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The k-functions in multiplicative number theory, III
Uniform distribution of zeta zeros; discrepancy

by

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1. Introduction and statement of results. As in part II of this cycle [2] let us denote by $\mathscr{A} = [a_{nk}]$ the positive Toeplitz matrix defined by

$$a_{nk} = \frac{1}{n!S_n} e^{-\gamma_k}\gamma_k^n, \quad n \geqslant 1, k \geqslant 1,$$

$$S_n = \frac{1}{n!}\sum_{k=1}^{\infty} e^{-\gamma_k}\gamma_k^n,$$

where $0 < \gamma_1 \leqslant \gamma_2 \leqslant \gamma_3 \leqslant \ldots$ denote the positive imaginary parts of the non-trivial zeros $\varrho_k = \beta_k + i\gamma_k$ of a fixed Dirichlet L-function corresponding to a primitive Dirichlet character $\chi(\bmod q)$, $q \geqslant 1$. Each γ_k occurs in this sequence according to the multiplicity of ϱ_k. This matrix defines a certain summation method and a kind of uniform distribution (mod 1) (matrix method $\mathscr{A}$ and $\mathscr{A}$-uniform distribution (mod 1) resp., cf. [5]), which were the subject of study in part II. The main result proved there states that for every real $x \neq 0$ the sequence $x\gamma_k$, $k = 1, 2, 3, \ldots$, is $\mathscr{A}$-uniformly distributed (mod 1). Our goal now is to pursue some further questions related to this theorem. Following the usual pattern denote by $D_n^*(x)$ the $\mathscr{A}$-discrepancy of the sequence $(x\gamma_k)$, i.e.

$$D_n^*(x) = \sup_{0 \leqslant t \leqslant 1} \left| \frac{1}{n!S_n} \sum_{\substack{k=1 \\ 0 \leqslant \{x\gamma_k\} < t}}^{\infty} e^{-\gamma_k}\gamma_k^n - t \right|, \tag{1.1}$$

where $\{u\}$ denotes the fractional part of a real number u.

THEOREM 1. *For every real $x \neq 0$ there exists a positive constant $c_0 = c_0(x)$ such that*

$$D_n^*(x) \leqslant c_0 \left(\frac{\log\log n}{\log n}\right)^{2/3} \quad \textit{for } n \geqslant 3. \tag{1.2}$$

Though this upper estimate is non-trivial, most probably it does not represent the true order of magnitude of $D_n^*(x)$. A better estimate is attained subject to a kind of density hypothesis. As in part II let us write

$$R_n(x) = \frac{1}{n!}\sum_{k=1}^{\infty} e^{-\gamma_k}\gamma_k^n e^{(\beta_k-1/2)x} e^{i\gamma_k x},$$

$$S_n(x) = S_n\mathscr{A}\left((e^{i\gamma_k x})_{k=1}^{\infty}\right) = \frac{1}{n!}\sum_{k=1}^{\infty} e^{-\gamma_k}\gamma_k^n e^{i\gamma_k x}.$$

THEOREM 2. *Suppose that for every real* $x \neq 0$ *there exists a positive constant* $c_1 = c_1(x)$ *such that for every natural* $n \geqslant 2$ *we have*

$$\sum_{h=1}^{[\log n]} \frac{1}{h}|R_n(hx) - S_n(hx)| \leqslant c_1. \tag{1.3}$$

Then there exists a positive $c_2 = c_2(x)$ *such that*

$$D_n^*(x) \leqslant \frac{c_2}{\log n} \quad \textit{for } n \geqslant 2. \tag{1.4}$$

Let us remark that (1.3) postulates, roughly speaking, that the density of non-trivial L-function zeros lying outside the critical line is "small". In the extremal case, when the Generalized Riemann Hypothesis is true, $R_n(y) = S_n(y)$ for every real y and natural n and the left-hand side of (1.3) is identically zero. It is obvious, therefore, that (1.3) is much less stringent than this hypothesis and one may hope it can be proved by the existing methods.

Estimate (1.4) seems to be the best possible. There are reasons to believe that for certain x the asymptotic relation of type $D_n^*(x) \sim c/\log n$ holds. More explicitly, we can put forward the following conjecture.

