for all $r \geq 2$, and hence if we choose $\mu < (m+n)^{-1}(m+n-1)^{-1}$, then

$$|C(q; \psi^*)| \ll \psi^*(q)^{m+n},$$

for all sufficiently large q. It now follows from the Borel–Cantelli lemma that the system of inequalities

$$\|q\xi_i(\boldsymbol{u})\| < \psi^*(q), \qquad i = 1, \ldots, m+n,$$

has at most finitely many solutions for almost all $\boldsymbol{u} \in \Omega$, which proves Theorem 1.3.

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Received on 9.5.1989 (1937)

ACTA ARITHMETICA
LVII (1991)

Representation of primes by the principal form of discriminant $-D$ when the classnumber $h(-D)$ is 3

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0. Notation and preliminary result. Throughout this paper p denotes a prime > 3. We shall be concerned with binary quadratic forms $ax^2+bxy+cy^2$, written (a, b, c), which are integral (that is, a, b, c are integers), positive-definite (that is, $a > 0$, $b^2-4ac < 0$) and primitive (that is, $\mathrm{GCD}(a, b, c) = 1$). The discriminant of the form (a, b, c) is the negative integer b^2-4ac. On the set of all such forms of fixed discriminant $-D$ $(D > 0)$, we define an equivalence relation $\sim$ as follows: we write $(a, b, c) \sim (a', b', c')$ if there exist integers p, q, r, s with $ps-qr = +1$ such that

$$a(px+qy)^2+b(px+qy)(rx+sy)+c(rx+sy)^2 = a'x^2+b'xy+c'y^2.$$

It is well known that there are only finitely many such equivalence classes. The number of classes is called the classnumber of forms of discriminant $-D$ and is denoted by $h(-D)$. The principal form of discriminant $-D$ is the form p_{-D} given by

$$p_{-D} = \begin{cases} (1, 0, D/4), & \text{if } D \equiv 0 \pmod 4, \\ (1, 1, (D+1)/4), & \text{if } D \equiv 3 \pmod 4. \end{cases} \tag{0.1}$$

A positive integer m is said to be represented by the form (a, b, c) if there exist integers x and y such that $m = ax^2+bxy+cy^2$. If the prime p (not dividing $2D$) is represented by a form of discriminant $-D$, it is well known that the Legendre symbol $\left(\frac{-D}{p}\right) = +1$. In this paper we shall be concerned with the representability of a prime p (> 3) by the principal form p_{-D} of discriminant $-D$ when $h(-D) = 3$.

Recent deep work of Goldfeld, Gross, Mestre, Oesterlé and Zagier (see [6], [7], [12], [13], [14], [20]) has led to the complete determination of all the imaginary quadratic fields with classnumber 3 [12: Théorème 4], namely,

* Research supported by Natural Sciences and Engineering Research Council of Canada Grant A-7233.

$$Q(\sqrt{-n}):\ n = 23,\ 31,\ 59,\ 83,\ 107,\ 139,\ 211,\ 283,\ 307,$$
$$331,\ 379,\ 499,\ 547,\ 643,\ 883,\ 907.$$

The complete list of all the imaginary quadratic fields with classnumber 1 has been known for over twenty years [15], namely,

$$Q(\sqrt{-n}):\ n = 1,\ 2,\ 3,\ 7,\ 11,\ 19,\ 43,\ 67,\ 163.$$

From these results we can deduce

PROPOSITION. *$h(-D) = 3$ if and only if*

(0.2) $D = 23, 31, 44, 59, 76, 83, 92, 107, 108, 124, 139, 172, 211, 243, 268, 283,$
$307, 331, 379, 499, 547, 643, 652, 883$ *or* 907.

Proof. Let d be the discriminant of the imaginary quadratic field given uniquely by

$$-D = f^2 d,$$

where f is a positive integer. Then, by a formula of Gauss, we have

$$h(-D) = h(f^2 d) = h(d)\psi_d(f)/u,$$

where

$$\psi_d(f) = f\prod_{q\mid f}\left(1-\left(\frac{d}{q}\right)\frac{1}{q}\right)$$

and

$$u = \begin{cases} 3, & \text{if } d = -3, \\ 2, & \text{if } d = -4, \\ 1, & \text{if } d < -4. \end{cases}$$

Note that q runs through the distinct primes dividing f and $\left(\frac{d}{q}\right)$ is the Kronecker symbol. As $\psi_d(f)$ is a positive integer and $h(-3) = h(-4) = 1$, we see that

$$h(-D) = 3 \Leftrightarrow \text{(a) } d < -4,\ h(d) = 3,\ \psi_d(f) = 1 \quad \text{or}$$
$$\text{(b) } d < -4,\ h(d) = 1,\ \psi_d(f) = 3 \quad \text{or}$$
$$\text{(c) } \psi_{-4}(f) = 6 \text{ or}$$
$$\text{(d) } \psi_{-3}(f) = 9.$$

Now it is easy to check that

$$\psi_d(f) = 1 \Leftrightarrow f = 1 \text{ or } f = 2,\ d \equiv 1 \pmod 8;$$
$$\psi_d(f) = 3 \Leftrightarrow f = 2,\ d \equiv 5 \pmod 8 \text{ or}$$
$$f = 3,\ d \equiv 0 \pmod 3 \text{ or}$$
$$f = 6,\ d \equiv 1 \pmod 8,\ d \equiv 0 \pmod 3;$$
$$\psi_{-4}(f) = 6 \quad \text{cannot occur};$$
$$\psi_{-3}(f) = 9 \Leftrightarrow f = 6 \text{ or } f = 9.$$

Thus, appealing to the lists of imaginary quadratic fields with classnumber 1 or 3, we see that:

(a) occurs if and only if $D = 23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907, 23\cdot 2^2, 31\cdot 2^2$;

(b) occurs if and only if $D = 11\cdot 2^2,\ 19\cdot 2^2,\ 43\cdot 2^2,\ 67\cdot 2^2,\ 163\cdot 2^2$;

(c) cannot occur;

(d) occurs if and only if $D = 3\cdot 6^2,\ 3\cdot 9^2$.

This gives the twenty-five values of D listed in (0.2).

1. Introduction. Gauss [5] showed that 2 is congruent to a cube modulo a prime $p \equiv 1 \pmod 3$ if and only if there exist integers x and y such that $p = x^2+27y^2$, that is, if and only if p is represented by the principal form of discriminant -108. Moreover, when 2 is a cube (mod p), where $p \equiv 1 \pmod 3$, 2 has three distinct cube roots (mod p). If $p \equiv 2 \pmod 3$ then $\left(\frac{-108}{p}\right) = \left(\frac{-3}{p}\right) = -1$ and p is not represented by any form of discriminant -108, and 2 has a unique cube root (mod p). Since every positive-definite, primitive, integral binary quadratic form of discriminant -108 is equivalent to exactly one of the three forms $(1, 0, 27)$, $(4, -2, 7)$, $(4, 2, 7)$, Gauss' theorem can be expressed as follows:

THEOREM (Gauss). *The polynomial x^3-2 is*

(i) *the product of three distinct linear polynomials* (mod p) *if* $\left(\frac{-3}{p}\right) = +1$ *and p is represented by* $(1, 0, 27)$;

(ii) *the product of a linear polynomial and an irreducible quadratic polynomial* (mod p) *if* $\left(\frac{-3}{p}\right) = -1$;

(iii) *irreducible* (mod p) *if* $\left(\frac{-3}{p}\right) = +1$ *and p is represented by* $(4, \pm 2, 7)$.

Clearly Gauss' theorem can be reformulated as a criterion for p to be represented by the principal form of discriminant -108, namely, THEOREM (Gauss). *The prime p is represented by* (1, 0, 27) *if and only if* $\left(\frac{-3}{p}\right) = +1$ *and* x^3-2 *is congruent to the product of three distinct linear polynomials* (mod p).

