\( \Omega \)-results for sums of Fourier coefficients of cusp forms

by

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1. Introduction. Let \( f \) be a normalized Hecke eigenform of weight \( k \) for the full modular group which is a cusp form and let \( f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z} \) be its Fourier expansion at the cusp \( i\infty \).

Hardy [5] and Rankin [9] showed

\[
 a(n) = \Omega(n^{k-1/2})
\]

and

\[
 \limsup_{n \to \infty} \frac{|a(n)|}{n^{k-1/2}} = +\infty
\]

described above.

R. Balasubramanian and M. Ram Murty [1] proved:

\[
 a(n) = \Omega(n^{k-1/2} \exp(c/\log n)^{1/2})
\]

Later, for an arbitrary cusp form, which is not necessarily an eigenfunction, Ram Murty [8] proved:

\[
 a(n) = \Omega\left(n^{k-1/2} \exp\left(\frac{c \log n}{\log\log n}\right)\right),
\]

which is best possible in view of Deligne's result.

In the same paper [8], Ram Murty conjectured that, if \( f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z} \) is an arbitrary cusp form of weight \( k \) for the full modular group and \( a(n) \in \mathbb{R} \), then

\[
 \sum_{p \leq x} a(p) p^{-k-1/2} = \Omega, \quad \frac{x^{1/2} \log\log\log x}{\log x}
\]

and proved that for a normalized eigenform \( f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z} \) of weight \( k \) for the full modular group, this is true, provided

\[
 L(s) = \prod_{p} \left(1 - \frac{e^{\text{th}(p)}}{p^s}\right)^{-1} \left(1 - \frac{e^{-\text{th}(p)}}{p^s}\right)^{-1}
\]

has no real zero in \( 1/2 \leq s \leq 1 \).
Here \( \theta(p) \) is given by
\[
a(p) = 2p^{k-1/2} \cos(\theta(p)).
\]
From the work of Deligne [2], we know that \( \theta(p) \) is real, which gives
\[
|a(p)| \leq 2p^{k-1/2}.
\]

We prove:

**Theorem 1.** If \( F(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z} \) is a cusp form of integral weight \( k \) for \( \Gamma_0(N) \) (for some integer \( N \geq 1 \)) with real character, \( F(z) \) is an eigenfunction of the Hecke operators and it does not vanish on \( \{iy \mid 0 < y < \infty\} \), then
\[
\sum_{p \leq x} c(p)p^{\frac{k-1}{2}} \log p \sim \Omega \left( \frac{1}{x^{1/2}} \log \log \log x \right).
\]

**Examples.** Recently, Dummit, Kislevsky and Mckay [4] have characterized the products of \( \eta \)-function whose Fourier coefficients are multiplicative. They are:
\[
\left( \prod_{i=1}^{t} \eta(n_i z) \right) \text{ with } \sum_{i=1}^{t} n_i = 24,
\]
where the corresponding partitions of 24 are given by:
- \((24), (8^3), (23, 1), (22, 2), (21, 3), (20, 4), (18, 6), (16, 8), (12^2), (15, 5, 3, 1), (14, 7, 2, 1), (12, 6, 4, 2), (11^2, 1^2), (10^2, 2^2), (9^2, 3^2), (8^2, 4^2), (6^4), (8^2, 4, 2, 1^2), (7^3, 1^2), (6^3, 2^3), (4^6), (6^2, 3^2, 2^1), (5^4, 1^4), (4^4, 2^4), (3^8), (4^2, 2^2, 1^4), (2^2)^2, (2^1)^8, (1^2)^4),
\]
where \((8^3)\) stands for the partition \((8, 8, 8)\) and so on.

If \((n_1, \ldots, n_t)\) is one of the above partitions, then \( \varphi(z) = \prod_{i=1}^{t} \eta(n_i z) \) is a cusp form of weight \( k \) for \( \Gamma_0(N) \) with real character, where \( k = t/2 \) and \( N = \text{min} n_i \text{max} n_i \).

For weight \( \geq 2 \), these functions are eigenfunctions of the Hecke operators.

Also, since \( \eta^{2k}(z) = \Delta(z) \) does not vanish on the upper half plane, \( \varphi(z) \) does not vanish there.

2. Some lemmas. Let \( S_k(N, \chi) \) denote the space of cusp forms of weight \( k \) for \( \Gamma_0(N) \) with a real character \( \chi \). Then the map
\[
f \mapsto f \left[ \begin{array}{cc} 0 & -1 \\ N & 0 \end{array} \right] := N^{k/2}(Nz)^{-k}f(-1/(Nz))
\]
is an isomorphism of the vector space \( S_k(N, \chi) \).