CONJECTURE A. *For* $x \in \boldsymbol{R}\setminus\{0\}$ *and* n *tending to infinity we have*

$$D_n^*(x) = \alpha(x)/\log n + o(1/\log n), \tag{1.5}$$

where

$$\alpha(x) = \frac{\log p}{\pi d}\left(\arcsin p^{-|a|/2} + \left|\operatorname{Arg}(1-\chi(p^a)p^{-|a|/2})\right|\right) \tag{1.6}$$

for

$$x = \frac{1}{2\pi}\frac{a}{d}\log p, \quad p\nmid q, \quad p \textit{ prime}, \ a, d \in \boldsymbol{Z}, \quad d > 0, \quad (a, d) = 1 \tag{1.7}$$

and $\alpha(x) = 0$ *otherwise.*

Introducing the discrepancy function $H_{n,x}$ by the formula

$$H_{n,x}(t) = \frac{\log n}{n!S_n} \sum_{\substack{k \geqslant 1 \\ 0 \leqslant \{x\gamma_k\} < t}} e^{-\gamma_k}\gamma_k^n - t\log n, \qquad 0 \leqslant t \leqslant 1, \tag{1.8}$$

we can state a still more general conjecture.

CONJECTURE B. *For every real* $x \neq 0$ *we have*

$$\lim_{n\to\infty} H_{n,x}(t) = H_{\infty,x}(t) \tag{1.9}$$

uniformly in $t \in [0, 1]$, *where*

$$H_{\infty,x}(t) = \begin{cases} \dfrac{\log p}{\pi d}\big(\operatorname{Arg}(1-\chi(p^{-a})p^{-|a|/2}e(dt)) + \operatorname{Arg}(1-\chi(p^{a})p^{-|a|/2})\big) \\ \qquad \text{if } x \text{ is given by (1.7)}, \\ 0 \quad \text{otherwise} \end{cases} \tag{1.10}$$

($e(u)$ *denotes* $\exp(2\pi i u)$).

It can easily be seen that Conjecture A follows from Conjecture B; in fact, $\alpha(x) = \|H_{\infty,x}\|_\infty = \sup_{0\leqslant t\leqslant 1}|H_{\infty,x}(t)|$ and $\|H_{n,x}\|_\infty \to \|H_{\infty,x}\|_\infty$ if $H_{n,x}\to H_{\infty,x}$ uniformly on [0, 1]. Moreover, let us write

$$h_{n,x} = \int_0^1 H_{n,x}(t)\,dt$$

and define the normalized discrepancy function $\tilde{H}_{n,x}$ as follows:

$$\tilde{H}_{n,x}(t) = H_{n,x}(t) - h_{n,x}, \qquad 0 \leqslant t \leqslant 1.$$

It can be observed that Conjecture B is equivalent to Conjectures B_1 and B_2 below.

CONJECTURE B_1. *For every real* $x \neq 0$ *we have*

$$\lim_{n\to\infty} \tilde{H}_{n,x}(t) = \tilde{H}_{\infty,x}(t)$$

uniformly in $t \in [0, 1]$, *where*

$$\tilde{H}_{\infty,x}(t) = \begin{cases} \dfrac{\log p}{\pi d}\operatorname{Arg}(1-\chi(p^{-a})p^{-|a|/2}e(dt)) \\ \qquad \text{if } x \text{ is given by (1.7)}, \\ 0 \quad \text{otherwise}. \end{cases}$$

CONJECTURE B_2. *For every real* $x \neq 0$ *we have*

$$\lim_{n\to\infty} h_{n,x} = h_{\infty,x},$$

where

$$h_{\infty,x} = \begin{cases} \dfrac{\log p}{\pi d}\operatorname{Arg}(1-\chi(p^{a})p^{-|a|/2}) \\ \qquad \text{if } x \text{ is given by (1.7)}, \\ 0 \quad \text{otherwise}. \end{cases}$$

Though we cannot prove any of these conjectures it is possible to obtain certain partial results. It is evident that the functions $H_{n,x}$, $\tilde{H}_{n,x}$, $H_{\infty,x}$ and $\tilde{H}_{\infty,x}$ belong to the space $L^\infty = L^\infty(0, 1)$ of essentially bounded, Lebesgue measurable functions defined on $[0, 1]$. L^∞ is a Banach space with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{0 \leqslant t \leqslant 1} |f(t)|.$$