Jacobi [10] showed that 3 is congruent to a cube modulo a prime $p \equiv 1 \pmod 3$ if and only if p can be written in the form $4p = A^2+243B^2$, where A and B are integers. If $4p = A^2+243B^2$ then we have $A \equiv B \pmod 2$ and $p = x^2+xy+61y^2$ with $x = \frac{1}{2}(A-B)$, $y = B$. Conversely, if $p = x^2+xy+61y^2$ then we have $4p = A^2+243B^2$ with $A = 2x+y$, $B = y$. Since every positive-definite, primitive, integral binary quadratic form of discriminant -243 is equivalent to exactly one of the three forms (1, 1, 61), (7, −3, 9), (7, 3, 9), Jacobi's theorem can be restated as follows:

THEOREM (Jacobi). *The prime p is represented by* (1, 1, 61) *if and only if* $\left(\frac{-3}{p}\right) = +1$ *and* x^3-3 *is congruent to the product of three distinct linear polynomials* (mod p).

In this paper we generalize the results of Gauss and Jacobi to all D (> 0) for which $h(-D) = 3$. These values of D are listed in (0.2). We prove

THEOREM 1. *Let D be a positive integer such that* $h(-D) = 3$. *Then the prime* p ($p > 3$, $p \nmid D$) *is represented by the principal form* p_{-D} *of discriminant* $-D$ *if and only if* $\left(\frac{-D}{p}\right) = +1$ *and* $f_{-D}(x)$ *is congruent to the product of three distinct linear polynomials* (mod p), *where* $f_{-D}(x)$ *is the monic cubic polynomial with integral coefficients listed in Table* 1. *Further we have*

$$\text{discriminant}(f_{-D}(x)) = \begin{cases} -D, & \text{if } D \equiv 3 \pmod 4 \text{ or } D \equiv 12 \pmod{32}, \\ -D/4, & \text{if } D \equiv 28 \pmod{32}. \end{cases}$$

Table 1

D	$f_{-D}(x)$	D	$f_{-D}(x)$
23	x^3-x+1	243	x^3-3
31	x^3+x+1	268	x^3+2x^2-2x+2
44	x^3+x^2-x+1	283	x^3+4x+1
59	x^3+2x+1	307	x^3-x^2+3x+2
76	x^3-2x+2	331	x^3-2x^2+4x+1
83	x^3+x^2+x+2	379	x^3+x^2+x+4
92	x^3-x+1	499	x^3+4x+3
107	x^3+x^2+3x+2	547	x^3+x^2-3x+4
108	x^3-2	643	x^3-2x+5
124	x^3+x+1	652	x^3+3x^2-5x+3
139	x^3-x^2+x+2	883	x^3+5x^2-5x+2
172	x^3-x^2-x+3	907	x^3+5x^2+x+2
211	x^3-2x+3		

The cases $D = 108$ and $D = 243$ of the theorem are the aforementioned results of Gauss and Jacobi respectively, so these two values of D will be excluded from further consideration. Furthermore, when $D = 92$ and $D = 124$, it is easy to check that p is represented by p_{-D} if and only if it is represented by $p_{-D/4}$, as $D/4 \equiv 7 \pmod 8$. Thus we can also exclude these two values of D from further consideration. We divide the remaining 21 values of D into two lists according as $D \equiv 3 \pmod 4$ or $D \equiv 0 \pmod 4$, namely,

(A) $D = 23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907$,

(B) $D = 44, 76, 172, 268, 652$.

The proof of Theorem 1 for the 16 values of D listed in (A) is based on a theorem of Weinberger [18] and is given in Section 2. For the 5 values of D listed in (B), Weinberger's theorem does not apply and we give a proof (in § 3) using Artin's reciprocity law instead. We remark that the existence of such a polynomial $f_{-D}(x)$ is known by class field theory (see [3: Theorem 9.2 and Ex. 9.3]). Our Theorem 1 gives such a polynomial $f_{-D}(x)$ explicitly for all D with $h(-D) = 3$, and furthermore shows that $f_{-D}(x)$ may be chosen with discriminant $-D/4$ or $-D$ according as $D \equiv 28 \pmod{32}$ or not. In future work it is planned to determine $f_{-D}(x)$ explicitly when $h(-D) = 4, 5, 6, 7$ and 8, assuming that the known lists of such D are complete. For general D not much is known about $f_{-D}(x)$ or its discriminant.

The case $D = 124$ of Theorem 1 was treated by Kronecker [11], who showed that p is represented by (1, 0, 31) if and only if the congruence

$$(x^3-10x)^2+31(x^2-1)^2 \equiv 0 \pmod p$$

is solvable. It is easy to check that this is equivalent to our result, namely, p ($\nmid 2\cdot3\cdot31$) is represented by (1, 0, 31) if and only if $\left(\frac{-31}{p}\right) = +1$ and the congruence $x^3+x+1 \equiv 0 \pmod p$ is solvable. Appealing to Theorem 1, a sextic polynomial analogous to that of Kronecker for $D = 124$ can be found for each D in (0.2).

In Section 4, we use Theorem 1 to construct explicitly some class fields. We prove

THEOREM 2. (i) *For those D in* (A), *the Hilbert class field over* $Q(\sqrt{-D})$ *is*

$$Q(\sqrt{-D}, \sqrt[3]{\varkappa_D}+\sqrt[3]{\varkappa'_D}),$$

where $\varkappa_D$ *is given as follows:*

D	$\varkappa_D$	D	$\varkappa_D$	D	$\varkappa_D$
23	$(-27+3\sqrt{69})/2$	139	$(-61+3\sqrt{417})/2$	379	$(-101+3\sqrt{1137})/2$
31	$(-27+3\sqrt{93})/2$	211	$(-81+3\sqrt{633})/2$	499	$(-81+3\sqrt{1497})/2$
59	$(-27+3\sqrt{177})/2$	283	$(-27+3\sqrt{849})/2$	547	$(-137+3\sqrt{1641})/2$
83	$(-47+3\sqrt{249})/2$	307	$(-79+3\sqrt{921})/2$	643	$(-135+3\sqrt{1929})/2$
107	$(-29+3\sqrt{321})/2$	331	$(-83+3\sqrt{993})/2$	883	$(-529+3\sqrt{2649})/2$
				907	$(-259+3\sqrt{2721})/2$

(ii) *For those D in* (B), *the ring class field of the order* $Z[\sqrt{-D/4}]$ *in* $Z[(-1+\sqrt{-D/4})/2]$ *is*

$$Q(\sqrt{-D/4}, \sqrt[3]{\varkappa_D}+\sqrt[3]{\varkappa'_D}),$$

where

$$\varkappa_{44} = -19+3\sqrt{33},$$
$$\varkappa_{76} = -27+3\sqrt{57},$$
$$\varkappa_{172} = -35+3\sqrt{129},$$
$$\varkappa_{268} = -53+3\sqrt{201},$$
$$\varkappa_{652} = -135+3\sqrt{489}.$$

