[3] P. Deligne and J. P. Serre, *Formes modulaires de poids 1*, Ann. Sci. École Norm. Sup. 7 (1974), 507–530.
[4] D. Dummit, H. Kisilevsky and J. Mckay, *Multiplicative product of η-functions*, in *Finite Groups – Coming of Age*, Montreal, Qué., 1982, 89–98; Contemp. Math. 45 (1985).
[5] G. H. Hardy, *Ramanujan*, Chelsea, New York 1940.
[6] A. Ingham, *The Distribution of Prime Numbers*, Cambridge University Press, 1932.
[7] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, 1984.
[8] M. Ram Murty, *Oscillations of Fourier coefficients of modular forms*, Math. Ann. 262 (1983), 241–252.
[9] R. A. Rankin, *An Ω-result for the coefficient of cusp forms*, ibid. 203 (1973), 239–250.

THE INSTITUTE OF MATHEMATICAL SCIENCES
Madras-600113, India

Present Adress:
SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
Homi Bhabha Road
Bombay-400005, India

Received on 5.5.1987
and in revised form on 30.8.1989 (1721)

ACTA ARITHMETICA
LVII (1991)

Mean value estimates for exponential sums with applications to *L*-functions

by

MATTI JUTILA (Turku)

1. Introduction

1.1. In our previous paper [J3], we studied the mean square of the exponential sum

$$S(M, M'; v, y) = \sum_{M}^{M'} d(m)\,g(m, v, y)\,e\big(f(m, v, y)\big)$$

with respect to v running over an interval $[0, V]$ and y running over a well-spaced system of real numbers. Here $d(m)$ is the usual divisor function, $e(\alpha) = e^{2\pi i\alpha}$, and the functions f and g are supposed to satisfy certain conditions. The main result, a general mean value theorem, was applied to the fourth moment of $\zeta(1/2+it)$ over a system of short intervals. In this way, we reproved a theorem of H. Iwaniec [Iw], which was in fact our principal motivation.

Our object in this paper is to generalize Iwaniec's theorem to *L*-functions. To this end, we need a mean value estimate for exponential sums

$$S_\chi(M, M'; v, y) = \sum_{M}^{M'} \chi(m)\,d(m)\,g(m, v, y)\,e\big(f(m, v, y)\big) \tag{1.1}$$

involving Dirichlet characters. If χ is a primitive character (mod D), then the sum S_χ can be written in terms of the Gaussian sum

$$\tau_\chi = \sum_{a=1}^{D} \chi(a)\,e(a/D)$$

and the exponential sum

$$S(M, M'; v, y, \alpha) = \sum_{M}^{M'} d(m)\,g(m, v, y)\,e\big(f(m, v, y)+m\alpha\big) \tag{1.2}$$

as follows:

$$S_\chi = (\tau_{\bar\chi})^{-1} \sum_{a=1}^{D} \bar\chi(a)\,S(M, M'; v, y, a/D).$$

Therefore, since $|\tau_\chi| = D^{1/2}$, we obtain

$$\sum_{\chi \bmod D}^{*} |S_\chi|^2 \leqslant D^{-1} \sum_{\chi \bmod D} \Big|\sum_{a=1}^{D} \bar{\chi}(a) S(\ldots, a/D)\Big|^2 = (\phi(D)/D) \sum_{a=1}^{D}{}^{*} |S(M, M'; v, y, a/D)|^2, \tag{1.3}$$

where $\sum^*$ denotes either a sum over primitive characters, or over a reduced system of residues (mod D), as the case may be.

The set of the numbers a/D in (1.3) is a D^{-1}-spaced system. More generally, let $\{\alpha_i\}$ be a finite set of numbers which is δ-spaced (mod 1), in other words $\|\alpha_i - \alpha_j\| \geqslant \delta$ if $i \neq j$, where $\|x\|$ denotes the distance of x from the nearest integer. Further, let $\{y_r\}_{r=1}^{R}$ be a Y_0-spaced system in the interval $[Y, 2Y]$, and consider the mean value

$$I = \sum_i \sum_{r=1}^{R} \int_0^V |S(M, M'; v, y_r, \alpha_i)|^2 \, dv. \tag{1.4}$$

It is our goal to estimate this quantity under certain conditions. The assumptions concerning the functions f and g will be similar to those in [J3], and we repeat these in the next section, retaining the previous notation.

1.2. For an interval $[a, b]$ and a positive number μ, we denote by $D(a, b; \mu)$ the set of all complex numbers z satisfying $|z - x| < \mu$ for some $x \in [a, b]$. Put, for short, $D_1 = D(M, M'; c_1 M)$ and $D_2 = D(0, V; c_2 V)$, where the c_i are positive constants. Also, write $M_1 = (1 - c_1) M$, $M_2 = M' + c_1 M$, $V_1 = -c_2 V$, $V_2 = (1 + c_2) V$. The notation $A \asymp B$ will mean that $A \ll B \ll A$.

The functions f and g are supposed to satisfy the following conditions.

(i) $f(x, v, y)$ is real for $(x, v, y) \in (M_1, M_2) \times (V_1, V_2) \times [Y, 2Y]$.

(ii) f is a holomorphic function of (x, v) in $D_1 \times D_2$ for fixed $y \in [Y, 2Y]$. Also, f_x is a continuously differentiable function of (x, y) in $(M_1, M_2) \times [Y, 2Y]$ for fixed $v \in (V_1, V_2)$.

(iii) There are positive numbers F and T such that

$$f_x \ll F M^{-1} \quad \text{in } D_1 \times (V_1, V_2) \times [Y, 2Y], \tag{1.5}$$

$$f_{vx} \ll F M^{-1} T^{-1} \quad \text{in } (M_1, M_2) \times D_2 \times [Y, 2Y], \tag{1.6}$$

and in the set $(M_1, M_2) \times (V_1, V_2) \times [Y, 2Y]$ we have

$$|f_{xx}| \asymp F M^{-2}, \tag{1.7}$$

$$|f_{xv}| \asymp F M^{-1} T^{-1}. \tag{1.8}$$

$$|f_{xy}| \asymp F M^{-1} Y^{-1}. \tag{1.9}$$

(iv) The function g, defined in $[M, M'] \times [0, V] \times [Y, 2Y]$, is continuous as a function of (x, v) for fixed y, and g_x is a continuous function of x for fixed (v, y). Also, $g \ll G$ and $g_x \ll G'$.

Remark. By the Hartogs theorem, the first part of (ii) is equivalent to the existence of the complex partial derivatives f_x and f_v in $D_1 \times D_2 \times [Y, 2Y]$.

1.3. The main result. We are now in a position to formulate our estimate for I, defined in (1.4). In the sequel, ε will stand for a positive constant (not necessarily the same at each occurrence) which may be supposed to be arbitrarily small. As in [J3], the assertion of the theorem in the case $R = 1$ should be understood in the sense that the parameter y and thus the conditions involving y_r, Y, and Y_0 are omitted.

THEOREM 1. *Suppose that the functions f and g satisfy the conditions (i)–(iv), where M and M' are sufficiently large positive numbers with $M < M' \leqslant 2M$. Let $Y \leqslant y_1 < \ldots < y_R \leqslant 2Y$, and suppose that $y_{r+1} - y_r \geqslant Y_0$, where*

$$Y_0 \geqslant T^{-1} V Y. \tag{1.10}$$

Let $\{\alpha_i\}$ be a finite set of real numbers such that $\|\alpha_i - \alpha_j\| \geqslant \delta \gg F^{-B}$ for $i \neq j$, where B is a positive constant. Suppose further that $\delta \ll 1$,

$$F^{2/3+\varepsilon} \ll M \ll \delta^{-1} F \tag{1.11}$$

and

$$(\min(F, M))^{-1/2+\varepsilon} T \ll V \ll (\min(F, M))^{-1/3} T. \tag{1.12}$$

Then

$$I \ll \delta^{-1} (G + M G')^2 M F^{\varepsilon} \{RV + R^{1/2} F^{-1/2} T^{3/2} V^{-1/2} \min(R^{1/2}, 1 + (F/M)^{1/2})\}. \tag{1.13}$$

The interesting range for M is $F \ll M \ll \delta^{-1} F$, for otherwise the assertion follows immediately from Theorem 1 of [J3]. Indeed, we may apply this theorem to the individual terms on the right of (1.4) for each i, and (1.13) follows, because the cardinality of the system $\{\alpha_i\}$ is $\ll \delta^{-1}$.

Owing to the inequality (1.3), we may infer an analogous result for S_χ.