Defining
\[
f^+ = \frac{1}{2} \left( f + i^k f \right) \left[ \begin{array}{cc} 0 & -1 \\ N & 0 \end{array} \right], \quad f^- = \frac{1}{2} \left( f - i^k f \right) \left[ \begin{array}{cc} 0 & -1 \\ N & 0 \end{array} \right],
\]
we see that \( f = f^+ + f^- \), where
\[
f^+ \left[ \begin{array}{cc} 0 & -1 \\ N & 0 \end{array} \right] = i^k f^+, \quad f^- \left[ \begin{array}{cc} 0 & -1 \\ N & 0 \end{array} \right] = -i^k f^-.
\]

Let \( F(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z} \) be as in Theorem 1. Then
\[
|c(p)| \leq 2p^{k-1/2}
\]
for primes \( p \) not dividing \( N \) (Deligne [2] for \( k \geq 2 \), Deligne and Serre [3] for \( k = 1 \)).

Let
\[
F^+(z) = \sum_{n=1}^{\infty} c_1(n) e^{2\pi i n z}, \quad F^-(z) = \sum_{n=1}^{\infty} c_2(n) e^{2\pi i n z}
\]
be the Fourier expansions of \( F^+ \) and \( F^- \) respectively.

Let
\[
L(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}, \quad L_1(s) = \sum_{n=1}^{\infty} \frac{c_1(n)}{n^s}, \quad L_2(s) = \sum_{n=1}^{\infty} \frac{c_2(n)}{n^s}
\]
be Dirichlet series corresponding to \( F(z) \), \( F^+(z) \), and \( F^-(z) \) respectively.

Using (2.1), by standard methods (see e.g. Kobitz [7], p. 140) one gets functional equations for \( L_1(s) \) and \( L_2(s) \), and hence the following lemma:

**Lemma 1.**
\[
V(s) = \left( \sqrt{N/(2\pi)} \right)^s \Gamma(s) [L_1(s) + L_2(s)],
\]
then we have
\[
V(s) = V^*(k-s).
\]

Also, \( L_1(s)+L_2(s) \) and \( L_1(s)-L_2(s) \) have analytic continuation to the whole complex plane as entire functions.

Now, we write
\[
L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad L_j(s) = \sum_{n=1}^{\infty} \frac{a_j(n)}{n^s},
\]
where \( a(n) = c(n) n^{-(k-1)/2} \) and \( a_j(n) = c_j(n) n^{-(k-1)/2}, \) \( j = 1, 2.\)
Therefore, writing
\[ A(s) = (\sqrt{N}/(2\pi))^{s+k/2-1/2} \Gamma(s+k/2-1/2) [\tilde{L}_1(s) + \tilde{L}_2(s)], \]
\[ A^*(s) = (\sqrt{N}/(2\pi))^{s+k/2-1/2} \Gamma(s+k/2-1/2) [\tilde{L}_1(s) - \tilde{L}_2(s)], \]
we have
\[ (2.2) \quad A(s) = A^*(1-s). \]
Now,
\[ L(s) = \prod_p (1 - c(p)p^{-s} + \chi(p)p^{k/2-1/2-s})^{-1} = \prod_p (1 - \beta_p p^{-s})^{-1} (1 - \chi(p) \beta_p p^{-s})^{-1}, \]
where $\beta_p$ is the complex conjugate of $\beta_p$. Therefore, we have
\[ (2.3) \quad \tilde{L}(s) = L(s+k/2-1/2) = \prod_p (1 - \gamma_p p^{-s})^{-1} (1 - \chi(p) \gamma_p p^{-s})^{-1}, \]
where
\[ \gamma_p = \beta_p p^{-k/2+1/2}, \quad \gamma_p + \chi(p) \gamma_p = a(p), \quad |\gamma_p| = |\gamma_p| = 1. \]
From (2.3), we have
\[ (2.4) \quad \frac{\tilde{L}}{L}(s) = \sum_p [((\gamma_p p^{-s} \log p + \chi(p) p^{-s} \log p + \ldots) \]
\[ + (\chi(p) \gamma_p p^{-s} \log p + \chi(p) p^{-s} \log p + \ldots)] \]
\[ = \sum_{n=1}^{\infty} Y(n)n^{-s}. \]
Here,
\[ Y(n) = \begin{cases} \gamma_p + \chi(p) \gamma_p \log p & \text{if } n = p^m \text{ (m > 1), for some prime } p, \\ 0 & \text{otherwise.} \end{cases} \]
We note that
\[ Y(p) = (\gamma_p + \chi(p) \gamma_p) \log p = a(p) \log p. \]
Now, the following lemma follows by standard methods (see e.g. Ingham [6], pp. 68–70).