We remark that Hasse [9] has shown that the Hilbert class field over $Q(\sqrt{-23})$ is

$$Q(\sqrt{-23}, \sqrt[3]{(25+3\sqrt{69})/2}+\sqrt[3]{(25-3\sqrt{69})/2})$$

and the Hilbert class field over $Q(\sqrt{-31})$ is

$$Q(\sqrt{-31}, \sqrt[3]{(29+3\sqrt{93})/2}+\sqrt[3]{(29-3\sqrt{93})/2}).$$

Our results for $D = 23$ and $D = 31$ agree with those of Hasse since $\beta = (\alpha-9)/\alpha$ for

$$\begin{cases}\alpha = \sqrt[3]{(-27+3\sqrt{69})/2}+\sqrt[3]{(-27-3\sqrt{69})/2} = -3.9741\ldots, \\ \beta = \sqrt[3]{(25+3\sqrt{69})/2}+\sqrt[3]{(25-3\sqrt{69})/2} = 3.2646\ldots;\end{cases}$$

and $\delta = (-\gamma-9)/\gamma$ for

$$\begin{cases}\gamma = \sqrt[3]{(-27+3\sqrt{93})/2}+\sqrt[3]{(-27-3\sqrt{93})/2} = -2.0469\ldots, \\ \delta = \sqrt[3]{(29+3\sqrt{93})/2}+\sqrt[3]{(29-3\sqrt{93})/2} = 3.3967\ldots\end{cases}$$

In Section 5, we use Theorem 1 and a theorem of Cauchy [2] to give a necessary and sufficient condition for the prime p to be represented by p_{-D} (D in list (A) or list (B)) in terms of integer sequences defined by a second order linear recurrence relation which need only be considered modulo p. When $D = 23$ our result agrees with that of Gurak [8]. We prove

THEOREM 3. *Let D denote one of the integers in list* (A) *or list* (B). *Let p be a prime* (>3) *such that* $\left(\frac{-D}{p}\right) = +1$. *Then*

$$p = \begin{cases} x^2+\frac{D}{4}y^2, & \text{if } D \equiv 0 \pmod 4, \\ x^2+xy+\left(\frac{1+D}{4}\right)y^2, & \text{if } D \equiv 3 \pmod 4,\end{cases}$$

is solvable in integers x and y if and only if

$$\begin{cases} u_{(p-1)/3} \equiv 2 \pmod p, & \text{if } p \equiv 1 \pmod 3, \\ u_{(p+1)/3} \equiv -2k \pmod p, & \text{if } p \equiv 2 \pmod 3,\end{cases}$$

where the sequence of integers $\{u_n\}_{n=0,1,2,\ldots}$ *is given by*

$$\begin{cases} u_0 = 2, \quad u_1 = l, \\ u_{n+2} = lu_{n+1}+k^3u_n, \quad n = 0, 1, 2, \ldots,\end{cases}$$

and the integers k, l are given in Table 2:

Table 2

D	k	l	D	k	l
23	−1	+25	283	+12	+27
31	−1	+29	307	+8	−79
44	−4	−38	331	+8	+83
59	−4	−43	379	+2	+101
76	+8	−2	499	+12	+81
83	+2	−47	547	−10	+137
107	+8	+29	643	−6	+135
139	+2	−61	652	+20	+196
172	−4	+70	883	−40	+529
211	−6	−81	907	−22	+259
268	−10	+106			

The identities

$$u_{2m} = u_m^2-2(-1)^m k^{3m}, \quad u_{3m} = u_m^3-3(-1)^m k^{3m} u_m,$$

are often useful in computing $u_{(p\pm1)/3} \pmod p$. We illustrate Theorem 3 with a simple example.

EXAMPLE. Is the prime 1297 represented by the form (1, 0, 19)? Here we have $p = 1297$, $(p-1)/3 = 432$, $D = 76$, $k = 8$, $l = -2$. Making use of the above identities, we obtain successively modulo 1297

$$u_0 \equiv 2, \quad u_1 \equiv -2, \quad u_2 \equiv 1028, \quad u_4 \equiv 726, \quad u_8 \equiv 889,$$
$$u_{16} \equiv 904, \quad u_{48} \equiv 544, \quad u_{144} \equiv 1296, \quad u_{432} \equiv 2,$$

so that, by Theorem 3, 1297 is represented by (1, 0, 19). Indeed we have $1297 = 1\cdot 9^2+19\cdot 8^2$.

2. Proof of Theorem 1 for those D listed in (A). Throughout this section, D denotes one of the integers listed in (A). Note that D is a prime $\equiv 3 \pmod 4$.

Let p be a prime >3 with $p \nmid D$. If $\left(\frac{-D}{p}\right) = -1$ then p is not represented by $p_{-D} = (1, 1, \frac{1}{4}(D+1))$ and, as $\operatorname{discrim}(f_{-D}(x)) = -D$, by a theorem of Stickelberger [16], $f_{-D}(x)$ is the product of a linear polynomial and an irreducible quadratic polynomial modulo p. Now suppose $\left(\frac{-D}{p}\right) = +1$. We must show that p is represented by $p_{-D} = (1, 1, \frac{1}{4}(D+1))$ if and only if $f_{-D}(x)$ is congruent to the product of three distinct linear polynomials (mod p).

We set

$$K_D = Q(\sqrt{3D}), \quad K_D^* = Q(\sqrt{3D})\backslash\{0\}. \tag{2.1}$$

Let G_D be the group defined by

$$G_D = \{\alpha \in K_D^* \colon (\alpha) = A^3 \text{ for some ideal } A \text{ of } K_D\} \tag{2.2}$$

and let H_D be the subgroup of G_D given by

$$H_D = \{\alpha \in K_D^* \colon \alpha = \beta^3 \text{ for some } \beta \in K_D^*\}. \tag{2.3}$$

Then G_D/H_D is a group isomorphic with the direct sum of r_D+1 groups of order 3, where r_D is the rank of the 3-Sylow subgroup of the classgroup $H(K_D)$ of K_D. Now

$$H(K_D) \simeq \begin{cases} Z_3, & \text{for } D = 107, 331, 643, \\ Z_5, & \text{for } D = 547, \\ Z_1, & \text{otherwise,} \end{cases} \tag{2.4}$$

so

$$r_D = \begin{cases} 1, & \text{for } D = 107, 331, 643, \\ 0, & \text{otherwise,} \end{cases} \tag{2.5}$$

and thus

$$G_D/H_D \simeq \begin{cases} Z_3 \times Z_3, & \text{if } D = 107, 331, 643, \\ Z_3, & \text{otherwise.} \end{cases} \tag{2.6}$$

Let ε_{3D} denote the fundamental unit (>1) of K_D. When $D \neq 107$, 331, 643 a basis for the group G_D/H_D is $\{\varepsilon_{3D}H_D\}$. When $D = 107$, 331 or 643, $H(K_D)$ is generated by the class containing the ideal $A_D = (2, \frac{1}{2}(1+\sqrt{3D}))$. Since

$$A_D^3 = \begin{cases} (\frac{1}{2}(17+\sqrt{321})), & \text{if } D = 107, \\ (\frac{1}{2}(31-\sqrt{993})), & \text{if } D = 331, \\ (\frac{1}{2}(4963-113\sqrt{1929})) = (\frac{1}{2}(1258562169097-28655537523\sqrt{1929})), & \\ & \text{if } D = 643. \end{cases}$$

a basis for G_D/H_D is given by $\{\varepsilon_{3D}H_D, \mu_{3D}H_D\}$, where

$$\mu_{3D} = \begin{cases} (\frac{1}{2}(17+\sqrt{321})), & \text{if } D = 107, \\ (\frac{1}{2}(31-\sqrt{993})), & \text{if } D = 331, \\ (\frac{1}{2}(1258562169097-28655537523\sqrt{1929})), & \text{if } D = 643. \end{cases}$$

Hence, for every nonzero integer α of K_D, there is a unique integer γ_{3D} of K_D, a unique integer r ($=0, 1, 2$), and, if $D = 107$, 331 or 643, a unique integer s ($=0, 1, 2$), such that

$$\begin{cases} \alpha\varepsilon_{3D}^r = \gamma_{3D}^3, & \text{if } D \neq 107, 331, 643, \\ \alpha\varepsilon_{3D}^r\mu_{3D}^s = \gamma_{3D}^3, & \text{if } D = 107, 331, 643. \end{cases} \tag{2.7}$$

The choice of generator μ_{3D} of A_D^3 with large coefficients in the case $D = 643$ is so that when α is taken to be α_D (see (2.12)) we have $r = 0$ and $s = 1$ (see Table 6 and (2.13)). The values of ε_{3D} for those D under consideration are taken from the table of Wada [17] and are listed in Table 3.