THEOREM 2. *Let D be a positive integer, and suppose that the assumptions of Theorem 1 on f, g, M, T, V, Y, and Y_0 are satisfied, except that (1.11) now reads*

$$F^{2/3+\varepsilon} \ll M \ll DF. \tag{1.14}$$

Also suppose that $D \ll F^B$ for a positive constant B. Then

$$\sum_{\chi \bmod D}^{*} \sum_{r=1}^{R} \int_0^V |S_\chi(M, M'; v, y_r)|^2 \, dv \ll D (G + M G')^2 M F^{\varepsilon} \{RV + R^{1/2} F^{-1/2} T^{3/2} V^{-1/2} \min(R^{1/2}, 1 + (F/M)^{1/2})\}. \tag{1.15}$$

If the functions f and g are of the form $f(x, t)$ and $g(x, t)$, where t is a parameter in the interval $[T, 2T]$, we write

$$S(M, M'; t, \alpha) = \sum_M^{M'} d(m) g(m, t) e\big(f(m, t) + m\alpha\big), \tag{1.16}$$

$$S_\chi(M, M'; t) = \sum_M^{M'} d(m) \chi(m) g(m, t) e\big(f(m, t)\big). \tag{1.17}$$

Then the mean values

$$\sum_i \sum_{r=1}^{R} \int_{t_r}^{t_r+T_0} |S(M, M'; t, \alpha_i)|^2\, dt, \tag{1.18}$$

$$\sum_{\chi \bmod D}^{*} \sum_{r=1}^{R} \int_{t_r}^{t_r+T_0} |S_\chi(M, M'; t)|^2\, dt, \tag{1.19}$$

where $T \leqslant t_1 < \ldots < t_R \leqslant 2T$ and $t_{r+1} - t_r \geqslant T_0$, can be interpreted as special cases of the analogous expressions in (1.4) and (1.15) if we define (as in [J3]) $f(x, v, y) = f(x, y+v)$, $g(x, v, y) = g(x, y+v)$, and put $y_r = t_r$, $Y = T$, and $V = Y_0 = T_0$. Then $f(x, v, y)$ and $g(x, v, y)$ satisfy the conditions (i)–(iv) if $f(x, t)$ and $g(x, t)$ are supposed to satisfy analogous conditions (see Remark 1 in [J3], § 1.3). Then, under the assumptions of Theorem 1 (where (1.10) is now trivial), we obtain the estimate (1.13) (with $V = T_0$) for the expression (1.18), and analogously the estimate (1.15) holds for (1.19). As was pointed out in [J3] (see Remark 4 in § 1.3), it is sometimes of advantage to apply the last mentioned results to a system of longer subintervals $[t_r', t_r' + T_0']$ which cover the original system. In this way, we obtain

$$\begin{aligned}\sum_i \sum_{r=1}^{R} \int_{t_r}^{t_r+T_0} &|S(M, M'; t, \alpha_i)|^2\, dt \\ &\ll \delta^{-1}(G+MG')^2 M F^\varepsilon \{R T_0 + R^{2/3} F^{-1/3} T \min(R^{1/3}, 1+(F/M)^{1/3})\}\end{aligned} \tag{1.20}$$

and

$$\begin{aligned}\sum_{\chi \bmod D}^{*} \int_{t_r}^{t_r+T_0} &|S_\chi(M, M'; t)|^2\, dt \\ &\ll D(G+MG')^2 M F^\varepsilon \{R T_0 + R^{2/3} F^{-1/3} T \min(R^{1/3}, 1+(F/M)^{1/3})\}.\end{aligned} \tag{1.21}$$

1.4. Applications to *L*-functions. The special case

$$f(x, t) = -t \log x, \qquad g(x, t) = x^{-1/2}$$

of (1.17) is the Dirichlet polynomial

$$S_\chi(M, M'; t) = \sum_M^{M'} d(m) \chi(m) m^{-1/2-it}.$$

We may choose now $F = T$, so the conditions corresponding to (1.14) and (1.12) read

$$T^{2/3+\varepsilon} \ll M \ll DT, \tag{1.22}$$

$$(\min(T, M))^{-1/2+\varepsilon} T \ll T_0 \ll (\min(T, M))^{-1/3} T, \tag{1.23}$$

and (1.21) gives

$$\begin{aligned}\sum_{\chi \bmod D}^{*} \sum_{r=1}^{R} \int_{t_r}^{t_r+T_0} &|S_\chi(M, M'; t)|^2\, dt \\ &\ll D T^\varepsilon \{R T_0 + (RT)^{2/3} \min(R^{1/3}, 1+(T/M)^{1/3})\}\end{aligned} \tag{1.24}$$

for $D \ll T^B$.

The case $M \asymp DT$ is of relevance for the estimation of Dirichlet *L*-functions, and we obtain the following generalization of a theorem of H. Iwaniec (see [Iw], Theorem 4), which is the special case $D = 1$.

THEOREM 3. *Let* $T \geqslant 1$, $T^{1/2+\varepsilon} \ll T_0 \ll T^{2/3}$, $T \leqslant t_1 < \ldots < t_R \leqslant 2T$, *and* $t_{r+1} - t_r \geqslant T_0$. *Then*

$$\sum_{\chi \bmod D} \sum_{r=1}^{R} \int_{t_r}^{t_r+T_0} |L(1/2+it, \chi)|^4\, dt \ll D(R T_0 + (RT)^{2/3})(DT)^\varepsilon. \tag{1.25}$$

In the first place, this follows from (1.24) only for primitive characters under the assumption $T \gg D^\varepsilon$. But the extension to all characters is easy once the primitive characters have been dealt with. Further, for $T \ll D^\varepsilon$ we may apply the well-known estimate

$$\sum_{\chi \bmod D} \int_T^{2T} |L(1/2+it, \chi)|^4\, dt \ll (DT)^{1+\varepsilon}$$

to verify (1.25) directly.

Two special cases of (1.25) are worth pointing out. Firstly, for $R = 1$ and $T_0 = T^{2/3}$, we have

$$\sum_{\chi \bmod D} \int_T^{T+T^{2/3}} |L(1/2+it, \chi)|^4\, dt \ll D T^{2/3} (DT)^\varepsilon. \tag{1.26}$$

Secondly, the estimate

$$\sum_{\chi \bmod D} \int_0^T |L(1/2+it, \chi)|^{12}\, dt \ll D^3 T^2 (DT)^\varepsilon \tag{1.27}$$

is an easy consequence of (1.25), as will be shown in Section 6. In the case $D = 1$, this implication was pointed out by H. Iwaniec [Iw]. While (1.26) is a new result, the estimate (1.27) was established by D. R. Heath-Brown [H1] in the case $D = 1$, and generally by T. Meurman [Me1].

1.5. Some analogues. In the exponential sums considered above, one may replace (as in [J1], [J2], and [J3]) the divisor function $d(m)$ throughout by $a(m)m^{-(\varkappa-1)/2}$, where the $a(m)$ are the Fourier coefficients of a holomorphic cusp form of weight $\varkappa$ for the full modular group. Owing to the work of T. Meurman [Me2], it is even possible to deal with sums involving Fourier coefficients of non-holomorphic cusp forms (Maass wave forms). As another class of arithmetical functions which may enter as coefficients in this context, we may mention $r_Q(m)$, the number of representations of m by the positive definite binary integral quadratic form Q (see [J5]). In particular, this class contains the classical arithmetical function $r(m)$.

The analogue of $L^2(s, \chi)$ in the theory of holomorphic cusp forms is the Dirichlet series

$$\varphi(s, \chi) = \sum_{n=1}^{\infty} a(n)\chi(n)n^{-s}$$

considered by G. Shimura [S]. Our results on $L(s, \chi)$ cannot be immediately carried over to $\varphi(s, \chi)$, for to estimate the latter functions, we have to deal with Dirichlet polynomials with $M \ll DT$, thus not only those with $M \asymp DT$. Nevertheless, the analogue of (1.26), *viz.*

$$\sum_{\chi \bmod D} \int_{T}^{T+T^{2/3}} |\varphi(s, \chi)|^2\, dt \ll DT^{2/3}(DT)^{\varepsilon}$$

is an easy consequence of the analogue of (1.24). For $D = 1$, this is a corollary of a result of A. Good [G].

1.6. Estimates for exponential sums. We may apply (1.20) to the estimation of exponential sums of the type

$$S = \sum_{M}^{M'} d(m)g(m)e\big(f(m)+m\alpha\big), \tag{1.28}$$

where f and g satisfy certain conditions analogous to (i)–(iv), where only those assumptions pertaining to x are taken into account. To this end, we interpret the sum S as the value at $t = F$ of the sum (1.16) with $f(x, t) = (t/F)f(x)$ and $g(x, t) = g(x)$. Now (1.20) with $\delta = \min(1, F/M)$, $R = 1$, $t_1 = F$, $T_0 = F^{2/3}$, and $\{\alpha_i\} = \{\alpha\}$ gives a mean value estimate, which implies a similar pointwise estimate for the integrand. Indeed, we may suppose that $f \ll F$ (on replacing f by $f-f(M)$ if necessary) and apply Lemma 1.4 in [Mo]. The resulting estimate is

$$S \ll (G+MG')(M^{1/2}F^{1/3}+MF^{-1/6})F^{\varepsilon} \quad \text{for } M \ll F^{B}; \tag{1.29}$$

note that the condition $M \gg F^{2/3+\varepsilon}$ in (1.11) can be omitted, for otherwise (1.29) is trivial.

If $M \ll F$, then the term $m\alpha$ in (1.28) can be absorbed into $f(m)$, and the term $MF^{-1/6}$ in (1.29) can be omitted. Then (1.29) amounts to the estimate (1.21) in [J3].

