{-|x|}).$$

The inner sum is non-empty when the class $\bar{a}(\bmod q_p)$ belongs to the cyclic group generated by $p(\bmod q_p)$. Moreover,

$$\sum_{\substack{m \\ p^m \equiv \bar{a}(\bmod q_p)}} \frac{1}{p^m} = \sum_{k=0}^{\infty}\frac{1}{p^{kg_p+l_0}} = \frac{1}{p^{l_0}}\left(1-\frac{1}{p^{g_p}}\right)^{-1}.$$

Hence

$$(9.8)\qquad R = \sum_{p^k\|q}\frac{\log p}{\varphi(p^k)p^{l_0}}\left(1-\frac{1}{p^{g_p}}\right)^{-1} + O(e^{-|x|}).$$

Combining (9.3), (9.5), (9.7) and (9.8) we get (9.2) with $g_2(x) = O(e^{-|x|/2})$. This concludes the proof.

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Received on 9.1.1989
and in revised form on 3.11.1989 (1897)