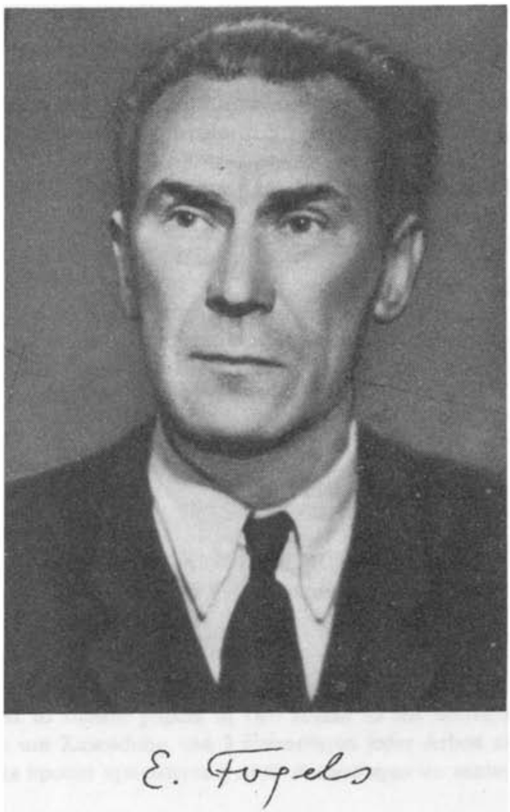

ACTA ARITHMETICA
LVII (1991)

Ernests Fogels (1910–1985)

by

J. KUBILIUS (Vilnius), L. REIZIŅŠ (Rīga) and R. RIEKSTIŅŠ (Rīga)

The Latvian mathematician Ernests Fogels died suddenly on February 22th, 1985. He was born on October 12th, 1910 in Līdzības, commune Nigrande, district Saldus. His parents were poor farmers. He attended the Second Gymnasium in Rīga, at that time one of the best schools in Latvia. At a mathematical competition he was rewarded with a book on number theory (G. Wertheim, *Anfangsgründe der Zahlenlehre*, Braunschweig, Vieweg 1902, 428 pp.). This was an extra stimulus to arouse his interest in mathematics.

In 1928 E. Fogels entered the Faculty of Mathematics and Natural Sciences of Latvian University. As he also had a bent for painting, he attended the Academy of Fine Arts too. In the meantime he worked as a clerk and somewhat later as a school teacher of mathematics.

E. Fogels intended to write his graduation (at that time called "candidate") thesis in number theory. But there was no professor who could supervise his research. Therefore his student's competitive paper "Space roulette" was accepted. In 1933 he graduated from the university. In 1935 he was invited to join the staff at the same university. During two years using his own lecture notes he edited two books by Lejnieks (E. Lejnieks, *Augstākā algebra* (*Higher algebra*), Rīga 1936, 158 pp.; *Skaitļu teorija* (*Number theory*), Rīga 1936, 289 pp.) and wrote two papers on Diophantine equations [2], [3].

In 1937 E. Fogels was appointed university private-docent and delivered lectures mainly in algebra and number theory. At the end of 1938 he went on probation course to England, to the Cambridge University. His plan to work under the supervision of G. H. Hardy failed. He was willingly accepted by A. E. Ingham who proposed to improve the estimate of the difference between two consecutive primes. In June 1939 he came home. World War II interrupted his probation course.

In 1940 E. Fogels was appointed associate professor at Latvian University. In 1947 he defended the candidate thesis "On mean values of arithmetical functions", the main part of which was written in the early forties [13].

The same year he went to work at the Institute of Physics and Mathematics of the Academy of Sciences of the Latvian SSR as a research fellow. In the course of three years he wrote 12 papers [14]–[25]. In 1950 he started working at the Rīga Pedagogical Institute. During eight years he gave almost all courses in mathematics and wrote about 30 lecture notes for students. Therefore he had practically no time for research. However, he wrote two papers [26], [27].

In 1958 the Pedagogical Institute was closed. Because of poor health, until 1961, E. Fogels held no official position. From 1961 to 1966 he was a research fellow at the Radioastrophysical Observatory of the Academy of Sciences of the Latvian SSR. This period was fruitful in his scientific activity. He obtained rather strong results on the density of zeros of different zeta-functions, on the distribution of primes in arithmetical progressions, on various algebraic fields and on binary and ternary quadratic forms.