An estimate for S in the case $M \gg F$ was needed in [J4] in connection with the problem of estimating the exponential sum

$$\tilde{A}(x, h/k) = \sum_{n \leqslant x} \tilde{a}(n)e(nh/k)$$

where $\tilde{a}(n) = a(n)n^{-(\varkappa-1)/2}$ is the "normalized" nth Fourier coefficient of a holomorphic cusp form of weight $\varkappa$. Using (1.29) in the proof of Lemma 2 in [J4], we obtain

$$\tilde{A}(x, h/k) \ll \min(k^{2/3}x^{1/3}, k^{1/4}x^{3/8})x^{\varepsilon} \quad \text{for } k \ll x^{1/4}, \tag{1.30}$$

which improves and simplifies the result of this lemma.

Actually we expect (1.30) to hold for all $k \ll x^{1/2}$, but the argument leading to (1.29) was quite wasteful, for the starting point was an estimate which holds primarily for a sum of $\asymp \delta^{-1} \asymp 1+M/F$ terms but it was applied just to a single term. For a more careful estimation, it might be of advantage to transform the sum S (by the method in [J1]) into a new form which would be more amenable to estimation. However, we do not go into details here.

1.7. In the proof of the main result, Theorem 1, we follow the general scheme of [J2] and [J3]. A new ingredient is another generalization of a lemma of E. Bombieri and H. Iwaniec (Theorem 4.2 in [B–I]). Previous generalizations were given in [H–W] and [J3], and the new variant is Lemma 3 below. Though our argument is analogous to that in [J2] and [J3], we prefer giving full details of the proof throughout, to make the paper reasonably self-contained and also to complement the somewhat sketchy presentation in [J3].

2. Smoothed exponential integrals and sums. The fact that a smoothed oscillating exponential integral or sum is small under suitable conditions, will be frequently used in the sequel. The following elementary lemma makes this argument more precise.

LEMMA 1. *Let f be a real and g a complex function in the interval (a,b). Suppose that the derivatives $f^{(j)}(x)$, $g^{(j)}(x)$ for $j \leqslant J$, where $J \geqslant 2$ is a fixed integer, are continuous in this interval and satisfy*

$$|f'(x)| \geqslant \Delta > 0, \tag{2.1}$$

$$f^{(j)}(x) \ll FX^{-j} \quad \text{for } j = 2, \ldots, J, \tag{2.2}$$

$$g^{(j)}(x) \ll GX^{-j} \quad \text{for } j = 0, \ldots, J, \tag{2.3}$$

$$g^{(j)}(a+) = g^{(j)}(b-) = 0 \quad \text{for } j = 0, \ldots, J-1. \tag{2.4}$$

Then

$$\int_a^b g(x)e(f(x))\,dx \ll (b-a)G(\Delta X)^{1-J}(1+(\Delta X)^{-1}F)^{J-1}. \tag{2.5}$$

If, in addition, we have

$$\Delta \leqslant |f'(x)| \leqslant \Delta' < 1 \tag{2.6}$$

and $J \geqslant 3$, *then*

$$\sum_a^b g(n)e(f(n)) \ll (b-a)G(\Delta_0 X)^{1-J}(1+(\Delta_0 X)^{-1}F)^{J-1}, \tag{2.7}$$

where $\Delta_0 = \min(\Delta, 1-\Delta')$.

Proof. The estimate (2.5) can be verified by repeated integration by parts (the argument is similar to that in the proof of Lemma 4 in [H2]). The integral in question equals

$$(2\pi i)^{-1}\int_a^b (g(x)/f'(x))\big(e(f(x))\big)'\,dx = -(2\pi i)^{-1}\int_a^b e(f(x))\big(g(x)/f'(x)\big)'\,dx,$$

which is

$$\ll (b-a)G(\Delta X)^{-1}(1+(\Delta X)^{-1}F)$$

by our assumptions. This settles the case $J=2$ of (2.5). Otherwise the same process is repeated $J-2$ times, and the last integral is estimated. Note that the integrated terms vanish by (2.4). At each step, the estimate of the integrand is changed by a factor $(\Delta X)^{-2}F$ if $\Delta \ll F/X$, and by a factor $(\Delta X)^{-1}$ otherwise. In any case, this factor is $\ll (\Delta X)^{-1}(1+(\Delta X)^{-1}F)$, so (2.5) follows by induction.

For a proof of (2.7), the exponential sum is written as

$$\sum_{m=-\infty}^{\infty}\int_a^b g(x)e\big(f(x)-mx\big)\,dx$$

by Poisson's summation formula, and the integrals here are estimated by (2.5).

3. A variant of Gallagher's lemma. The following lemma (Lemma 1 in [J3]) is an analogue of a well-known lemma due to P. X. Gallagher (Lemma 1.9 in [Mo]).

LEMMA 2. *Let $f(x, v)$ be a function of the real variable $x \in [M, M']$ and of the complex variable $v \in D = D(0, V; cV)$, where $1 \leqslant M < M' \leqslant 2M$, $V > 0$, and $c > 0$ is a constant. Put $V_1 = -cV$, $V_2 = (1+c)V$. Suppose that f is a holomorphic function of v in D for given x, f_{vx} exists in $[M, M'] \times D$, and*

$$f_{vx} \ll \lambda M^{-1}V^{-1} \quad \text{in } [M, M'] \times D, \tag{3.1}$$

$$|f_{vx}| \asymp \lambda M^{-1}V^{-1} \quad \text{in } [M, M'] \times (V_1, V_2), \tag{3.2}$$

where $\lambda \gg 1$. Let a_m for $m \in [M, M']$ be any complex numbers with $a_m \ll A$, and set $a_m = 0$ for $m \notin [M, M']$. Write $X = \lambda^{-1}M$. Then

$$\int_0^V \Big|\sum_M^{M'} a_m e(f(m, v))\Big|^2 dv \ll M^{\varepsilon}X^{-1}\int_{V_2}^{V_1} dv \int \Big|\sum_{\xi}^{\xi+X} a_m e(f(m, v))\Big|^2 d\xi + A^2 V. \tag{3.3}$$

4. A generalization of a lemma of Bombieri and Iwaniec. The original lemma of E. Bombieri and H. Iwaniec [B–I] was concerned with the question how often $\|\bar{h}_1/k_1 - \bar{h}_2/k_2\|$ and $|h_1k_1 - h_2k_2|$ can be simultaneously small. Here $\bar{h}_i$ is defined by $h_i\bar{h}_i \equiv 1 \pmod{k_i}$. This lemma was generalized in [H–W] and [J3]. A further generalization is given in the next lemma. We write $h \sim H$ to mean that $H \leqslant h \leqslant 2H$.

LEMMA 3. *Let $H, K \geqslant 2$, $0 < \delta \ll 1$, and let $J_1, J_2, \ldots$ be disjoint intervals of length $\ll \delta H/K$ in the interval $[H/2K, 2H/K]$ with the union $J = \bigcup_i J_i$. Let $Y > 0$, Z a subinterval of $[Y, 2Y]$, and $\omega(x, y)$ a positive function in the set $J \times Z$ which is everywhere of the same order of magnitude. Suppose that ω_x and ω_y are continuous, and that*

$$|\omega_x/\omega| \ll \delta^{-1}K/H, \tag{4.1}$$

$$|\omega_y/\omega| \gg Y^{-1}. \tag{4.2}$$

Let $\{y_r\}_{r=1}^R \subset Z$ be a Y_0-spaced set of numbers. Then for $0 < \Delta_1 \leqslant 1/2$ and $0 < \Delta_2 \leqslant 1/4$ the number of quadruples $(h_1/k_1, h_2/k_2, y_{r_1}, y_{r_2})$ such that $(h_i, k_i) = 1$, $h_i/k_i \in J$, $h_i \sim H$, $k_i \sim K$, and

$$\|\bar{h}_1/k_1 - \bar{h}_2/k_2\| \leqslant \Delta_1, \tag{4.3}$$

$$|k_1/k_2 - \omega(h_2/k_2, y_{r_2})/\omega(h_1/k_1, y_{r_1})| \leqslant \Delta_2 \tag{4.4}$$

is at most

$$\ll (H^2+K^2+\delta^{-1}HK)(1+\Delta_2 Y_0^{-1}Y)R + \Delta_1(\Delta_1+\Delta_2)(HKR)^2. \tag{4.5}$$

Proof. We follow the argument of the proof of Lemma 2.4 in [H–W] and that of Lemma 2 in [J3] with appropriate modifications. For simplicity, we may assume that $\omega \asymp 1$.