Table 3

D	ε_{3D}
23	$(25+3\sqrt{69})/2$
31	$(29+3\sqrt{93})/2$
59	$62423+4692\sqrt{177}$
83	$8553815+542076\sqrt{249}$
107	$215+12\sqrt{321}$
139	$85322647+4178268\sqrt{417}$
211	$440772247+17519124\sqrt{633}$
283	$1501654712948695+51536656330476\sqrt{849}$
307	$2522057712835735+83104627139412\sqrt{921}$
331	$2647+84\sqrt{993}$
379	$650468934487+19290626292\sqrt{1137}$
499	$22516718751127+581961430932\sqrt{1497}$
547	$4375+108\sqrt{1641}$
643	$126794455+2886916\sqrt{1929}$
883	$99736649218553790682248535+1937821608115448210697276\sqrt{2649}$
907	$5231287949706796270736288215+100286934195999623391686388\sqrt{2721}$

Next we define $g_{-D}(x)$ to be the monic cubic polynomial

$$g_{-D}(x) = x^3 + \frac{a_D}{3}x + \frac{b_D}{27}, \tag{2.8}$$

where the integers a_D and b_D are listed in Table 4.

Table 4

D	a_D	b_D	D	a_D	b_D
23	−1	−25	307	+8	+79
31	−1	−29	331	+8	−83
59	−4	+43	379	+2	−101
83	+2	+47	499	+12	−81
107	+8	−29	547	−10	−137
139	+2	+61	643	−6	−135
211	−6	+81	883	−40	−529
283	+12	−27	907	−22	−259

The integers a_D and b_D were chosen so that the polynomials $f_{-D}(x)$ and $g_{-D}(x)$ have the same discriminant as well as the same number of roots (mod p). It is clear that

$$\operatorname{discrim}(f_{-D}(x)) = \operatorname{discrim}(g_{-D}(x))$$

as

$$\operatorname{discrim}(f_{-D}(x)) = -D, \quad \operatorname{discrim}(g_{-D}(x)) = (-4a_D^3 - b_D^2)/27,$$

and

(2.9) $$4a_D^3 + b_D^2 = 27D.$$

It is also clear that $f_{-D}(x)$ and $g_{-D}(x)$ have the same number of roots (mod p) as

(2.10) $$f_{-D}(x) = (-1)^d x^e g_{-D}\left(\frac{tx+u}{vx+w}\right),$$

where the integers d $(=0, 1)$, e $(=0, 3)$, t, u, v, w are given in Table 5.

Table 5

D	d	e	t	u	v	w	D	d	e	t	u	v	w
23	1	3	1	−3	3	0	307	0	0	3	−1	0	3
31	1	3	−1	−3	3	0	331	1	0	−3	2	0	3
59	0	3	2	3	3	0	379	1	0	−3	−1	0	3
83	0	0	3	1	0	3	499	1	0	−1	0	0	1
107	1	0	−3	−1	0	3	547	1	0	−3	−1	0	3
139	0	0	3	−1	0	3	643	1	0	−1	0	0	1
211	0	0	1	0	0	1	883	1	0	−3	−5	0	3
283	1	0	−1	0	0	1	907	1	0	−3	−5	0	3

We can also see that $\operatorname{discrim}(f_{-D}(x)) = \operatorname{discrim}(g_{-D}(x))$ from (2.10) and

Table 5, as in each case we have

(2.11) $$\left(t^3 + \frac{a_D}{3}tv^2 + \frac{b_D}{27}v^3\right)^2 = \pm(tw-uv)^3.$$

Set

(2.12) $$\alpha_D = \tfrac{1}{2}(b_D + 3\sqrt{3D}),$$

so that by (2.9) α_D is of norm $(-a_D)^3$. For each D, we determine the values of r, s and $\gamma_{3D} = \frac{1}{2}(u_D + v_D\sqrt{3D})$ in (2.7) when $\alpha = \alpha_D$. These are listed in Table 6.

Table 6

D	r	s	u_D	v_D
23	1		−2	0
31	1		−2	0
59	1		+173	+13
83	1		+931	+59
107	1	0	+17	+1
139	1		+2185	+107
211	1		+4101	+163
283	1		+449331	+15421
307	1		+754117	+24849
331	1	0	+31	+1
379	1		+4687	+139
499	1		+92433	+2389
547	1		−41	−1
643	0	1	−55164	+1256
883	1		−3343018627	−64952791
907	1		−8124416167	−155749941

It is no coincidence that $r = 1$ for $D \neq 643$, this is a consequence of the choice of sign of b_D.

Summarizing we have

(2.13) $$\begin{cases} \alpha_D \varepsilon_{3D} = \gamma_{3D}^3, & \text{for } D \neq 643, \\ \alpha_D \mu_{3D} = \gamma_{3D}^3, & \text{for } D = 643. \end{cases}$$

In view of (2.10), $f_{-D}(x)$ is the product of three distinct linear polynomials (mod p) if and only if $g_{-D}(x)$ is the product of three distinct linear polynomials (mod p). By a theorem of Dickson [4], as $\operatorname{discrim}(g_{-D}(x)) = -D$ and $\left(\frac{-D}{p}\right) = +1$, the polynomial $g_{-D}(x)$ is the product of three distinct linear polynomials (mod p) if and only if α_D is congruent to a cube (mod p), where p is a prime ideal of the ring of integers of K_D which divides p. We note that $\alpha_D \not\equiv 0 \pmod{\mathfrak{p}}$, otherwise $p|a_D$, which is seen to be impossible from Table 4 remembering that $p>3$ and $\left(\frac{-D}{p}\right)=+1$. In view of (2.13), α_D is a cube (mod $\mathfrak{p}$) if and only if ε_{3D} (if $D\neq 643$), μ_{3D} (if $D=643$) is a cube (mod $\mathfrak{p}$).

Let $H(-9D)$ denote the group of classes of primitive, positive-definite, binary quadratic forms of discriminant $-9D$, so that, for those D under consideration, $H(-9D)$ is cyclic of order 12 (resp. 6) if $D\equiv 1 \pmod 3$ (resp. $D\equiv 2 \pmod 3$). As the 3-Sylow subgroup of $H(-9D)$ is of order 3, by a theorem of Weinberger [18], ε_{3D} (if $D\neq 643$), μ_{3D} (if $D=643$) is a cube (mod $\mathfrak{p}$) if and only if $N(\mathfrak{p})$ is represented by one of the forms in the subgroup of sixth powers in $H(-9D)$, that is, by

$$\text{(2.14)}\qquad \begin{cases}(1,1,\tfrac14(9D+1)) \text{ or } (9,9,\tfrac14(D+9)), & \text{if } D\equiv 1 \pmod 3,\\ (1,1,\tfrac14(9D+1)), & \text{if } D\equiv 2 \pmod 3.\end{cases}$$