E. Fogels made reports at seminars in Moscow and Leningrad. Yu. V. Linnik and other colleagues suggested that he should prepare a doctoral thesis on the basis of his most important results. A special permission for this was given by the Higher Certifying Commission. However, E. Fogels did not want to fill in the necessary documents and take up all sorts of other routine work.

In 1966 E. Fogels retired. He continued his scientific work until his death.

For some time E. Fogels was a reviewer for review journals. He was extremely careful and verified each formula. It took him too much time. Therefore he soon gave up this work. He was a member of the editorial board of "Acta Arithmetica" from 1967.

The first published research papers by E. Fogels [2]–[5] concern the theory of Diophantine equations. He found new cases of solvable Diophantine equations in quadratic and relative quadratic fields.

The papers [6], [9], [15] were written under the influence of A. E. Ingham. The first two deal with the problem of finding functions $h = h(x)$ for which the asymptotic relation

$$\sum_{1 \leqslant n \leqslant x} a(n) \sim A\psi(x), \qquad x \to \infty,$$

can be extended to

$$\sum_{x \leqslant n \leqslant x+h} a(n) \sim A\psi(x+h) - A\psi(x), \qquad x \to \infty, \tag{1}$$

where $a(n)$ is one of the functions $\Lambda(n)$, $\lambda(n)$ or $\mu(n)$. By using the method of Heilbronn and the convexity theorem of Ingham, E. Fogels proved that (1) holds when $h(x) = x^{\theta}$, $\theta > (1+4c)/(2+4c)$, where c is a constant for which $\zeta(1/2+it) = O(t^c)$, $t \to \infty$. Here ζ is the Riemann zeta-function. Similar results were obtained for the arithmetical functions $q(n)$, $2^{\omega(n)}$, $d(n^2)$, $d^2(n)$, where $q(h) = 1$ if n is not divisible by a kth power of a prime, $k \geqslant 2$, and $q(n) = 0$ otherwise, $\omega(n)$ is the number of different prime factors of n and $d(n)$ is the number of divisors of n. In [15] the author considered an analogous problem in case n runs over an arithmetical progression.

In [10] E. Fogels proved that almost all integers of the field $Q(\sqrt{-5})$ have a unique factorization into indecomposable factors. In [17], [18] he showed that any arithmetic of a countable set whose elements have uniqũe factorizations into primes is isomorphic to the ordinary arithmetic of natural numbers.

In [14], [26] E. Fogels gave two interesting examples of sequences of positive integers which are not linear progressions but still contain infinitely many prime numbers.

In 1948–1952 E. Fogels grew interested in elementary methods of number theory. The papers [19], [21]–[25] are devoted to the so-called finite methods. According to this point of view a finite proof should use only rational numbers with bounded denominators. There should be no differentiation and integration; the elementary functions $\cos x$, $\sin x$ ect. should be replaced by their rational approximations coming from power series expansions. He proved in this way some classical theorems of prime number theory. For example, in [19] he showed the prime number theorem in the form

$$\pi(x) \sim x\Big(\sum_{n \leqslant x} n^{-1}\Big)^{-1}.$$

In [20] an analogue of the Brun–Titchmarsh theorem was proved. Let $\pi(x, F)$ be the number of primes not exceeding x which are representable by a positive definite quadratic form F with integer coefficients and discriminant $-d$. Let $h(-d)$ be the number of classes of forms with discriminant $-d$. Then for each $\varepsilon > 0$ there is a number $d_0(\varepsilon) > 0$ such that if $d > d_0(\varepsilon)$ and if $x > d^5$ then

$$\pi(x, F) < \frac{\alpha x}{h(-d)\log x};$$

$$\alpha = (1+\varepsilon)(0.5 - 2.1\log d/\log x - \log\log x)^{-1}.$$

The principal tool in the proof was an extension of A. Selberg's sieve method to the case of ideals in $Q(\sqrt{-d})$.