Conjecture B asserts that $H_{n,x}$, $n = 1, 2, \ldots$, tends to $H_{\infty,x}$ in the (strong) topology of L^∞. Similarly, Conjecture B_1 postulates the same upon $\tilde{H}_{n,x}$ and $\tilde{H}_{\infty,x}$, respectively. From elementary functional analysis we know that L^∞ is the dual space to $L^1 = L^1(0, 1)$, the space of Lebesgue integrable functions on $[0, 1]$; every $f \in L^\infty$ defines a unique functional $f^* \in (L^1)^*$ by the formula

$$f^*(g) = \int_0^1 f(t) g(t)\, dt, \qquad g \in L^1.$$

We can endow L^∞ with the weak L^1-topology, i.e. the topology with the basic neighbourhoods of zero defined by

$$V(g_1, \ldots, g_n; \varepsilon) = \{f \in L^\infty : \ |f^*(g_j)| < \varepsilon, \ j = 1, \ldots, n\}$$

$(g_j \in L^1, \ j = 1, \ldots, n, \ \varepsilon > 0)$.

THEOREM 3. *Suppose that for a real $x \neq 0$ the condition* (1.3) *is satisfied. Then the sequence of functions $\tilde{H}_{n,x} \in L^\infty$, $n = 1, 2, \ldots$, tends to $\tilde{H}_{\infty,x}$ in weak L^1-topology.*

In the following corollaries we assume that both $x \neq 0$ and (1.3) is satisfied.

COROLLARY 1. *Suppose that the sequence $\tilde{H}_{n,x}$, $n = 1, 2, \ldots$, is convergent almost everywhere in $[0, 1]$ to a function $\tilde{f}$. Then $\tilde{f} = \tilde{H}_{\infty,x}$ almost everywhere.*

COROLLARY 2. *Suppose that the sequence $H_{n,x}$, $n = 1, 2, \ldots$, is pointwise convergent to a continuous function f. Then $f = H_{\infty,x}$ and $h_{n,x} \to h_{\infty,x}$ as $n \to \infty$.*

COROLLARY 3. *We have*

$$\liminf_{n \to \infty} \tilde{H}_{n,x}(t) \leqslant \tilde{H}_{\infty,x}(t) \leqslant \limsup_{n \to \infty} \tilde{H}_{n,x}(t)$$

for almost all $t \in [0, 1]$.

COROLLARY 4. *Define the normalized discrepancy function of the sequence $x\gamma_k$, $k = 1, 2, \ldots$, by the formula*

$$\tilde{D}_n^*(x) = \left\| \frac{\tilde{H}_{n,x}}{\log n} \right\|_\infty, \qquad n \geqslant 2.$$

Then

$$\tilde{D}_n^*(x) \geqslant \tilde{\alpha}(x)/\log n + o(1/\log n),$$

where $\tilde{\alpha}(x) = \dfrac{\log p}{\pi d} \arcsin p^{-|a|/2}$ *if* x *is of the form* (1.7) *and* $\tilde{\alpha}(x) = 0$ *otherwise.*

COROLLARY 5. *Suppose that Conjecture* B_2 *is true. Then*

$$D_n^*(x) \geqslant \alpha(x)/\log n + o(1/\log n), \qquad n \geqslant 2,$$

where $\alpha(x)$ *is defined as in the formulation of Conjecture* A.

Theorem 3 and its corollaries strongly support Conjectures A and B, showing in particular that $H_{\infty,x}$ and $\tilde{H}_{\infty,x}$ are the only reasonable candidates for the limits of the discrepancy functions $H_{n,x}$ and $\tilde{H}_{n,x}$ respectively.

The foregoing conjectures and results have obvious connections with the problem of E. Landau who, on the basis of the classical explicit formulae, wondered about the arithmetical relations between prime numbers and zeros of the zeta function (cf. [4], § 89). As we have already seen the sequences $x\gamma_k$, $k = 1, 2, \ldots$, behave differently according as x is of the form (1.7) or not. This can be considered as a partial answer to Landau's question. However, the described relation between primes and zeta zeros seems to be more "diophantine" than "arithmetic" in character. Another approach to Landau's problem has been proposed by P. Turán [6].