Choosing the class $\bar{h}_i \pmod{k_i}$ suitably, we may assume that (4.3) holds in the stronger form

$$|\bar{h}_1/k_1 - \bar{h}_2/k_2| \leqslant \Delta_1 \tag{4.6}$$

for a given pair of fractions h_1/k_1, h_2/k_2 occurring in the lemma. Now, if $\bar{k}_i$ is defined by $h_i\bar{h}_i + k_i\bar{k}_i = 1$, then

$$\begin{pmatrix} h_2 \\ k_2 \end{pmatrix} = \begin{pmatrix} h_2 & -\bar{k}_2 \\ k_2 & \bar{h}_2 \end{pmatrix}\begin{pmatrix} \bar{h}_1 & \bar{k}_1 \\ -k_1 & h_1 \end{pmatrix}\begin{pmatrix} h_1 \\ k_1 \end{pmatrix} = \begin{pmatrix} h_2\bar{h}_1 + k_1\bar{k}_2 & h_2\bar{k}_1 - h_1\bar{k}_2 \\ k_2\bar{h}_1 - k_1\bar{h}_2 & k_2\bar{k}_1 + h_1\bar{h}_2 \end{pmatrix}\begin{pmatrix} h_1 \\ k_1 \end{pmatrix}.$$

Hence to each pair h_1/k_1, h_2/k_2 satisfying (4.3) we may assign a unimodular matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

(4.7) $$\begin{pmatrix} h_2 \\ k_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} h_1 \\ k_1 \end{pmatrix},$$

where

$$|c| \leqslant \Delta_1 k_1 k_2.$$

By a little calculation, it can be verified that

(4.8) $$a = ch_2/k_2 + k_1/k_2, \quad d = -ch_1/k_1 + k_2/k_1.$$

The pairs $(h_1/k_1, h_2/k_2)$ are now classified according to the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This induces a classification for the quadruples $(h_1/k_1, h_2/k_2, y_{r_1}, y_{r_2})$ as well.

The matrices with $b = 0$ or $c = 0$ can be dealt with as in [J3], for the condition (4.1), which is the principal novelty of our lemma, is not used at this point. The number of such quadruples is

$$\ll (H^2 + K^2)(1 + \Delta_2 Y_0^{-1} Y) R.$$

Consider now matrices with $bc \neq 0$. We may suppose that $c > 0$, on interchanging the roles of h_1/k_1 and h_2/k_2 if necessary. In [H–W], an iterative argument is given which shows, under the assumption (4.1), that if y_{r_1} and y_{r_2} are given, then h_1/k_1 must lie in an interval of length $\ll \Delta_2/c$ if

(4.9) $$c \gg \delta^{-1} K/H,$$

where the implied constant is supposed to be sufficiently large. Actually, in [H–W], there is a constant c_0 in place of δ^{-1} in (4.1) and (4.9), but it is irrelevant whether c_0 is understood as a constant or as a parameter. As in [H–W], the number of admissible pairs $(h_1/k_1, h_2/k_2)$ (for given y_{r_1} and y_{r_2}) such that the corresponding matrix satisfies (4.9), is at most

$$\ll \Delta_1(\Delta_1 + \Delta_2)(HK)^2$$

if $H \geqslant K$. The number of the pairs (y_{r_1}, y_{r_2}) being R^2, we end up with the last term in (4.5). Also, it is shown in [H–W] how the cases $H \geqslant K$ and $H \leqslant K$ can be "reflected" to each other (the rôles of H and K can be interchanged). Hence we may restrict ourselves to the case $H \geqslant K$.

It remains to estimate the number of admissible quadruples for which the corresponding matrix satisfies

(4.10) $$1 \leqslant c \ll \delta^{-1} K/H.$$

Let c and h_2/k_2 be fixed for a moment. Since $k_1 = -ch_2 + ak_2$, the condition (4.4) can be written as

$$|a - ch_2/k_2 - \omega(h_2/k_2, y_{r_2})/\omega(h_1/k_1, y_{r_1})| \leqslant \Delta_2.$$

Hence there are only finitely many possibilities for a. The congruence $ad \equiv 1 \pmod c$ determines $d \pmod c$ if a is given. Since d lies in an interval of length $\ll cH/K$, by (4.8), there are $\ll H/K$ possibilities for d. Accordingly, for given c and h_2/k_2, there are $\ll H/K$ possible matrices, and each matrix then determines h_1/k_1. Further, for each choice of y_{r_2}, h_1/k_1, and h_2/k_2, there are $\ll 1 + \Delta_2 Y_0^{-1} Y$ possibilities for y_{r_1}. Because the triplet $(c, h_2/k_2, y_{r_2})$ can be chosen in $\ll \delta^{-1} K^2 R$ ways, we end up with at most

$$\ll \delta^{-1} HKR(1 + \Delta_2 Y_0^{-1} Y)$$

quadruples. This completes the proof of (4.5).

5. Proof of Theorem 1

5.1. As we pointed out in § 1.3, the assertion of Theorem 1 for $M \ll F$ follows immediately from Theorem 1 of [J3]. By the following simple device, the remaining case $F \ll M \ll \delta^{-1} F$ can be reduced to $M \asymp \delta^{-1} F$.

Suppose that the theorem holds for $M \asymp \delta^{-1} F$, and consider the case $F \ll M \ll \delta^{-1} F$. We may write $M \asymp \delta_0^{-1} F$, where $\delta \ll \delta_0 \ll 1$. The numbers α_i can be subdivided into $\asymp \delta_0 \delta^{-1}$ classes, each of which is δ_0-spaced (mod 1). Then, by our assumption, the theorem is applicable to each of these classes, and summation over all classes yields the desired result.

5.2. Before going into the proof, we make a couple of simplifications. Firstly, we may suppose that g is a constant, for the general case then follows by partial summation. To emphasize the analogy between our sum S and Dirichlet polynomials, we choose $g = G = M^{-1/2}$.

Secondly, as was pointed out in [J3], we may change the value of the parameter T, occurring in (1.6) and (1.8), on replacing v by a new variable Av. Let us suppose that $T = F$.

We introduce the simplifying notation $A \lll B$ to mean that $|A| \leqslant c(\varepsilon) T^{\varepsilon} B$ for arbitrarily small positive numbers ε. Then the assertion (1.13) to be proved can be written as

(5.1) $$I \lll \delta^{-1}(RV + R^{1/2} T V^{-1/2})$$

for $M \asymp \delta^{-1} T$ and $T^{1/2+\varepsilon} \ll V \ll T^{2/3}$.

5.3. As the first step in the proof of (5.1), we apply Lemma 2 to the integral in (1.4) and simplify the resulting inequality. By our assumptions on the function f, the parameters λ and X in Lemma 2 are now $\lambda \asymp V$ and $X \asymp MV^{-1}$. For a certain number $v_0 \in (V_1, V_2)$, we have

(5.2) $$I \lll M^{-2} V^2 \sum_i \sum_r \int \Big| \sum_{\xi}^{\xi+X} d(m) e(f(m, v_0, y_r) + m\alpha_i) \Big|^2 d\xi + \delta^{-1} M^{-1} RV,$$

where the sum over m is understood in the sense that only those numbers m lying in the interval $[M, M']$ are taken into account.

Next we want to ignore the "incomplete" sums on the right of (5.2). This simplification is not as trivial as it was in [J2] and [J3], for X may be relatively large if δ is small. Therefore we appeal to the large sieve inequality for exponential sums ([Mo], Corollary 2.2), which implies that

$$\sum_i \Big|\sum_{\xi}^{\xi+X}\Big|^2 \lll (\delta^{-1}+X)X \ll X^2 \ll M^2 V^{-2}.$$

Hence the contribution of the incomplete sums is $\lll RX \ll \delta^{-1}RTV^{-1} \ll \delta^{-1}RV$, which can be omitted. Accordingly, the integral over ξ in (5.2) can be restricted to the interval $[M, M'']$, where $M'' = M'-X$.

As a further simplification, we replace the integral over ξ by the corresponding sum over integers, with an error $\lll \delta^{-1}RV$. Then (5.2) implies that

$$I \lll M^{-2}V^2 \sum_i \sum_r \sum_{M \leqslant m \leqslant M''} \Big|\sum_{0 \leqslant n \leqslant X} d(m+n)e\big(f(m+n, v_0, y_r)+n\alpha_i\big)\Big|^2 + \delta^{-1}RV. \tag{5.3}$$

5.4. Next we linearize the function in $e(\dots)$ and recast the right hand side of (5.3) in terms of integrals involving exponential sums. This step is essentially a repetition of Section 3 in [J2] in a more general context.

We want to have an approximation to $f(m+n, v_0, y_r)$, which is linear in n and independent of m when m lies in an interval of length $\mu \asymp \delta^{-1}V$. We subdivide the interval $[M, M'']$ into segments $[M_{q-1}, M_q]$ with $M_q = M+q\mu$ for $1 \leqslant q \leqslant Q \asymp TV^{-1}$. Then, putting

$$\alpha_{q,r,i} = f_x(M_q, v_0, y_r)+\alpha_i, \tag{5.4}$$

we write

$$e\big(f(m+n, v_0, y_r)+n\alpha_i\big) = e(n\alpha_{q,r,i})\, e\big(f(m+n, v_0, y_r)-n\alpha_{q,r,i}+n\alpha_i\big)$$

for $M_{q-1} \leqslant m \leqslant M_q$. Recalling the assumption (1.7) on f_{xx}, it is easily seen that the derivative of the latter factor on the right with respect to n is $\lll X^{-1}$. Therefore this factor can be eliminated by partial summation, and redefining the symbol X suitably (the new X being possibly smaller than the old one), we may write (5.3) as

$$I \lll M^{-2}V^2 \sum_i \sum_r \sum_{q=1}^{Q} \sum_{M_{q-1} \leqslant m \leqslant M_q} \Big|\sum_{0 \leqslant n \leqslant X} d(m+n)e\big((m+n)\alpha_{q,r,i}\big)\Big|^2 + \delta^{-1}RV, \tag{5.5}$$

where $X \ll MV^{-1}$.