In 1944 Yu. V. Linnik proved the existence of an absolute constant $c_1 > 0$ such that the smallest prime in any arithmetical progression $Dn+l$, $(D, l) = 1$, $n = 0, 1, 2, \ldots$, does not exceed D^{c_1}. In 1954 K. A. Rodosskiĭ simplified Linnik's method. By supplying this method with two more parameters and using another dissection of the critical strip, E. Fogels proved [27]–[29] the following result. There are absolute constants $c_2 > 0$, $c_3 > 0$ such that for any positive $\varepsilon \leqslant c_2$, for all $x \geqslant D_0(\varepsilon)$ and all $x \geqslant D^{c_3\log(c_2/\varepsilon)}$, there is at least one prime $p \equiv l \bmod D$, $(D, l) = 1$, in the interval (x, xD^{ε}). There are more than $x/\varphi(D)D^{2\varepsilon}$ primes $p \equiv l \bmod D$ for $x < \exp(D^{\varepsilon})$, $D > D_1(\varepsilon)$. Here φ is the Euler totient function. A corollary of this result is $\pi(x, D, l) > x/\varphi(D)D^{3\varepsilon}$ for all $x \in (D^{c_3\log(c_2/\varepsilon)+\varepsilon}, \exp(D^{\varepsilon}))$, $D > D_2(\varepsilon)$. Here $\pi(x, D, l)$ is the number of primes not exceeding x and $\equiv l \bmod D$. The constant $D_0(\varepsilon)$ depends on Siegel's zero. This theorem filled a gap between Linnik's theorem and a generalization of Hoheisel's theorem stating that there is a prime $p \equiv l \bmod D$ between x and $x+x^{\theta}$ if $x > \exp(D^{\varepsilon_1})$, $D > D_0(\varepsilon_1)$, and θ, $0 < \theta < 1$, is an appropriate constant.

Later on E. Fogels extended [30], [31], [35] these results to algebraic number fields. Let K be an algebraic number field of degree n and discriminant Δ, let f be an ideal in K, and let H be a class of ideals mod f. Set $w = |\Delta| Nf$ and let $x \geqslant 1$. E. Fogels showed that there is then a positive constant c_4, which depends only on n, such that the interval (x, xw^{c_4}) contains at least one rational prime that is the norm of an ideal in H. In the special case $n = 2$, this implies an appropriate assertion on rational primes representable by a primitive quadratic form. The proof requires information about the zeros of the Hecke L-functions near the line $\sigma = 1$. This information was proved in [32]–[34]. In [32], [38] an estimate for the "exceptional" zero was given.

These investigations led E. Fogels to the abstract theory of primes [36], [37], [39], [43], [45] introduced by A. Beurling and studied by many other authors. An abstract analogue of primes in arithmetical progressions is obtained by considering an infinite commutative semigroup G with a countable number of generators. The elements of G are divided into h classes H_i, $1 \leqslant i \leqslant D$, forming an Abelian group (for any $a \in H_i$ and $a' \in H_j$ we have $aa' \in H_k$ where k depends only on i and j). A homomorphism N of G into the multiplicative semigroup of reals $\geqslant 1$ leads to analogues $N(a)$ of absolute values. An asymptotic density law is assumed in the form

$$\sum_{\substack{a \in H_j \\ N(a) \leqslant x}} 1 = D^{\eta} x + O(D^{c_5} x^{1-\theta})$$

with $0 < \theta \leqslant 1$ and η, c_5 and θ independent of j.

If $\theta > 1/2$, then

(*) there is a positive constant c_6 such that for any $x \geqslant 1$ and any H_i there is at least one generator $b \in H_i$ with $x \leqslant N(b) \leqslant xD^{c_6}$.

For h (the class number) odd this assertion holds as well in the case of $\theta \leqslant 1/2$.

For even h let K_j denote any subgroup of G with index 2. Then there exists the limit

$$\lim_{x \to \infty} \Bigg(\sum_{\substack{x \geqslant N(a) \\ a \in G}} \frac{1}{a} - \sum_{\substack{x \geqslant N(a) \\ a \in K_j}} \frac{1}{a} \Bigg) = C_j.$$

Let $\theta \leqslant 1/2$. If there is a constant c_7 such that $C_j > D^{-c_7}$ for any j, then the above assertion (*) holds (with the constant c_6 depending also on c_7).

Let $\pi(x, H_i)$ be the number of generators $b \in H_i$ with $N(b) \leqslant x$. For appropriate constants $c_8, c_9 > 0$ and any $x > D^{c_8}$ we have $\pi(x, H_i) > x/D^{c_9} \log x$.