2. Some auxiliary results.

LEMMA 1 (see [2], Lemma 4). *We have*

$$\left|\frac{1}{n!} \sum_{|\gamma_k - n| \geqslant 2\sqrt{n \log n}} e^{-\gamma_k} \gamma_k^n e^{(\beta_k - 1/2)x}\right| \ll \frac{e^{|x|/2}}{n}.$$

LEMMA 2 (see [2], Lemma 7). *For* $|x| \geqslant 1/2$,

$$R_n(x) = -\frac{1}{2\pi i^{n+1}} \sum_{\substack{p,k \\ |k \log p - x| \leqslant 1/8}} \frac{\chi(p^k) \log p}{\sqrt{p^{|k|}}\,(k \log p - (x+i))^{n+1}} + O(e^{2|x|} n^{-1/2} \log^2 n).$$

For $0 < |x| < 1/2$,

$$R_n(x) = O(n^{-1/2} \log^2 n),$$

where the implied constant depends on x. *Moreover,*

$$S_n = R_n(0) = \frac{1}{2\pi} \log \frac{qn}{2\pi} + O(n^{-1/2} \log^2 n).$$

LEMMA 3 (see [2], Lemma 8). *For real* σ, T, H *satisfying* $1/2 \leqslant \sigma \leqslant 1$, $T > 0$, $0 < H < T$ *let us denote by* $N(\sigma, T, H, \chi)$ *the number of* $L(s, \chi)$ *zeros*

$\varrho = \beta + i\gamma$ such that $\beta \geqslant \sigma$, $|\gamma - T| \leqslant H$. Then for every $T \geqslant e$, $H \geqslant T^{a+\varepsilon}$, $\varepsilon > 0$ and $a = 1/3$ we have

$$N(\sigma, T, H, \chi) \ll H^{4(1-\sigma)/(3-2\sigma)} \log^7 T.$$

LEMMA 4. *For $|x| \leqslant \frac{1}{4}\log n$, $n \geqslant 2$, we have*

$$|R_n(x) - S_n(x)| \ll n^{-7/8} + |x|^2 D^2 \log n + |x| n^{-D/4} \log^{6.5} n,$$

where D is an arbitrary real number such that $D|x| \leqslant 1$.

Proof. Lemma 1 yields

$$\begin{aligned} (2.1)\qquad R_n(x) - S_n(x) &= \frac{1}{n!} \sum_{|\gamma_k - n| < 2\sqrt{n \log n}} e^{-\gamma_k} \gamma_k^n \left(e^{(\beta_k - 1/2)x} - 1\right) + O(n^{-7/8}) \\ &= \frac{2}{n!} \sum_{\substack{|\gamma_k - n| < 2\sqrt{n \log n} \\ \beta_k > 1/2}} e^{-\gamma_k} \gamma_k^n \left(\cosh\left((\beta_k - 1/2)x\right) - 1\right) + O(n^{-7/8}) \\ &= 2x \int_0^{1/2} \frac{1}{n!} \sum_{\substack{|\gamma_k - n| < 2\sqrt{n \log n} \\ \beta_k > 1/2 + u}} e^{-\gamma_k} \gamma_k^n \sinh(ux)\, du + O(n^{-7/8}). \end{aligned}$$

We break the last integral into two integrals I_1 and I_2 for which $0 \leqslant u \leqslant \min(1/2, D)$ and $\min(1/2, D) \leqslant u \leqslant 1/2$, respectively. Since $D|x| \leqslant 1$ we have

$$\sinh(u|x|) \ll u|x| \quad \text{for } 0 \leqslant u \leqslant D$$

and hence,

$$(2.2)\qquad |I_1| \ll |x| \log n \int_0^D u\, du \ll |x| D^2 \log n.$$

Furthermore, we have $I_2 = 0$ for $D \geqslant 1/2$. Let $0 < D < 1/2$. Since $e^{-\gamma_k} \gamma_k^n \leqslant e^{-n} n^n \ll n!/\sqrt{n}$, we have

$$|I_2| \ll n^{-1/2} \int_D^{1/2} N(1/2 + u, n, 2\sqrt{n \log n}, \chi) \sinh(u|x|)\, du.$$

An application of Lemma 3 gives $n^{-1/2} N(1/2 + u, n, 2\sqrt{n \log n}, \chi) \ll n^{-u/2} \log^{7.5} n$ and thus

$$\begin{aligned} |I_2| &\ll \int_D^{1/2} \exp\left(-u\left(\tfrac{1}{2}\log n - |x|\right)\right) du \log^{7.5} n \\ &\ll \int_D^{1/2} \exp\left((-u/4)\log n\right) du \log^{7.5} n \ll n^{-D/4} \log^{7.5} n, \end{aligned}$$

which, together with (2.1) and (2.2), ends the proof.