For each q, we define somehow a function $v_q(x)$ which is J times continuously differentiable (with J sufficiently large) and satisfies

$$0 \leqslant v_q(x) \leqslant 1 \quad \text{for all } x,$$
$$v_q(x) = 1 \quad \text{for } M_{q-1} \leqslant x \leqslant M_q+X,$$
$$v_q(x) = 0 \quad \text{for } x \notin [M_{q-1}-\mu, M_q+X+\mu],$$
$$v^{(j)}(x) \ll \mu^{-j} \quad \text{for } j = 0, 1, \dots, J.$$

Define further

$$S_q(\alpha) = \sum_m v_q(m)\, d(m)\, e(m\alpha), \qquad U(\alpha) = \sum_0^X e(-n\alpha). \tag{5.6}$$

Then the sum over m in (5.5) is at most

$$\int_0^1 |S_q(\alpha+\alpha_{q,r,i})|^2\, |U(\alpha)|^2\, d\alpha = \int_0^1 |S_q(\alpha)|^2\, |U(\alpha-\alpha_{q,r,i})|^2\, d\alpha;$$

to verify this, write the product $S_q(\alpha+\alpha_{q,r,i})\,U(\alpha)$ as an exponential sum and calculate its mean square over the unit interval.

The range of integration here is split up into Farey intervals ("arcs") of the order K with

$$K = (\delta^{-1}V)^{1/2}T^{-\varepsilon}. \tag{5.7}$$

The Farey interval $A(a/k)$ corresponding to a fraction a/k is of length $\asymp (kK)^{-1}$. We shall restrict ourselves to fractions with $k \sim K_0$ (i.e., $K_0 \leqslant k \leqslant 2K_0$). Then, since obviously

$$U(\alpha) \ll \min(X, \|\alpha\|^{-1}) \ll (M^{-1}V+\|\alpha\|)^{-1},$$

the assertion (5.1) follows if we show that

$$M^{-2}V^2 \sum_i \sum_r \sum_q \sum_{k \sim K_0} \sideset{}{^*}\sum_{a \bmod k} \int_{A(a/k)} |S_q(\alpha)|^2 (M^{-1}V+\|\alpha-\alpha_{q,r,i}\|)^{-2}\, d\alpha \lll \delta^{-1}(RV+R^{1/2}TV^{-1/2}). \tag{5.8}$$

The critical values of the integrand are those with α near to $\alpha_{q,r,i}$. The numbers $\alpha_{q,r,i}$ carry information on the function f and on the systems $\{y_r\}$ and $\{\alpha_i\}$.

5.5. Because the sums $S_q(\alpha)$ in (5.8) involve values of $d(n)$ in different ranges, they are not immediately comparable. Therefore we transform them by the following summation formula (Theorem 1.7 in [J1]) into sums over essentially the same range.

LEMMA 4. *Let $0 < A < B$ and let the function f be continuously differentiable in the interval $[A, B]$. Let $k \geqslant 1$ and $(a, k) = 1$. Then*

$$\sideset{}{'}\sum_{A \leqslant n \leqslant B} d(n)\, e(na/k) f(n) = k^{-1}\int_A^B (\log x+2\gamma-2\log k) f(x)\, dx \tag{5.9}$$

$$+k^{-1}\sum_{n=1}^{\infty} d(n)\int_A^B f(x)\{-2\pi e(-n\bar{a}/k)\,Y_0(4\pi\sqrt{nx}/k)$$

$$+4e(n\bar{a}/k)\,K_0(4\pi\sqrt{nx}/k)\}\,dx,$$

where γ is Euler's constant, Y_0 and K_0 are Bessel functions (in the usual notation), and $\sum'$ means that if A or B is an integer, then the term corresponding to $n = A$ or $n = B$ is to be halved.

We apply this to $S_q(\alpha)$ for $\alpha \in A(a/k)$ with $k \sim K_0$. Thus $\alpha = a/k+\beta$, where $\beta \ll (kK_0)^{-1}$, and $v_q(x)\,e(\beta x)$ stands for the function f in the lemma. Since the function $K_0(x)$ is exponentially decreasing for $x \geqslant 1$, it is clear that the terms involving $K_0(\ldots)$ in (5.9) are negligible.

Consider next the leading term

$$k^{-1}\int v_q(x)(\log x+2\gamma-2\log k)\,e(\beta x)\,dx. \tag{5.10}$$

By Lemma 1, this is small if $|\beta| \gg \mu^{-1}T^{\varepsilon} \asymp \delta V^{-1}T^{\varepsilon}$. In any case, we have the trivial estimate $\ll K_0^{-1}\mu\log T \ll \delta^{-1}K_0^{-1}V\log T$. So, to estimate the contribution of the terms (5.10) to the left hand side of (5.8), we may replace $S_q(\alpha)$ by $K_0^{-1}\mu\log T$ and restrict the integration to the interval $|\alpha-a/k| \leqslant \mu^{-1}T^{\varepsilon}$. The result is

$$\ll \delta^{-1}K_0^{-2}M^{-2}V^3\sum_i\sum_r\sum_q\sum_k\sum_a(M^{-1}V+\|a/k-\alpha_{q,r,i}\|)^{-2}. \tag{5.11}$$

It should be noted that since $\mu^{-1}\asymp\delta V^{-1} \ll \delta T^{-1}V \asymp M^{-1}V$, it was possible to replace α by a/k in $M^{-1}V+\|\alpha-\alpha_{q,r,i}\|$. The most significant terms in (5.11) are those with

$$\|a/k-\alpha_{q,r,i}\| \leqslant M^{-1}V; \tag{5.12}$$

the contribution of the others can be estimated similarly.

How often (5.12) can be fulfilled? Since the numbers α_i are δ-spaced and $f_x \ll \delta$ by (1.5), there are only finitely many possible indices i for a given fraction a/k. Further, since

$$\alpha_{q+1,r,i}-\alpha_{q,r,i}\asymp M^{-1}V, \tag{5.13}$$

by (1.7) and our choice of μ, there are finitely many possible values of q for given a/k, r, and i. Hence (5.12) can hold at most $\ll K_0^2R$ times, and the contribution of these terms to (5.11) is $\ll \delta^{-1}RV$.

The terms involving Y_0 in (5.9) constitute the nontrivial part of the transformation formula for $S_q(\alpha)$. By the asymptotic expansion

$$Y_0(x) \sim e^{ix}\sum_{r\geqslant 1} a_r x^{1/2-r}+e^{-ix}\sum_{r\geqslant 1} b_r x^{1/2-r}, \tag{5.14}$$

these terms are essentially exponential integrals. If sufficiently many terms here are taken into account, the error will be negligible. The most significant terms in (5.14) are those with $r = 1$, so let us consider the contribution of one of these, say $a_1 x^{-1/2}e^{ix}$.

The corresponding part of the transformation formula for $S_q(\alpha)$ equals, up to a constant factor, the following series:

$$k^{-1/2}\sum_{n=1}^{\infty} d(n)\,n^{-1/4}e(-n\bar{a}/k)\int v_q(x)\,x^{-1/4}e(\beta x+2\sqrt{nx}/k)\,dx. \tag{5.15}$$

Next we show, by Lemma 1, that the tail of this series can be omitted. Indeed, if $n > N$, where N is a suitable number satisfying

$$N\asymp K^{-2}M\asymp T^{1+2\varepsilon}V^{-1} \tag{5.16}$$

with ε as in (5.7), then

$$\left|\frac{d}{dx}(\beta x+2\sqrt{nx}/k)\right| \gg n^{1/2}M^{-1/2}K_0^{-1}$$

for $n > N$ (recall that $\beta \ll (KK_0)^{-1}$). Lemma 1 is now applied with $\Delta\asymp n^{1/2}M^{-1/2}K_0^{-1}$ and $X\asymp\delta^{-1}V$. Then

$$\Delta X \gg (n/N)^{1/2}(KK_0)^{-1}\delta^{-1}V \gg (n/N)^{1/2}T^{2\varepsilon},$$

so that taking J sufficiently large, we may omit the terms with $n > N$ in (5.15).

The partial sum over $n \leqslant N$ of the series (5.15) is subdivided into subsums, in which the condition of summation is $n \sim N_0$ for a certain number $N_0 < N$. It suffices to consider one of these sums, say $\tilde{S}_q(\alpha)$. This is substituted into (5.8) in place of $S_q(\alpha)$. However, for technical reasons, we replace the integral over the Farey interval $A(a/k)$ by a weighted integral over a wider interval $[a/k-\eta,\ a/k+\eta]$. We suppose that $A(a/k) \subseteq [a/k-\eta/2,\ a/k+\eta/2]$, and let $w(x)$ be a smooth weight function (similar to $v_q(x)$ above) with support $[-\eta,\ \eta]$. The integrand in (5.8) is now provided with the weight $w(\alpha-a/k) = w(\beta)$, and the integration is taken over the interval $|\beta| \leqslant \eta$.