**Lemma 2.** For $T > 0$, let $N(T)$ denote the number of zeros of $\tilde{L}(s)$ in the rectangle $0 \leq \sigma \leq 1$, $0 \leq t \leq T$. Then, as $T \to \infty$,
\[ N(T) = T \log T + (\log(\sqrt{N}/(2\pi)) - 1) T + O(\log T). \]

The following are easy consequences of Lemma 2.

**Corollary 2.1.** If $h$ is a fixed positive number, then
\[ N(T+h) - N(T) = O(\log T). \]

**Corollary 2.2.** If $\beta = \beta + \gamma i$, $0 \leq \beta \leq 1$ are zeros of $\tilde{L}(s)$ in the critical strip, then
\[ \sum_{0 \leq \gamma \leq \frac{T}{2}} \frac{1}{\gamma^2} = O(\log^2 T), \quad \sum_{0 \leq \gamma \leq \frac{T}{2}} \frac{1}{\gamma^2} = O\left(\frac{\log T}{T}\right). \]

**Definitions.** We define
\[ \Psi(x) = \sum_{n=1}^{\infty} Y(n) \]
where $Y(n)$ is defined in (2.4) and
\[ \Psi_0(x) = (\Psi_1(x) + 1 + \Psi_1(x)) / 2. \]

**Remark 2.1.** $\Psi_0(x)$ differs from $\Psi_1(x)$ only when $x$ is a prime-power $p^m$, the difference then being $\frac{1}{2} (\gamma^2 + \chi(p) \gamma^2) \log p$.

Now, by standard methods (see Ingham [6], Theorems 26–29) one gets the explicit formula
\[ (2.5) \quad \Psi_0(x) = -\sum_{q \leq Q} \frac{x^q}{q \log(1 + 1/q)}. \]
We have $F(iy) \neq 0$ for all $y > 0$. Hence, the equation
\[ (-2\pi)^{-s} \Gamma(s) L(s) = \int_0^\infty F(z) z^{s-1} \, dz \]
implies that $L(s) \neq 0$ for $s > 0$ and hence the following lemma:

**Lemma 3.** $\tilde{L}(s)$ has no zeros on the part of the real axis given by $s > (-k+1)/2$.

**Lemma 4.** If $\theta$ denotes the upper bound of the real parts of the complex zeros of $\tilde{L}(s)$, then $\Psi_1(x) = \Omega(x^{1-\delta})$ for any fixed positive number $\delta$.

By Abel's identity,
\[ \frac{\tilde{L}}{L}(s) = \int_1^\infty \frac{\Psi_1(x)}{x^{s+1}} \, dx \quad (s > 1). \]
Writing
\[ c(x) = (\Psi_1(x) - x^\alpha) / x, \]
for some $0 < \alpha < \theta$,
\[ (2.6) \quad \int_1^\infty \frac{c(x)}{x^s} \, dx = - \frac{1}{s} \int_1^\infty \frac{\tilde{L}}{L}(s) \, ds - \frac{1}{s - \alpha} \quad (s > 1) \]
\[ = f(s), \quad \text{say.} \]
If \( \sigma_0 \) is the abscissa of convergence of the Dirichlet integral in (2.6), then \( \sigma_0 \geq \theta \).
Also, \( f(s) \) has no singularities on the stretch \( s > \alpha \), since \( \bar{I}(s) \) is regular and has no zeros on the positive real axis by Lemma 3.

Since \( \sigma_0 \geq \theta > \alpha \), \( s = \sigma_0 \) is not a singularity of \( f(s) \).
Therefore, by Landau’s theorem, we cannot have either \( c(x) \geq 0 \) or \( c(x) \leq 0 \) for sufficiently large \( x \), which proves the lemma.

Remark 2.2. If \( \theta > \frac{1}{2} \), Lemma 4 would imply

\[
\Psi_1(x) = \Omega_\pm(x^{1/2}).
\]

If \( \theta = \frac{1}{2} \), writing

\[
c(x) = \frac{\Psi_1(x) - cx^{1/2}}{x}
\]

and

\[
f(s) = \left(\frac{1}{s} \right) \overline{\bar{I}}(s) + \frac{s}{s-1/2}
\]

for some \( c > 0 \), we have

\[
\int_1^\infty \frac{c(x)}{x} \, dx = f(s) \quad (\sigma > 1).
\]

If \( f(s) \) has no singularities on the real axis to the right of \( 1/2 \), Landau’s theorem would imply that the abscissa of convergence of the integral in (2.8) is \( \sigma_0 = 1/2 \) and hence (2.8) is valid for \( \sigma > 1/2 \).