LEMMA 5. *Let $F(t)$ be nondecreasing on $[0, 1]$ with $F(0) = 0$ and $F(1) = 1$ and let $G(t)$ satisfy a Lipschitz condition on $[0, 1]$, i.e.*

$$|G(t)-G(u)| \leqslant M|t-u|$$

for all $0 \leqslant t, u \leqslant 1$. Suppose that $G(0) = 0$ and $G(1) = 1$. Then for any positive integer m

$$\sup_{0\leqslant t\leqslant 1} |F(t)-G(t)| \leqslant \frac{4M}{m+1}+\frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h}-\frac{1}{m+1}\right)|\hat{F}(h)-\hat{G}(h)|$$

where

$$\hat{F}(h) = \int_0^1 \exp(2\pi iht)\,dF(t) \quad \text{and} \quad \hat{G}(h) = \int_0^1 \exp(2\pi iht)\,dG(t).$$

This is a generalized version of the well-known Erdős–Turán inequality proved by H. Niederreiter and W. Philipp [5] (compare also [1]).

3. Proof of Theorems 1 and 2. We apply Lemma 5 to the functions

$$F(t) = \frac{1}{n!S_n}\sum_{0\leqslant\{x\gamma_k\}<t} e^{-\gamma_k}\gamma_k^n \quad \text{and} \quad G(t) = t, \quad 0\leqslant t\leqslant 1.$$

Then for $h \geqslant 1$,

$$\hat{F}(h) = S_n(2\pi hx)/S_n \quad \text{and} \quad \hat{G}(h) = 0.$$

Hence for $m \geqslant 1$

$$\text{(3.1)} \quad D_n^*(x) \ll \frac{1}{m}+\frac{1}{\log n}\sum_{h=1}^{m}\frac{1}{h}|S_n(2\pi hx)|$$

$$\ll \frac{1}{m}+\frac{1}{\log n}\sum_{h=1}^{m}\frac{1}{h}|R_n(2\pi hx)|+\frac{1}{\log n}\sum_{h=1}^{m}\frac{1}{h}|R_n(2\pi hx)-S_n(2\pi hx)|$$

$$= 1/m+A+B, \quad \text{say.}$$

Let us consider A first. Suppose $m \leqslant (\log n)/(16\pi|x|)$. The sum A contains $O_x(1)$ terms of type $|R_n(y)|$ with $0<|y|\leqslant 1/2$. Their total contribution is at most $O_x(n^{-1/2}\log^2 n)$, by Lemma 2. For $(4\pi|x|)^{-1} \leqslant h \leqslant m$ we have, using the same lemma,

$$|R_n(2\pi hx)| \ll \sum_{\substack{p,k \\ |k\log p-2\pi hx|\leqslant 1/8}} \frac{\log p}{\sqrt{p^{|k|}}\,|1+(k\log p-2\pi hx)^2|^{n/2}}+O(n^{-1/4}\log^2 n).$$

If $p^{|k|} < P^{|K|}$ denote two prime-powers such that

$$|k\log p-2\pi hx| \leqslant 1/8 \quad \text{and} \quad |K\log P-2\pi hx| \leqslant 1/8,$$

then

$$|k\log p - K\log P| = \log(P^{|K|}/p^{|k|}) \geqslant \log(1+p^{-|k|}) \gg p^{-|k|} \gg e^{-2\pi h|x|} \gg n^{-1/8}.$$

Hence, there exists at most one number $N_h = p_h^{|k_h|}$ such that

$$|\pm\log N_h - 2\pi hx| \leqslant b_0 n^{-1/8}$$

with a sufficiently smali $b_0 > 0$. Therefore,

$$\sum_{\substack{p,k \\ |k\log p - 2\pi hx| \leqslant 1/8}} \frac{\log p}{\sqrt{p^{|k|}}\,|1+(k\log p - 2\pi hx)^2|^{n/2}}$$

$$\ll \frac{\log N_h}{N_h} + \psi(e^{2\pi h|x|})\exp\left(-\frac{n}{2}\log(1-b_1 n^{-1/4})\right) \ll he^{-h|x|} + \exp(-b_2 n^{5/8})$$

with sufficiently small positive b_1 and b_2. Hence

$$\frac{1}{\log n}\sum_{(4\pi|x|)^{-1}\leqslant h\leqslant m} \frac{1}{h}|R_n(2\pi hx)|$$

$$\ll \frac{1}{\log n}\Big(\sum_{h=1}^{\infty} e^{-h|x|} + (\log m)\exp(-b_2 n^{5/8})\Big) \ll (\log n)^{-1}.$$

Finally, we get

$$A \ll_x (\log n)^{-1}. \tag{3.2}$$

We can now complete the proof of Theorem 2. Taking

$$m = [\min(1,\ 1/16\pi|x|)\log n],$$

using (1.3), we get $B \ll (\log n)^{-1}$ and thus an application of (3.1) and (3.2) ends the proof.