The integrand in (5.8) depends heavily on the size of $\|\alpha-\alpha_{q,r,i}\|$. We split up the range for α so as to control this quantity. Let a/k, q, r, and i be given. Since $\eta\asymp(KK_0)^{-1}$, there exists a nonnegative integer ϱ such that

$$(2^{\varrho}-1)\,M^{-1}V \ll \|\alpha-\alpha_{q,r,i}\| \ll 2^{\varrho}M^{-1}V+(KK_0)^{-1} \tag{5.17}$$

for all $\alpha\in[a/k-\eta,\ a/k+\eta]$. Then

$$\int_{A(a/k)} |\tilde{S}_q(\alpha)|^2(M^{-1}V+\|\alpha-\alpha_{q,r,i}\|)^{-2}\,d\alpha \ll 2^{-2\varrho}M^2V^{-2}\int w(\beta)\,|\tilde{S}_q(\alpha)|^2\,d\alpha,$$

where $\beta = \alpha-a/k$. From now on, we restrict ourselves to quadruples $\tau = (a/k,\ q,\ r,\ i)$ such that (5.17) holds for a certain fixed value of ϱ.

Returning to (5.8), we now see that its proof is reduced to showing that

$$(5.18)\qquad K_0^{-1}2^{-2\varrho}\sum_{\tau}\iint v_q(x)v_q(y)(xy)^{-1/4}\sum_{m,n\sim N_0} d(m)d(n)(mn)^{-1/4}$$
$$\times e((n-m)\bar{a}/k)\,e(2k^{-1}(\sqrt{mx}-\sqrt{ny}))\,\lambda(x-y)\,dxdy$$
$$\lll \delta^{-1}(RV+R^{1/2}TV^{-1/2}),$$

where

$$\lambda(\xi)=\int w(\beta)e(\xi\beta)\,d\beta.$$

If $|\xi|\geqslant KK_0T^{\varepsilon}$, then $\lambda(\xi)$ is small, by Lemma 1. Hence the domain of integration in the double integral in (5.18) can be restricted by the condition $|x-y|\leqslant KK_0T^{\varepsilon}$. In this case, we estimate trivially $\lambda(x-y)\ll(KK_0)^{-1}$.

Accordingly, we set $y=x+\xi$ with $|\xi|\leqslant KK_0T^{\varepsilon}$, and consider the integral with respect to x. Again, we may apply Lemma 1 to eliminate a number of unimportant terms. More exactly, those pairs (m, n) for which

$$(5.19)\qquad K_0^{-1}|\sqrt{m/x}-\sqrt{n/(x+\xi)}|\gg\mu^{-1}T^{\varepsilon}$$

can be omitted. Observe that the validity of this condition is independent of ξ, for the variation of the left hand side as a function of ξ is $\lll KM^{-3/2}N^{1/2}\ll M^{-1}$ by (5.16). Now (5.19) with $\xi=0$ yields the condition $|m-n|\leqslant P$, where

$$(5.20)\qquad \delta K_0M^{1/2}N_0^{1/2}V^{-1}\ll P\lll\delta K_0M^{1/2}N_0^{1/2}V^{-1}.$$

We dispose of the diagonal terms with $m=n$ in (5.18). For given values of a/k and r, the number of pairs (q, i) satisfying (5.17) is $\ll 2^{\varrho}+M(KK_0V)^{-1}$ by (5.13) and the spacing condition for the α_i. Hence a little calculation shows that the contribution of the diagonal terms is

$$\lll R\delta^{-1}V(1+TV^{-2})\ll R\delta^{-1}V.$$

In the rest of the proof, we impose an additional condition on our quadruples τ satisfying (5.17). Since (5.17) holds, in particular, for $\alpha=a/k$, there are integers h and ν such that $(h, k)=1$, $h/k\equiv a/k \pmod 1$, and

$$(5.21)\qquad \alpha_{q,r,i}=h/k+(\nu+O(1))\,2^{\varrho}M^{-1}V.$$

We may suppose that $h>0$ and $h\asymp K_0$ on specifying the numbers α_i suitably (mod 1). Denote the number of possible values of ν by ν_0. Then

$$(5.22)\qquad \nu_0\ll 1+2^{-\varrho}(KK_0)^{-1}MV^{-1}.$$

In the sequel, we restrict ourselves to quadruples $\tau=(a/k, q, r, i)$ such that besides (5.17) also (5.21) is satisfied for given values of ϱ and ν.

To reformulate the assertion (5.18), we write $x=M_q+u$ and $y=x+\xi$. Then it suffices to prove that for given u and ξ with $u\ll\mu$ and $\xi\lll KK_0$ we have

$$(5.23)\qquad K_0^{-1}2^{-2\varrho}\nu_0V\sum_{\tau}\sum_{\substack{m,n\sim N_0\\0<|m-n|\leqslant P}} v_q(x)v_q(y)d(m)d(n)(xymn)^{-1/4}$$
$$\times e((n-m)\bar{a}/k+2(\sqrt{mx}-\sqrt{ny})/k)$$
$$\lll RV+R^{1/2}TV^{-1/2}.$$

5.6. To estimate the expression on the left of (5.23), we appeal to the following inequality of E. Bombieri and H. Iwaniec ([B–I], Lemma 2.4). In the rest of the proof, we follow closely the argument in [J2] and [J3].

LEMMA 5. *Let $\mathscr{X}$ and $\mathscr{Y}$ be two sets of points $\bar{x}\in\mathbf{R}^K$ and $\bar{y}\in\mathbf{R}^K$ with components x_k and y_k, respectively, and let $a(\bar{x})$ for $\bar{x}\in\mathscr{X}$ and $b(\bar{y})$ for $\bar{y}\in\mathscr{Y}$ be arbitrary complex numbers. Let $X_1, \ldots, X_K$ and $Y_1, \ldots, Y_K$ be positive numbers. Define*

$$A=\sum_{\substack{\bar{x},\bar{x}'\in\mathscr{X}\\|x_k-x_k'|\leqslant(2Y_k)^{-1}}}|a(\bar{x})a(\bar{x}')|,\qquad B=\sum_{\substack{\bar{y},\bar{y}'\in\mathscr{Y}\\|y_k-y_k'|\leqslant(2X_k)^{-1}}}|b(\bar{y})b(\bar{y}')|,$$
$$C=\sum_{\substack{\bar{x}\in\mathscr{X}\\|x_k|\leqslant X_k}}\sum_{\substack{\bar{y}\in\mathscr{Y}\\|y_k|\leqslant Y_k}} a(\bar{x})b(\bar{y})e(\bar{x}\cdot\bar{y}).$$

Then

$$|C|^2\leqslant(2\pi^2)^K AB\prod_{k=1}^{K}(1+X_kY_k).$$

Now we interpret the expression $e(\ldots)$ in (5.23) as $e(\bar{x}\cdot\bar{y})$, where

$$\bar{x}=(n-m,\ \sqrt{m}-\sqrt{n},\ \sqrt{n}),\qquad \bar{y}=(\bar{a}/k,\ 2\sqrt{x}/k,\ 2(\sqrt{x}-\sqrt{y})/k).$$

Here x and y depend on q as indicated above. Further, with

$$a(\bar{x})=d(m)d(n)(mn)^{-1/4},\qquad b(\bar{y})=v_q(x)v_q(y)(xy)^{-1/4},$$

the sum over τ, m, and n in (5.23) equals the sum C in Lemma 5. We may choose

$$(X_1, X_2, X_3)=(P,\ O(P/\sqrt{N_0}),\ O(\sqrt{N_0})),$$
$$(Y_1, Y_2, Y_3)=(1,\ O(\sqrt{M}/K_0),\ O(KT^{\varepsilon}/\sqrt{M})).$$

Then, noting that $X_3Y_3\lll 1$, we have

$$(5.24)\qquad \prod_{k=1}^{3}(1+X_kY_k)\lll K_0^{-1}M^{1/2}N_0^{-1/2}P^2.$$

Next we have to estimate the quantities A and B in Lemma 5. It is easily seen, as in [J2] and [J3], that

$$(5.25)\qquad A\lll P.$$

The estimation of B is a deeper problem and requires the arithmetic information embodied in Lemma 3.

In the definition of B, there are three conditions of summation, two first of which read

$$|\bar{a}_1/k_1 - \bar{a}_2/k_2| \ll P^{-1}, \tag{5.26}$$

$$|\sqrt{x_1}/k_1 - \sqrt{x_2}/k_2| \ll \sqrt{N_0}/P; \tag{5.27}$$

here x_1 and x_2 should be understood as two values of the variable x, not as components of the vector $\bar{x}$. The third condition will be ignored.