In view of the identities

$$x^2+xy+\frac{(9D+1)}{4}y^2 \equiv (x-y)^2+(x-y)(3y)+\frac{(D+1)}{4}(3y)^2,$$

$$9x^2+9xy+\frac{(D+9)}{4}y^2 \equiv (3x+y)^2+(3x+y)\,y+\frac{(D+1)}{4}y^2,$$

it is clear that if $N(\mathfrak{p})$ is represented by $(1,1,\frac14(9D+1))$ or $(9,9,\frac14(D+9))$ it is represented by $p_{-D}=(1,1,\frac14(D+1))$. In order to treat the converse, we first show that $N(\mathfrak{p})\equiv 1 \pmod 3$. We have

$$N(\mathfrak{p})=\begin{cases}p, & \text{if } \left(\frac{3D}{p}\right)=1.\\ p^2, & \text{if } \left(\frac{3D}{p}\right)=-1.\end{cases}$$

Recalling that $\left(\frac{-D}{p}\right)=1$, the condition $\left(\frac{3D}{p}\right)=1$ (resp. -1) is equivalent to $p\equiv 1$ (resp. 2) (mod 3). Hence we have $N(\mathfrak{p})\equiv 1 \pmod 3$. Thus, if $N(\mathfrak{p})$ is represented by $p_{-D}=(1,1,\frac14(D+1))$, then

$$N(\mathfrak{p})=x^2+xy+\tfrac14(D+1)\,y^2,$$

with either (i) $y\equiv 0 \pmod 3$, or (ii) $x\equiv y\not\equiv 0 \pmod 3$, $D\equiv 1 \pmod 3$. If (i) holds then $N(\mathfrak{p})$ is represented by $(1,1,\frac14(9D+1))$ as

$$N(\mathfrak{p})=\left(x+\frac{y}{3}\right)^2+\left(x+\frac{y}{3}\right)\left(\frac{y}{3}\right)+\frac{(9D+1)}{4}\left(\frac{y}{3}\right)^2.$$

If (ii) holds then $N(\mathfrak{p})$ is represented by $(9,9,\frac14(D+9))$ as

$$N(\mathfrak{p})=9\left(\frac{x-y}{3}\right)^2+9\left(\frac{x-y}{3}\right)y+\frac{(D+9)}{4}y^2.$$

This completes the proof when $p\equiv 1 \pmod 3$ as in this case $N(\mathfrak{p})=p$. When $p\equiv 2 \pmod 3$, we have $N(\mathfrak{p})=p^2$, and since there are exactly three inequivalent forms of discriminant $-D$, p^2 is represented by p_{-D} if and only if p is represented by p_{-D}.

This completes the proof of Theorem 1 for those D listed in (A).

We conclude this section by noting that when $D=44$, and p is a prime $\equiv 1 \pmod 3$ with $\left(\frac{-44}{p}\right)=1$, Weinberger's theorem [18] gives a necessary and sufficient condition for p to be represented by the form (1, 1, 223), namely

p is represented by (1, 1, 223) if and only if $\varepsilon_{33}=23+4\sqrt{33}$ is a cube (mod $\mathfrak{p}$), where $\mathfrak{p}$ is a prime ideal of $Q(\sqrt{33})$ with $N(\mathfrak{p})=p$.

This result is not relevant to Theorem 1. Similar remarks apply to the other values of D in (B). Thus a different approach is needed to prove Theorem 1 for those D in (B), and this is done in the next section.

3. Proof of Theorem 1 for those D listed in (B). Throughout this section, D is one of the five integers listed in (B). Note that $D=4D^*$, where D^* is a prime $\equiv 3 \pmod 8$. Let L_D denote the bicyclic biquadratic field $Q(\sqrt{-3},\sqrt{-D^*})$. If $\theta\in L_D$ the conjugates of θ are θ, θ', $\bar\theta$, $\bar\theta'$, where

$$\text{(3.1)}\qquad \begin{cases}\theta = a+b\sqrt{-3}+c\sqrt{-D^*}+d\sqrt{3D^*},\\ \theta' = a-b\sqrt{-3}+c\sqrt{-D^*}-d\sqrt{3D^*},\\ \bar\theta = a-b\sqrt{-3}-c\sqrt{-D^*}+d\sqrt{3D^*},\\ \bar\theta' = a+b\sqrt{-3}-c\sqrt{-D^*}-d\sqrt{3D^*},\end{cases}$$

where $a, b, c, d\in Q$. The ring of integers of L_D is denoted by R_D. It is known that R_D is a unique factorization domain [1].

Let p be a prime >3 not dividing D. If $\left(\frac{-D}{p}\right)=-1$, p is not represented by $p_{-D}=(1,0,D/4)$, and, as discrim $(f_{-D}(x))=-D$, by a theorem of Stickelberger [16], $f_{-D}(x)$ is the product of a linear polynomial and an irreducible quadratic (mod p).

Suppose now that $\left(\frac{-D}{p}\right)=+1$. We must show that p is represented by $p_{-D}=(1,0,D/4)$ if and only if $f_{-D}(x)$ is congruent to the product of three distinct linear polynomials (mod p). Define

$$\text{(3.2)}\qquad g_{-D}(x)=x^3+\frac{a_D}{3}x+\frac{b_D}{27},$$

where the integers a_D and b_D are given in Table 7.

Table 7

D	a_D	b_D
44	-4	$+38$
76	$+8$	$+2$
172	-4	-70
268	-10	-106
652	$+20$	-196

We note that

(3.3). $$\operatorname{discrim}(g_{-D}(x)) = (-4a_D^3 - b_D^2)/27 = \begin{cases} -D, & \text{if } D \neq 652, \\ -4D, & \text{if } D = 652, \end{cases}$$

and that

(3.4) $$f_{-D}(x) = \frac{1}{d}(vx+w)^e g_{-D}\left(\frac{tx+u}{vx+w}\right),$$

where the integers d, e $(=0, 3)$, t, u, v, w are given in Table 8.

Table 8

D	d	e	t	u	v	w
44	$+1$	0	$+3$	$+1$	0	$+3$
76	$+27$	$+3$	$+1$	$+2$	$+3$	-3
172	-1	0	-3	$+1$	0	$+3$
268	-1	0	-3	-2	0	$+3$
652	-108	$+3$	-4	-2	-3	$+3$

From (3.4) we see that $f_{-D}(x)$ is congruent to the product of three distinct linear polynomials (mod p) if and only if $g_{-D}(x)$ is the product of three distinct linear polynomials (mod p). By (3.3) we have

$$\left(\frac{\operatorname{discrim}(g_{-D})}{p}\right) = \left(\frac{-D}{p}\right) = +1,$$

so that by a theorem of Dickson [4], $g_{-D}(x)$ is the product of three distinct linear polynomials (mod p) if and only if

(3.5) $$\left[\frac{\mu_D}{\lambda_D}\right]_3 = 1,$$

where

(3.6) $$\mu_D = \begin{cases} 19+3\sqrt{33}, & \text{if } D = 44, \\ 1+3\sqrt{57}, & \text{if } D = 76, \\ -35+3\sqrt{129}, & \text{if } D = 172, \\ -53+3\sqrt{201}, & \text{if } D = 268, \\ -98+6\sqrt{489}, & \text{if } D = 652, \end{cases}$$

and λ_D is a prime divisor of p in R_D. (The symbol $\left[\frac{\mu}{\lambda}\right]_3$ in (3.5) is the cubic Legendre symbol.) The prime factorization of the prime 3 in R_D is given as follows:

(3.7) $$3 = \begin{cases} -\pi_D^2 \bar{\pi}_D'^2, & \text{if } D = 44, \\ -\pi_D^2, & \text{if } D = 76, 172, 268, 652, \end{cases}$$

where

(3.8) $$\pi_D = \begin{cases} \frac{1}{2}(1+2\sqrt{-3}+\sqrt{-11}), & \text{if } D = 44, \\ \sqrt{-3}, & \text{if } D = 76, 172, 268, 652. \end{cases}$$