For the proof of Theorem 1 we have to estimate B without the use of any unproved hypotheses. Put $D = 32(\log\log n)/\log n$ in Lemma 4. For

$$h \leqslant \min\left(\frac{1}{8\pi|x|}\log n,\ \frac{1}{64\pi|x|}\cdot\frac{\log n}{\log\log n}\right)$$

it yields

$$|R_n(2\pi hx) - S_n(2\pi hx)| \ll_x n^{-7/8} + h^2\frac{(\log\log n)^2}{\log n} + h(\log n)^{-1.5}.$$

Hence for $m \leqslant n$

$$B \ll n^{-7/8} + m^2\frac{(\log\log n)^2}{\log^2 n} + m(\log n)^{-2.5} \ll \frac{1}{\log n} + m^2\frac{(\log\log n)^2}{\log^2 n}.$$

Now (3.1) and (3.2) imply

$$D_n^*(x) \ll \frac{1}{m} + \frac{1}{\log n} + m^2\frac{(\log\log n)^2}{\log^2 n}.$$

We make the optimal choice of m by putting

$$m = \left[\left(\frac{\log n}{\log\log n}\right)^{2/3}\right].$$

This yields $D_n^*(x) \ll ((\log\log n)/\log n)^{2/3}$, as required.

4. Proof of Theorem 3. For a Lebesgue integrable 1-periodic function f denote by $a_m(f)$, $m \in \boldsymbol{Z}$, its mth Fourier coefficient:

$$a_m(f) = \int_0^1 f(t)e(-tm)\,dt.$$

To show Theorem 3 it is sufficient to prove that

$$\lim_{n\to\infty} a_m(\tilde{H}_{n,x}) = a_m(\tilde{H}_{\infty,x}) \tag{4.1}$$

for every $m \in \boldsymbol{Z}$. This is so because the sequence of norms $\|\tilde{H}_{n,x}\|_\infty$, $n = 1, 2, \ldots$, is bounded (Theorem 2) and the set of trigonometric polynomials is dense in $L^1(0, 1)$.

(4.1) is obvious for $m = 0$. For $m \neq 0$ we have

$$a_m(\tilde{H}_{n,x}) = a_m(H_{n,x}) = \frac{\log n}{n!S_n} \sum_{k=1}^{\infty} e^{-\gamma_k}\gamma_k^n \int_{\{\gamma_k x\}}^{1} e(-mt)\,dt - \log n \int_0^1 te(-mt)\,dt$$

$$= \frac{\log n}{2\pi i S_n m} S_n(-2\pi m x).$$

Lemma 4 implies $\lim_{n\to\infty}(S_n(y) - R_n(y)) = 0$ for every $y \neq 0$. Moreover, Lemma 2 yields for $y \neq 0$

$$\lim_{n\to\infty} R_n(y) = \begin{cases} \dfrac{-\chi(p^k)\log p}{2\pi\sqrt{p^{|k|}}} & \text{if } y = k\log p,\ p \text{ prime},\ k \in \boldsymbol{Z}\setminus\{0\},\ p \nmid q, \\ 0 & \text{otherwise}. \end{cases}$$

Hence we get

$$\lim_{n\to\infty} a_m(\tilde{H}_{n,x}) = \begin{cases} -\dfrac{1}{2\pi i m}\dfrac{\chi(p^k)\log p}{\sqrt{p^{|k|}}} & \text{if } -2\pi m x = k\log p,\ p \text{ prime},\ k \in \boldsymbol{Z}\setminus\{0\},\ p \nmid q, \\ 0 & \text{otherwise}. \end{cases}$$

Suppose that $x \neq 0$ is not of the form (1.7). Then $\lim_{n\to\infty} a_m(\tilde{H}_{n,x}) = 0$ for all $m \in \boldsymbol{Z}$ and thus $\tilde{H}_{n,x} \to 0$ in weak L^1-topology.