In (5.26), we may replace $\bar{a}_i/k_i$ by $\bar{h}_i/k_i$ (mod 1). Henceforth we are going to work with h_i/k_i instead of a_i/k_i, and the condition (5.26) is accordingly replaced by

$$\|\bar{h}_1/k_1 - \bar{h}_2/k_2\| \ll P^{-1}. \tag{5.28}$$

Next we estimate how many pairs of quadruples $(h_j/k_j, q_j, r_j, i_j)$ $(j = 1, 2)$ satisfy (5.21), (5.27), and (5.28). Recall that $x_j = M_{q_j} + u$, so that (5.27) involves implicitly the indices q_j. Let W be an upper bound for the number of such pairs of quadruplets. Then $B \ll M^{-1} W$, and combining this with (5.24) and (5.25) we may, by Lemma 5, reduce the proof of (5.23) to showing that

$$K_0^{-3/2} 2^{-2\varrho} M^{-1/4} N_0^{-1/4} v_0 P^{3/2} V W^{1/2} \lll RV + R^{1/2} T V^{-1/2}. \tag{5.29}$$

To analyze the condition (5.27), we recall the equations (5.4) and (5.21), which give together

$$f_x(M_{q_j}, v_0, y_{r_j}) + \alpha_{i_j} = h_j/k_j + v 2^{\varrho} M^{-1} V + O(2^{\varrho} M^{-1} V) \tag{5.30}$$

for $j = 1, 2$. If we ignore the error term, this appears to be an implicit equation for M_{q_j}. Therefore, with an approximation to M_{q_j} in mind, we define the function $\Omega(\beta, y)$ implicitly by

$$f_x(\Omega(\beta, y), v_0, y) = \beta + v 2^{\varrho} M^{-1} V. \tag{5.31}$$

If x runs over the interval $[M, M']$, then $f_x(x, v_0, y)$ runs over an interval of length $\asymp \delta$. This interval depends on y but not too strongly if we limit y temporarily to a sufficiently short subinterval $[Y_1, Y_2]$ of $[Y, 2Y]$ with $Y_2 - Y_1 \asymp Y$. Let $[\beta_1, \beta_2]$ be the set of the values of $f_x(x, v_0, y) - v 2^{\varrho} M^{-1} V$ for all $x \in [M, M']$, $y \in [Y_1, Y_2]$, and let $[\beta_1', \beta_2']$ be a concentric interval which is longer by a small proportion. Then $\Omega(\beta, y) \in \big((1-c_1) M, M' + c_1 M\big)$ (with c_1 as in § 1.2) is defined for all $\beta \in [\beta_1', \beta_2']$ and $y \in [Y_1, Y_2]$. Without essential loss of generality, we may suppose that $y_r \in [Y_1, Y_2]$ for all r.

By the implicit function theorem and our assumptions on the function f, we have

$$|\Omega_\beta / \Omega| \asymp \delta^{-1}, \tag{5.32}$$

$$|\Omega_y / \Omega| \asymp Y^{-1}. \tag{5.33}$$

The equation (5.30) gives information on the dependence of M_{q_j} on h_j/k_j, y_{r_j} and α_{i_j} only if the error term is at most $O(\delta)$. Therefore we assume now that

$$2^{\varrho} M^{-1} V \ll \delta, \tag{5.34}$$

where the implied constant is sufficiently small, and consider the opposite case later. Then $h_j/k_j - \alpha_{i_j} \in [\beta_1', \beta_2']$ for $j = 1, 2$.

By (5.30)–(5.32) and (5.34), we have

$$M_{q_j} = \Omega(h_j/k_j - \alpha_{i_j}, y_{r_j}) + O(2^{\varrho} \delta^{-1} V). \tag{5.35}$$

Since $x_j - M_{q_j} \ll \delta^{-1} V$, the same expression is valid for x_j as well. Hence, substituting x_j in this form into (5.27), we obtain the inequality

$$|k_1/k_2 - \Omega(h_1/k_1 - \alpha_{i_1}, y_{r_1})^{1/2} / \Omega(h_2/k_2 - \alpha_{i_2}, y_{r_2})^{1/2}| \ll \Delta_2, \tag{5.36}$$

where

$$\Delta_2 = K_0 M^{-1/2} N_0^{1/2} P^{-1} + 2^{\varrho} T^{-1} V \ll 2^{\varrho} T^{-1} V. \tag{5.37}$$

Since the numbers $h_j/k_j - \alpha_{i_j}$ in (5.36) lie in the interval $[\beta_1', \beta_2']$ of length $\asymp \delta$, the rationals h_j/k_j determine the numbers α_{i_j} uniquely, at least if we suppose, as we may, that the system $\{\alpha_i\}$ is $c\delta$-spaced for a sufficiently large constant c. If we fix α_i, then $\Omega(h/k - \alpha_i, y)$ is defined for $y \in [Y_1, Y_2]$ and for those rationals h/k lying in the interval $[\alpha_i + \beta_1', \alpha_i + \beta_2']$. This observation, (5.36), and Lemma 3 motivate the following definition:

$$\omega(x, y) = \Omega(x - \alpha_i, y)^{-1/2} \quad \text{for } x \in [\alpha_i + \beta_1', \alpha_i + \beta_2'],\ y \in [Y_1, Y_2]. \tag{5.38}$$

Then (5.36) amounts essentially to the condition (4.4) in Lemma 3. Moreover, the assumptions (4.1) and (4.2) of the lemma are satisfied by (5.32) and (5.33) (now $H \asymp K$). Further, the condition (5.28) coincides with (4.3), where

$$\Delta_1 \asymp P^{-1}. \tag{5.39}$$

Lemma 3 now gives for the number X of the pairs of admissible triplets $(h_j/k_j, r_j, i_j)$ $(j = 1, 2)$ the estimate

$$X \ll \delta^{-1} K_0^2 (1 + \Delta_2 Y_0^{-1} Y) R + \Delta_1 (\Delta_1 + \Delta_2) K_0^4 R^2. \tag{5.40}$$

Further, for each triplet $(h_j/k_j, r_j, i_j)$, the number of possible values of q_j is $\ll 2^{\varrho}$, by (5.35). Hence $W \ll 2^{2\varrho} X$, and (5.29) follows if we prove that

$$K_0^{-3/2} 2^{-\varrho} M^{-1/4} N_0^{-1/4} v_0 P^{3/2} V X^{1/2} \lll RV + R^{1/2} T V^{-1/2}. \tag{5.41}$$

Next we estimate X using (5.37), (5.39), and (5.40). Note that $P \lll T V^{-1}$ by (5.20) and (5.16), whence $\Delta_2 \lll 2^{\varrho} \Delta_1$. Also, $\Delta_2 Y_0^{-1} Y \ll 2^{\varrho}$ by (1.10) and (5.37). Hence

$$X \lll 2^{\varrho} (\delta^{-1} K_0^2 R + K_0^4 P^{-2} R^2).$$

The quantity v_0 was estimated in (5.22). Then, using the definitions (5.7), (5.16), and (5.20) of K, N, and P, we find that the left hand side of (5.41) is

$$\ll K_0^{-3/2} 2^{-\varrho/2} M^{-1/4} N_0^{-1/4} P^{3/2} V(\delta^{-1/2} K_0 R^{1/2} + K_0^2 P^{-1} R) \times (1 + 2^{-\varrho}(KK_0)^{-1} MV^{-1})$$
$$\ll (K_0 K^{-1} R^{1/2} TV^{-1/2} + \delta^{1/2} K_0 RV^{1/2})(1 + (KK_0)^{-1} MV^{-1})$$
$$\ll (K^{-1} R^{1/2} TV^{-1/2} + \delta^{1/2} RV^{1/2})(K + K^{-1} MV^{-1})$$
$$\ll (R^{1/2} TV^{-1/2} + RV)(1 + TV^{-2})$$
$$\ll R^{1/2} TV^{-1/2} + RV,$$

as desired.

5.7. To complete the proof of Theorem 1, we still have to verify (5.29) when (5.34) fails to hold, i.e. if

$$2^{\varrho} M^{-1} V \gg \delta. \tag{5.42}$$

Recall that W denotes the number of pairs of quadruples $(h_j/k_j, q_j, r_j, i_j)$ $(j = 1, 2)$ satisfying (5.21), (5.27), and (5.28). We ignore the condition (5.27), and letting q_j and r_j run over all possible values, we estimate the number of possible pairs $(h_j/k_j, i_j)$ $(j = 1, 2)$.

If h_1/k_1 is given, then there are $\ll 1 + K_0^2 P^{-1}$ possibilities for h_2/k_2, by (5.28). Further, for given h_j/k_j, q_j, and r_j, there are $\ll \delta^{-1} 2^{\varrho} M^{-1} V$ possibilities for i_j, since the α_i are δ-spaced. The number of all pairs (q_j, r_j) $(j = 1, 2)$ being $\ll (TV^{-1})^2 R^2$, we have

$$W \ll 2^{2\varrho} K_0^2 (1 + K_0^2 P^{-1}) R^2.$$

Further, by (5.22) and (5.42), we have

$$v_0 \ll 1 + \delta^{-1}(KK_0)^{-1}.$$

Hence the left hand side of (5.29) is

$$\ll K_0^{-1/2} 2^{-\varrho} M^{-1/4} N_0^{-1/4} P^{3/2} RV(1 + \delta^{-1}(KK_0)^{-1})(1 + K_0 P^{-1/2})$$
$$\ll K_0^{-1/2} M^{-1/4} N_0^{-1/4} P^{3/2} RT^{-1} V^2 (1 + \delta^{-1}(KK_0)^{-1})(1 + K_0 P^{-1/2})$$
$$\ll \delta^{3/2} K_0 M^{1/2} N_0^{1/2} RT^{-1} V^{1/2} + \delta^{1/2} K^{-1} M^{1/2} N_0^{1/2} RT^{-1} V^{1/2}$$
$$+ \delta K_0^{3/2} M^{1/4} N_0^{1/4} RT^{-1} V + K^{-1} K_0^{1/2} M^{1/4} N_0^{1/4} RT^{-1} V$$
$$\ll RT^{-1/2} V^{3/2} \ll RV,$$

so that (5.29) holds even in the case (5.42).