By Artin's reciprocity law, we have

(3.9) $$\left[\frac{\mu_D}{\lambda_D}\right]_3 = \begin{cases} \left(\frac{\mu_D, \lambda_D}{\pi_D}\right)_3 \left(\frac{\mu_D, \lambda_D}{\bar{\pi}_D'}\right)_3 \left[\frac{\lambda_D}{\mu_D}\right]_3, & \text{if } D = 44, \\ \left(\frac{\mu_D, \lambda_D}{\pi_D}\right)_3 \left[\frac{\lambda_D}{\mu_D}\right]_3, & \text{if } D \neq 44, \end{cases}$$

where $\left(\frac{\alpha, \beta}{\pi}\right)_3$ is the cubic Hilbert symbol. From (3.6) we see that

(3.10) $$\mu_D \equiv 1 \pmod{(\sqrt{-3})^3},$$

so that

(3.11) $$\left(\frac{\mu_D, \lambda_D}{\pi_D}\right)_3 = \left(\frac{\mu_D, \lambda_D}{\bar{\pi}_D'}\right)_3 = 1.$$

Thus (3.9) reduces to

(3.12) $$\left[\frac{\mu_D}{\lambda_D}\right]_3 = \left[\frac{\lambda_D}{\mu_D}\right]_3.$$

Next we observe that

(3.13) $$\mu_D = \omega_D \theta_D \bar{\theta}_D'^2 \gamma_D^3,$$

where $\gamma_D \in R_D$, ω_D is a unit of R_D, and θ_D is the prime divisor of 2 in R_D given by

$$(3.14)\qquad \theta_D = \begin{cases} \tfrac{1}{2}(\sqrt{-3}+\sqrt{-11}), & \text{if } D = 44,\\ \tfrac{1}{2}(3\sqrt{-3}+\sqrt{-19}), & \text{if } D = 76,\\ \tfrac{1}{2}(19\sqrt{-3}+5\sqrt{-43}), & \text{if } D = 172,\\ \tfrac{1}{2}(5\sqrt{-3}+\sqrt{-67}), & \text{if } D = 268,\\ \tfrac{1}{2}(715\sqrt{-3}+97\sqrt{-163}), & \text{if } D = 652. \end{cases}$$

We note that

$$(3.15)\qquad \theta_D\bar{\theta}_D = \begin{cases} 2, & \text{if } D = 44,\\ -2, & \text{if } D \neq 44. \end{cases}$$

Appealing to (3.13) we see that

$$(3.16)\qquad \left[\frac{\lambda_D}{\mu_D}\right]_3 = \left[\frac{\lambda_D}{\theta_D}\right]_3 \left[\frac{\lambda_D}{\bar{\theta}_D}\right]_3^2.$$

Thus we have shown:

$$(3.17)\qquad p \text{ is represented by } p_{-D} \Leftrightarrow \left[\frac{\lambda_D}{\theta_D}\right]_3 = \left[\frac{\lambda_D}{\bar{\theta}_D}\right]_3.$$

From (3.14) and (3.15) we obtain

$$\pm\theta_D^3\bar{\theta}_D = 2\theta_D^2 = \begin{cases} -7-\sqrt{33}, & \text{if } D = 44,\\ -23-3\sqrt{57}, & \text{if } D = 76,\\ -1579-95\sqrt{129}, & \text{if } D = 172,\\ -71-5\sqrt{201}, & \text{if } D = 268,\\ -1533671-69355\sqrt{489}, & \text{if } D = 652, \end{cases}$$

from which we see that

$$(3.18)\qquad \begin{cases} \sqrt{3D^*} \equiv r_D \pmod{\theta_D^3},\\ \sqrt{3D^*} \equiv -r_D \pmod{\bar{\theta}_D^3}, \end{cases}$$

where

$$(3.19)\qquad r_D = \begin{cases} 1, & \text{if } D = 44,\\ 3, & \text{if } D = 76, 172, 652,\\ 5, & \text{if } D = 268. \end{cases}$$

Multiplying (3.18) by $\sqrt{-3}$, we obtain

$$(3.20)\qquad \begin{cases} \sqrt{-D^*} \equiv 3r_D\sqrt{-3} \pmod{\theta_D^3},\\ \sqrt{-D^*} \equiv -3r_D\sqrt{-3} \pmod{\bar{\theta}_D^3}. \end{cases}$$

Next, as λ_D is a prime divisor of p in R_D, we have

$$(3.21)\qquad p = \begin{cases} \lambda_D\bar{\lambda}_D\lambda'_D\bar{\lambda}'_D, & \text{if } p \equiv 1 \pmod 3,\\ \lambda_D\bar{\lambda}_D, & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

As λ_D is an integer of $Q(\sqrt{-3}, \sqrt{-D^*})$, if $p \equiv 1 \pmod 3$, and of $Q(\sqrt{-D^*})$, if $p \equiv 2 \pmod 3$, there are integers x_0, x_1, x_2, x_3, if $p \equiv 1 \pmod 3$, and integers x_0, x_1, if $p \equiv 2 \pmod 3$, such that

$$(3.22)\qquad \lambda_D = \begin{cases} \tfrac{1}{4}(x_0+x_1\sqrt{-3}+x_2\sqrt{-D^*}+x_3\sqrt{3D^*}), & \text{if } p \equiv 1 \pmod 3,\\ \tfrac{1}{2}(x_0+x_1\sqrt{-D^*}), & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

with

$$(3.23)\qquad \begin{cases} \begin{Bmatrix} x_0 \equiv x_1 \equiv x_2 \equiv x_3 \pmod 2\\ x_0-x_1+x_2+x_3 \equiv 0 \pmod 4 \end{Bmatrix}, & \text{if } p \equiv 1 \pmod 3,\\ x_0 \equiv x_1 \pmod 2, & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

see [14]. (Note that $\sqrt{m_1 n_1}$ should be replaced by $\sqrt{m_1}\sqrt{n_1}$ in Theorem 1 of [19].) Set

$$(3.24)\qquad \tfrac{1}{2}(u+v\sqrt{-D^*}) = \begin{cases} \lambda_D\lambda'_D, & \text{if } p \equiv 1 \pmod 3,\\ \lambda_D, & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

so that u and v are integers such that

$$(3.25)\qquad u = \begin{cases} (x_0^2+3x_1^2-D^*x_2^2-3D^*x_3^2)/8, & \text{if } p \equiv 1 \pmod 3,\\ x_0, & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

$$(3.26)\qquad v = \begin{cases} (x_0x_2-3x_1x_3)/4, & \text{if } p \equiv 1 \pmod 3,\\ x_1, & \text{if } p \equiv 2 \pmod 3, \end{cases}$$

and

$$(3.27)\qquad 4p = u^2+D^*v^2, \quad u \equiv v \pmod 2.$$

Clearly p is represented by p_{-D} if and only if $u \equiv v \equiv 0 \pmod 2$. Thus, in view of (3.17), we must show that

$$(3.28)\qquad \left[\frac{\lambda_D}{\theta_D}\right]_3 = \left[\frac{\lambda_D}{\bar{\theta}_D}\right]_3 \Leftrightarrow \begin{cases} x_0x_2-3x_1x_3 \equiv 0 \pmod 8, & \text{if } p \equiv 1 \pmod 3,\\ x_1 \equiv 0 \pmod 2, & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Next, as θ_D is a prime divisor of 2 and λ_D is a prime divisor of the odd prime p, we have $\lambda_D \nmid \theta_D$ and

$$(3.29)\qquad \lambda_D^3 \equiv \lambda_D^{N(\theta_D)-1} \equiv 1 \pmod{\theta_D},$$

showing that

$$(3.30)\qquad \lambda_D \equiv 1, \omega \text{ or } \omega^2 \pmod{\theta_D},$$