In [43] E. Fogels used another homomorphism Φ of G into the multiplicative semigroup of complex numbers, $\Phi\colon a \to (N(a))^{1/2} \exp(2\pi i \alpha(a))$, $0 \leqslant \alpha(a) < 1$. In addition to the asymptotic density law above, he assumed that

$$\sum_{\substack{a \in H_j \\ N(a) \leqslant x,\, 0 \leqslant \alpha(a) < \varkappa}} 1 = D^{\eta} \varkappa x + O(D^{c_{10}} x^{1-\delta})$$

uniformly in $\varkappa$, $0 < \varkappa \leqslant 1$, and with δ independent of j and satisfying $0 < \delta \leqslant \vartheta$. He proved a theorem analogous to (*) on the generators lying in the region $\{\alpha(a) \equiv \alpha_0 + \beta\Gamma \bmod 1,\ 0 \leqslant \beta < 1,\ x < N(a) < x D^{c_{11} \log\log(8/\Gamma)}\}$, $D^{-c_{12}} < \Gamma \leqslant 1$.

In [45] E. Fogels considered some interesting particular semigroups. Specifically, the representation of primes by binary quadratic forms $F(x_1, x_2) = Ax_1^2 + Bx_1x_2 + Cx_2^2$ was studied. Let $d = B^2 - AC$. The following theorem is obtained. For appropriate absolute constants c_{13}, c_{14}, $0 < \gamma < 1$ and for all $x \geqslant |d|^{c_{13}}$ in the region $x < |F(x_1, x_2)| < x + x^{\gamma}$ in the (x_1, x_2)-plane between any two straight lines starting from the origin and forming an angle with the non-Euclidean measure $\geqslant x^{-c_{14}}$ there is a lattice point (x_1, x_2) for which $|F(x_1, x_2)|$ is a prime.

In [40], [44] a strengthening of a theorem of Linnik was proved. Let $0 \leqslant \lambda \leqslant \log T$ where $T \geqslant D$; the number of zeros of $\prod_{\chi \bmod D} L(s, \chi)$ in $1 - \lambda/\log T \leqslant \sigma \leqslant 1$, $|t| \leqslant T$, $s = \sigma + it$, does not exceed $e^{c_{15}}$. Yu. V. Linnik proved this theorem in case $T = D$. A similar theorem holds for the product of Hecke's L-functions over an arbitrary algebraic number field as well as for L-functions of a semigroup.

This result led E. Fogels to some improvements of the previous ones. For example, he proved that there exists an absolute constant $\delta < 1$ such that if $D > D_0$, then for any $x \geqslant D^{c_{16}}$ there is a prime $p \equiv l \bmod D$ in the interval $(x, x + x^{\delta})$. This improves the theorem of [29] where the interval was (x, xD^{ε}) and where the restrictions $D > D_0(\varepsilon)$, $x > D^{c_3 \log(c_2/\varepsilon)}$ were used.

In [46] E. Fogels gave an estimate for the number of zeros in a rectangle of the critical strip of zeta functions on Gauss' field with Hecke's characters (Grössencharaktere). From this theorem he deduced that for any $\varepsilon > 0$ there is a Gaussian prime in any circle S in the complex plane whenever the distance R of the centre of S from the origin is large enough and the radius of S exceeds $R^{2/3+\varepsilon}$. It follows that there are infinitely many primes $p = a^2 + b^2$ such that $b^2 < p^{2/3+\varepsilon}$. These investigations were in line with those of Vilnius mathematicians (J. Kubilius, K. Bulota, M. Maknys).

The papers [47], [48], [50], [51], [52] were devoted to the Hecke L-functions of a quadratic number field. E. Fogels proved an approximate functional equation in case $\sigma = 1/2$ and a Bombieri-type density theorem. As an application, an analogue of Bombieri's theorem on large sieve for prime ideals of a quadratic field was stated.

E. Fogels devoted the last years of his life to the Riemann hypothesis. He constructed many variants of possible proofs, though no one of them was successful. The papers [53], [54] contain one of such attempts. The author himself noted a gap in the proof. However, it presented some new interesting connections of the Riemann hypothesis with the theory of prime numbers.

E. Fogels had a strong personality. He was very hard-working and energetic. He was one of those men who devoted all their lives to science. He was also a good teacher. His personality will remain in the memory of his colleagues for ever.