6. Applications to L-functions. We comment briefly on the proofs of the estimates (1.25) and (1.27).

Let χ be a primitive character (mod D). The fact that $|L(1/2 + iT, \chi)|^2$, for

$D \ll T^B$, can be expressed in terms of a Dirichlet polynomial $S_\chi(M, M'; t)$ with $|t - T| \ll \log^2 T$ and $M, M' \asymp DT$, follows from the estimate

$$|L(1/2 + iT, \chi)|^2 \ll \log T \Big(1 + \int_{-\log^2 T}^{\log^2 T} |S_\chi(M, M'; T + v)|^2 \, dv\Big), \tag{6.1}$$

where $M = a(DT/2\pi)$, $M' = b(DT/2\pi)$, and $0 < a < 1 < b$. This, in turn, is a consequence of the estimate

$$|L(1/2 + iT, \chi)|^2 \ll \log T \Big(1 + \int_{-\log^2 T}^{\log^2 T} e^{-|v|} |L(1/2 + i(T + v), \chi)|^2 \, dv\Big). \tag{6.2}$$

In the case of the zeta-function, this is due to D. R. Heath-Brown (for a proof, see [Iv], Lemma 7.1), and the generalization to L-functions is straightforward. By the functional equation $L(s, \chi) = \Phi(s, \chi) L(1 - s, \bar{\chi})$, we have

$$|L(1/2 + i(T + v), \chi)|^2 = L^2(1/2 + i(T + v), \chi)\, \Phi(1/2 + i(T + v), \chi)^{-1}.$$

We replace $L^2(s, \chi)$ by its Dirichlet series (or by a suitable partial sum if χ is the principal character), multiply this by $e^{-|v|}$, and integrate term by term over the interval $-\log^2 T \leqslant v \leqslant \log^2 T$. It is easily seen (by arguments similar to those in [Iv], §7.4) that the contribution of those terms for which the variable of summation lies outside the interval $[M, M']$ is negligible, so (6.1) follows. Then (1.25) is a consequence of (1.24).

Note that the above argument cannot be carried over to functions $\varphi(s, \chi)$, so that the analogy between $L^2(s, \chi)$ and $\varphi(s, \chi)$ is not quite perfect.

Finally we show how the twelfth moment estimate (1.27) follows from Theorem 3.

Consider the number of pairs (t_j, χ_j), $j = 1, \ldots, J$ with $T \leqslant t_j \leqslant 2T$ such that

$$|L(1/2 + it_j, \chi_j)| \geqslant V \tag{6.3}$$

and $|t_i - t_j| \geqslant 1$ if $\chi_i = \chi_j$ but $i \neq j$. It suffices to show that

$$J \ll D^3 T^2 V^{-12} (DT)^{\varepsilon}. \tag{6.4}$$

We may suppose that $V \gg D^{1/4} T^{1/8} (DT)^{\varepsilon}$, for otherwise the assertion follows from the bound $J \ll (DT)^{1+\varepsilon} V^{-4}$, which is a consequence of the standard estimate for the fourth moment of L-functions.

We construct a system of R non-overlapping intervals of length $T^{1/2+\varepsilon}$ covering the points t_j. We omit those points lying in a $\log^2 T$-neighbourhood of one of the endpoints of some interval, but a similar argument can be applied to these points afterwards, with a new system of intervals. All the powers $|L(1/2 + it, \chi)|^4$, in particular those with $\chi = \chi_j$, are now integrated over our system of intervals. Then it follows from (6.2), (6.3), and (1.24) that

$$JV^4 \ll D\big(RT^{1/2+\varepsilon} + (RT)^{2/3}\big)(DT)^{\varepsilon}.$$

Now $R \leqslant J$, so by our assumption on V we may omit the first term on the right. Then

$$JV^4 \ll DJ^{2/3}T^{2/3}(DT)^{\varepsilon},$$

which implies the assertion (6.4), in the case $D \ll T^B$. But otherwise (1.27) follows directly from the fourth moment and the estimate $L(1/2+it, \chi) \ll (DT)^{1/4+\varepsilon}$.

References

[B-I] E. Bombieri and H. Iwaniec, *On the order of* $\zeta(1/2+it)$, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. 13 (1986), 449–472.

[G] A. Good, *The square mean of Dirichlet series associated with cusp forms*, Mathematika 29 (1982), 278–295.

[H1] D. R. Heath-Brown, *The twelfth power moment of the Riemann zeta-function*, Quart. J. Math. Oxford 29 (1978), 443–462.

H2] — *The growth rate of the Dedekind Zeta-function on the critical line*, Acta Arith. 49 (1988), 323–339.

[H-W] M. N. Huxley and N. Watt, *Exponential sums and the Riemann zeta function*, Proc. London Math. Soc. (3) 57 (1988), 1–24.

[Iv] A. Ivić, *The Riemann Zeta-Function*, John Wiley & Sons, 1985.

[Iw] H. Iwaniec, *Fourier coefficients of cusp forms and the Riemann zeta-function*, Séminaire de Théorie des Nombres, Univ. Bordeaux 1979/80, exposé no 18, 36 pp.

[J1] M. Jutila, *Lectures on a method in the theory of exponential sums*, Tata Institute of Fundamental Research, Lectures in Mathematics and Physics 80, Bombay 1987.

[J2] — *The fourth power moment of the Riemann zeta-function over a short interval*, Coll. Soc. János Bolyai 52, Number Theory (Budapest 1987), North-Holland, Amsterdam 1990, 221–244.

[J3] — *Mean value estimates for exponential sums*, Number Theory, Ulm 1987, Lecture Notess in Mathematics 1380, Springer, 1989, 120–136.

[J4] — *On exponential sums involving the Ramanujan function*, Proc.Indian Acad. Sci. (Math. Sci.) 97 (1987), 157–166.

[J5] — *Exponential sums connected with quadratic forms*, Proc. of the First Conference of the Canadian Number Theory Association, Banff, Alberta, 1988, Walter de Gruyter, 1990, 271–286.

[Me1] T. Meurman, *The mean twelfth power of Dirichlet L-functions on the critical line*, Ann. Acad. Sci. Fenn. Ser. A I, Dissertationes, 52 (1984), 44 pp.

[Me2] — *On exponential sums involving the Fourier coefficients of Maass wave forms*, J. Reine Angew. Math. 384 (1988), 192–207.

[Mo] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Lecture Notes in Mathematics 227, Springer, 1971.

[S] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press, 1971.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TURKU
SF-20500 Turku 50
Finland

Received on 21.2.1989 (1909)

ACTA ARITHMETICA
LVII (1991)

Khintchine-type theorems on manifolds

by

M. M. Dodson (York), B. P. Rynne and J. A. G. Vickers (Southampton)

To the memory of Professor V. G. Sprindžuk

1. Introduction. Sprindžuk made fundamental contributions to the difficult problem of extending classical results on metric Diophantine approximation to submanifolds, or in his terminology, from the case of "independent variables" to that of "dependent variables" [9]. In this paper some Khintchine-type theorems are obtained for a fairly general class of manifolds.

For any vectors $\boldsymbol{x} = (x_1, \ldots, x_k)$, $\boldsymbol{y} = (y_1, \ldots, y_k)$ in $\boldsymbol{R}^k$ we write

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{k} x_i y_i \quad \text{and} \quad |\boldsymbol{x}| = \max\{|x_i|: i = 1, \ldots, k\}.$$

For any real number t let

$$\|t\| = \inf\{|t-p|: p \in \boldsymbol{Z}\}.$$

Let $\psi(r)$, $r = 1, 2, \ldots$, be a sequence of numbers with $\psi(r) \in [0, 1/2]$. It follows from Groshev's generalisation of Khintchine's theorem ([9], Chap. 1, Theorem 12) that for almost all $\boldsymbol{x} \in \boldsymbol{R}^k$ the inequality

$$\|\boldsymbol{q} \cdot \boldsymbol{x}\| < \psi(|\boldsymbol{q}|) \tag{1.1}$$

has finitely many solutions $\boldsymbol{q} \in \boldsymbol{Z}^k$ if the series

$$\sum_{r=1}^{\infty} \psi(r) r^{k-1} \tag{1.2}$$

converges and infinitely many solutions if the series diverges (providing $\psi(r)$ satisfies certain monotonicity conditions when $k = 1$ or 2). Khintchine's theorem on simultaneous Diophantine approximation ([9], Chap. 1, Theorem 8) asserts that the dual system of inequalities

$$\|qx_i\| < \psi(|q|), \quad i = 1, \ldots, k, \tag{1.3}$$

has finitely many solutions $q \in \boldsymbol{Z}$ for almost all $\boldsymbol{x} \in \boldsymbol{R}^k$ if the series

$$\sum_{r=1}^{\infty} \psi(r)^k \tag{1.4}$$