where $\omega=(-1+\sqrt{-3})/2$. Appealing to (3.18) and (3.20), we obtain for $p\equiv 1 \pmod 3$

$$\lambda_D \equiv \begin{cases} 1 \pmod{\theta_D}, & \text{if } E\equiv 0 \pmod 4,\ F\equiv 4 \pmod 8,\\ \omega \pmod{\theta_D}, & \text{if } E\equiv 2 \pmod 4,\ F\equiv 4 \pmod 8,\\ \omega^2 \pmod{\theta_D}, & \text{if } E\equiv 2 \pmod 4,\ F\equiv 0 \pmod 8, \end{cases} \tag{3.31}$$

where

$$E=x_0+rx_3,\qquad F=x_0-x_1-3rx_2+rx_3; \tag{3.32}$$

and for $p\equiv 2 \pmod 3$

$$\lambda_D \equiv \begin{cases} 1 \pmod{\theta_D}, & \text{if } x_0\equiv x_1\equiv 0 \pmod 2,\ x_0+rx_1\equiv 2 \pmod 4,\\ \omega \pmod{\theta_D}, & \text{if } x_0\equiv x_1\equiv 1 \pmod 2,\ x_0+rx_1\equiv 2 \pmod 4,\\ \omega^2 \pmod{\theta_D}, & \text{if } x_0\equiv x_1\equiv 1 \pmod 2,\ x_0+rx_1\equiv 0 \pmod 4. \end{cases} \tag{3.33}$$

We now treat the two cases $p\equiv 1 \pmod 3$ and $p\equiv 2 \pmod 3$ separately.

Case (i): $p\equiv 1 \pmod 3$. We have by (3.31)

$$\left[\frac{\lambda_D}{\theta_D}\right]_3=\left[\frac{\lambda_D}{\bar\theta_D}\right]_3$$

$$\Leftrightarrow \begin{Bmatrix}\lambda_D\equiv 1 \pmod{\theta_D}\\ \lambda_D\equiv 1 \pmod{\bar\theta_D}\end{Bmatrix} \text{ or } \begin{Bmatrix}\lambda_D\equiv \omega \pmod{\theta_D}\\ \lambda_D\equiv \omega \pmod{\bar\theta_D}\end{Bmatrix} \text{ or } \begin{Bmatrix}\lambda_D\equiv \omega^2 \pmod{\theta_D}\\ \lambda_D\equiv \omega^2 \pmod{\bar\theta_D}\end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} x_0\equiv -rx_3 \pmod 4\\ x_0-x_1-3rx_2+rx_3\equiv 4 \pmod 8\\ x_0\equiv rx_3 \pmod 4\\ x_0+x_1-3rx_2-rx_3\equiv 4 \pmod 8\end{Bmatrix} \text{ or }$$

$$\begin{Bmatrix} x_0+2\equiv -rx_3 \pmod 4\\ x_0-x_1-3rx_2+rx_3\equiv 4 \pmod 8\\ x_0+2\equiv rx_3 \pmod 4\\ x_0+x_1-3rx_2-rx_3\equiv 0 \pmod 8\end{Bmatrix} \text{ or }$$

$$\begin{Bmatrix} x_0+2\equiv -rx_3 \pmod 4\\ x_0-x_1-3rx_2+rx_3\equiv 0 \pmod 8\\ x_0+2\equiv rx_3 \pmod 4\\ x_0+x_1-3rx_2-rx_3\equiv 4 \pmod 8\end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} x_0\equiv x_1\equiv x_2\equiv x_3\equiv 0 \pmod 2, \text{ say } x_i=2y_i\ (i=0,1,2,3)\\ \text{and}\\ y_0\equiv y_3 \pmod 2,\ y_0-y_1-3ry_2+ry_3\equiv 2 \pmod 4,\ y_0+y_1-3ry_2-ry_3\equiv 2 \pmod 4\\ \text{or}\\ y_0+1\equiv y_3 \pmod 2,\ y_0-y_1-3ry_2+ry_3\equiv 2 \pmod 4,\ y_0+y_1-3ry_2-ry_3\equiv 0 \pmod 4\\ \text{or}\\ y_0+1\equiv y_3 \pmod 2,\ y_0-y_1-3ry_2+ry_3\equiv 0 \pmod 4,\ y_0+y_1-3ry_2-ry_3\equiv 2 \pmod 4\end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} y_0\equiv y_1\equiv y_2\equiv y_3 \pmod 2,\ y_0-y_1+ry_2+ry_3\equiv 2 \pmod 4\\ \text{or}\\ y_0\equiv y_3+1 \pmod 2,\ y_0-y_1-3ry_2+ry_3\equiv y_0+y_1-3ry_2-ry_3+2 \pmod 4\end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} y_0\equiv y_1\equiv y_2\equiv y_3 \pmod 2,\ y_0-y_1-y_2-y_3\equiv 2 \pmod 4\\ \text{or}\\ y_0\equiv y_1\equiv y_2+1\equiv y_3+1 \pmod 2\end{Bmatrix}.$$

It should be noted that if $x_0\equiv x_1\equiv x_2\equiv x_3\equiv 0 \pmod 2$, with $x_i=2y_i$ $(i=0,1,2,3)$, then by (3.23), we have

$$y_0+y_1+y_2+y_3\equiv 0 \pmod 2. \tag{3.34}$$

In view of (3.28) we must show that the assertion

$$x_0x_2-3x_1x_3\equiv 0 \pmod 8 \tag{3.35}$$

is equivalent to

$$\begin{cases} x_i=2y_i\ (i=0,1,2,3) \text{ and}\\ y_0\equiv y_1\equiv y_2\equiv y_3 \pmod 2,\ y_0-y_1-y_2-y_3\equiv 2 \pmod 4, \text{ or}\\ y_0\equiv y_1\equiv y_2+1\equiv y_3+1 \pmod 2,\end{cases} \tag{3.36}$$

under (3.23). It is clear that (3.36) implies (3.35) as

$$x_0x_2-3x_1x_3=4(y_0y_2-3y_1y_3)\equiv 4(y_0y_2-3y_0y_2)\equiv 0 \pmod 8.$$

Next we assume that (3.35) holds and begin by showing that the x_i are all even. We suppose that this is not the case, so that by (3.23) the x_i are all odd, say $x_i=2z_i+1$ $(i=0,1,2,3)$. Then, from (3.35), we have

$$2(z_0z_2+z_1z_3)+(z_0+z_1+z_2+z_3)\equiv 1 \pmod 4. \tag{3.37}$$

Further, as $u\equiv v\equiv 0 \pmod 2$, by (3.27) we see that $u+v\equiv 2 \pmod 4$, and so by (3.25) and (3.26), we have

$$(x_0^2+3x_1^2-D^*x_2^2-3D^*x_3^2)+2(x_0x_2-3x_1x_3)\equiv 16 \pmod{32},$$

and so (as $D^*\equiv 3 \pmod 8$) we obtain

$$(z_0^2+3z_1^2-3z_2^2-z_3^2)+2(z_0z_2+z_1z_3)+2(z_0-z_2+2z_3)\equiv 7 \pmod 8. \tag{3.38}$$

From (3.37) we deduce

(3.39) $$(2z_1+1)z_3 \equiv 1-z_0-z_1-z_2+2z_0z_2 \pmod 4.$$

Multiplying (3.39) by $(2z_1+1)$, we obtain

(3.40) $$z_3 \equiv 1-(z_0+z_1+z_2)+2(z_0z_1+z_1z_2+z_2z_0) \pmod 4,$$

so that

(3.41) $$\begin{cases} z_3 \equiv 1-A+2B \pmod 4, \\ z_3^2 \equiv 1+A^2-2A+4AB \pmod 8, \end{cases}$$

where

(3.42) $$A = z_0+z_1+z_2, \quad B = z_0z_1+z_1z_2+z_2z_0.$$

Using (3.41) in (3.38), we obtain

$$3+4(z_0+z_2)\big((z_0z_1+z_1z_2+z_2z_0)-z_1\big) \equiv 7 \pmod 8,$$

that is

$$(z_0+z_2)(z_0z_1+z_1z_2+z_2z_0-z_1) \equiv 1 \pmod 2,$$

showing that

$$z_0+z_2 \equiv z_0z_1+z_1z_2+z_2z_0-z_1 \equiv 1 \pmod 2,$$

which gives the contradiction

$$z_0+z_2 \equiv z_0z_2 \equiv 1 \pmod 2.$$

This completes the proof that (3.35) implies that all the x_i are even, say $x_i = 2y_i$ $(i = 0, 1, 2, 3)$. We complete the proof in the case $p \equiv 1 \pmod 3$ by showing that we must have either

$$y_0 \equiv y_1 \equiv y_2 \equiv y_3 \pmod 2, \quad y_0-y_1-y_2-y_3 \equiv 2 \pmod 4$$

or

$$y_0 \equiv y_1 \equiv y_2+1 \equiv y_3+1 \pmod 2.$$

As $u \equiv 0 \pmod 2$, $v \equiv 0 \pmod 2$, $u+v \equiv 2 \pmod 4$ we have

(3.43) $$y_0^2-y_1^2+y_2^2-y_3^2 \equiv 0 \pmod 4,$$

(3.44) $$y_0y_2+y_1y_3 \equiv 0 \pmod 2,$$

(3.45) $$y_0^2+3y_1^2-3y_2^2-y_3^2+2y_0y_2+2y_1y_3 \equiv 4 \pmod 8.$$

We begin by showing that $y_0 \equiv y_1 \pmod 2$. Suppose not, so that we have $y_0 \equiv y_1+1 \pmod 2$. Next (3.34) gives $y_2 \equiv y_3+1 \pmod 2$. Then, from either (3.43) or (3.44), we deduce that $y_1 \equiv y_3+1 \pmod 2$. Thus we have

(3.46) $$y_0 \equiv y_1+1 \equiv y_2+1 \equiv y_3 \pmod 2.$$

If $y_0 \equiv 0 \pmod 2$ then (3.45) and (3.46) give

$$y_0^2-y_3^2+2y_0+2y_3 \equiv 4 \pmod 8,$$

which gives the contradiction

$$0 \equiv (y_0+1)^2-(y_3-1)^2 \equiv 4 \pmod 8.$$

If $y_0 \equiv 1 \pmod 2$ then (3.45) and (3.46) give

$$y_1^2+y_2^2+2y_1+2y_2 \equiv 4 \pmod 8,$$

which gives the contradiction

$$2 \equiv (y_1+1)^2+(y_2+1)^2 \equiv 6 \pmod 8.$$

Hence we must have

$$y_0 \equiv y_1 \pmod 2,$$

and so, by (3.34), we also have

$$y_2 \equiv y_3 \pmod 2.$$

If $y_1 \equiv y_2+1 \pmod 2$ we are finished. Otherwise $y_1 \equiv y_2 \pmod 2$ and we must show that $y_0-y_1-y_2-y_3 \equiv 2 \pmod 4$. We have

$$y_0 \equiv y_1 \equiv y_2 \equiv y_3 \pmod 2.$$

If $y_0 \equiv y_1 \equiv y_2 \equiv y_3 \equiv 1 \pmod 2$ then (3.45) gives

$$y_0y_2+y_1y_3 \equiv 2 \pmod 4,$$

and thus

$$\begin{aligned} y_0-y_1-y_2-y_3 &\equiv 2y_0-(y_0+y_1+y_2+y_3) \pmod 4 \\ &\equiv 2-(y_0+1)(y_2+1)-(y_1+1)(y_3+1)+(y_0y_2+y_1y_3) \\ &\quad +2 \pmod 4 \\ &\equiv 2-0-0+2+2 \pmod 4 \\ &\equiv 2 \pmod 4, \end{aligned}$$

as required. If $y_0 \equiv y_1 \equiv y_2 \equiv y_3 \equiv 0 \pmod 2$ then (3.45) gives (remembering that $n^2 \equiv 2n \pmod 8$ when n is even)

$$y_0-y_1+y_2-y_3 \equiv 2 \pmod 4,$$

and thus

$$y_0-y_1-y_2-y_3 \equiv (y_0-y_1+y_2-y_3)-2y_2 \equiv 2 \pmod 4,$$

as required. This completes the proof when $p \equiv 1 \pmod 3$.

Case (ii): $p \equiv 2 \pmod 3$. As $\lambda'_D = \lambda_D$ and $\bar{\theta}_D = -\theta'_D$, we have $\left[\frac{\lambda_D}{\bar{\theta}'_D}\right]_3 = \left[\frac{\lambda_D}{\theta_D}\right]_3^2$, and so $\left[\frac{\lambda_D}{\bar{\theta}'_D}\right]_3 = \left[\frac{\lambda_D}{\theta_D}\right]_3$ holds if and only if $\left[\frac{\lambda_D}{\theta_D}\right]_3 = 1$, that is, if and only if $\lambda_D \equiv 1 \pmod{\theta_D}$. By (3.33) this condition is equivalent to $x_0 \equiv x_1 \equiv 0 \pmod 2$, $x_0 + rx_1 \equiv 2 \pmod 4$, which by (3.25), (3.26) and (3.27) is equivalent to $u \equiv v \equiv 0 \pmod 2$ as required.

The proof of Theorem 1 is now complete.

4. Proof of Theorem 2. Since $\sqrt[3]{\varkappa_D} + \sqrt[3]{\varkappa'_D}$ is the real root of $27f_{-D}((x-r)/3)$, where r is the coefficient of x^2 in $f_{-D}(x)$, Theorem 2 follows immediately from Theorem 1 and [3: Theorem 9.2, Exercise 9.3].

5. Proof of Theorem 3. Theorem 3 follows from Theorem 1 and the following theorem (which is essentially due to Cauchy [2]) with $k = A_1 = a_D$, $l = -B = -b_D$ (see (2.8) and (3.2)).

THEOREM (Cauchy). *Let A and B be integers and let p be a prime such that*

$$p > 3, \quad p \nmid AB, \quad \left(\frac{-4A^3 - 27B^2}{p}\right) = +1.$$

Define an integer A_1 by $A \equiv 3A_1 \pmod p$. Let $\{u_n\}_{n=0,1,2,\ldots}$ be the sequence of integers defined by

$$u_{n+2} + Bu_{n+1} - A_1^3 u_n = 0,$$

$$u_0 = 2, \quad u_1 = -B.$$

Then $x^3 + Ax + B$ is congruent to the product of three distinct linear polynomials $\pmod p$ *if*

$$\begin{cases} u_{(p-1)/3} \equiv 2 \pmod p, & p \equiv 1 \pmod 3, \\ u_{(p+1)/3} \equiv -2A_1 \pmod p, & p \equiv 2 \pmod 3, \end{cases}$$

and $x^3 + Ax + B$ is irreducible $\pmod p$ *if*

$$\begin{cases} u_{(p-1)/3} \equiv -1 \pmod p, & p \equiv 1 \pmod 3, \\ u_{(p+1)/3} \equiv A_1 \pmod p, & p \equiv 2 \pmod 3. \end{cases}$$

6. Acknowledgement. The authors would like to thank Dr. Kenneth Hardy (Carleton University) and Mr. Nicholas Buck (College of New Caledonia) for doing some computing for them in connection with this research.

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Received on 22.5.1989
and in revised form on 19.9.1989 (1938)