If possible, let \( c(x) \geq 0 \) for all \( x > X \) (\( X > 1 \)). Then for \( \sigma > 1/2 \),

\[
|f(\sigma + it)| \leq \frac{1}{\sqrt{\sigma}} \| c(x) \|_{1/2} + \int_1^\infty \frac{c(x)}{x} \, dx = \frac{1}{\sqrt{\sigma}} \| c(x) \|_{1/2} - c(x) \, dx + f(\sigma)
\]

\[
\leq 2 \int_1^\infty \frac{c(x)}{x^{1/2}} \, dx + f(\sigma) = K + f(\sigma)
\]

where \( K \) is independent of \( \sigma \) and \( t \).

If \( 1/2 + \gamma_1 i \) is the zero with least positive \( \gamma \), let \( t = \gamma_1 \) and then multiplying both sides by \( \sigma - 1/2 \) and making \( \sigma \rightarrow 1/2 + 0 \), we get from above

\[
\left| \frac{m_1}{1/2 + \gamma_1 i} \right| \leq c,
\]

where \( m_1 \) is the order of multiplicity of the zero \( 1/2 + \gamma_1 i \). But we could have chosen \( 0 < c < m_1/1/2 + \gamma_1 i \) and that shows that the supposition \( c(x) \geq 0 \) for \( x > X \) leads to a contradiction.

So \( c(x) < 0 \) for arbitrary large \( x \). Similarly, one can show that \( c(x) > 0 \) for arbitrary large \( x \), i.e., (2.7) holds in the case \( \theta = 1/2 \), as well.

3. Proof of Theorem 1. We multiply the explicit formula (2.5) by \( x^{-3/2} \), make the change of variable \( x = e^u \) and integrate the resulting expression in \( u \) from \( w-\eta \) to \( w+\eta \) (where \( w, \eta \) are parameters to be chosen). This gives

\[
\int_{w-\eta}^{w+\eta} e^{-u/2} \left( \psi_0(e^u) + \frac{P}{L}(0) + 2\log(1 - e^{-u/2}) \right) \, du = -\sum_{\nu > 0} \int_{w-\eta}^{w+\eta} e^{\nu(1/2-1/2)} \, du.
\]

We put

\[
G(u) = e^{-u/2} \left( \psi_0(e^u) + P(0) + 2\log(1 - e^{-u/2}) \right).
\]

Clearly,

\[
G(u) = \Omega_\pm \left( \log \log u \right) \Leftrightarrow \psi_0(x) = \Omega_\pm \left( x^{1/2} \log \log \log x \right).
\]

If the Riemann hypothesis is false for \( \bar{I} \), i.e., \( \theta > 1/2 \), then from Lemma 4 (see Remark 2.1) a result stronger than

\[
\left(3.2\right) \quad \psi_0(x) = \Omega_\pm \left( x^{1/2} \log \log \log x \right)
\]

is true.

So, we can assume the Riemann hypothesis.

Putting \( q = 1/2 + it \) in the formula (3.1) and integrating

\[
\frac{1}{2\pi w-\eta} \sum_{\eta \leq \nu \leq T} G(u) \, du = -\sum_{\nu > 0} \frac{\sin \eta \gamma e^{\nu w}}{\nu \eta}
\]

Let \( 1/2 + \gamma_1 i \) be the first zero of \( \bar{I}(s) \) on the line \( 1/2 \) and let \( T > \max \{ \epsilon^2, \gamma_1 \} \).

Then

\[
\sum_{\nu \leq T} \frac{\sin \eta \gamma e^{\nu w}}{\nu \eta} = \sum_{\nu \leq T} \frac{\sin \eta \gamma e^{\nu w}}{\nu \eta} + \sum_{\nu > T} \frac{\sin \eta \gamma e^{\nu w}}{\nu \eta}
\]

Now,

\[
\sum_{\nu \leq T} \frac{\sin \eta \gamma e^{\nu w}}{\nu \eta} = \sum_{\nu \leq T} \frac{\sin \eta \gamma e^{\nu w}}{\nu \eta} + O \left( \sum_{\nu \leq T} \gamma^{-2} \right)
\]

\[
= \sum_{\nu \leq T} \frac{\sin \eta \gamma \cos \nu w}{\nu \eta} + \sum_{\nu \leq T} \frac{\sin \eta \gamma e^{\nu w}}{\nu \eta} \gamma + O \left( \sum_{\nu \leq T} \gamma^{-2} \right)
\]

\[
= \sum_{\nu \leq T} \frac{\sin \eta \gamma \sin \nu w}{\nu \eta} + O(1).
\]

On the other hand, by Corollary 2.1

\[
\sum_{\nu > T} \frac{\sin \eta \gamma e^{\nu w}}{\nu \eta} = O \left( \sum_{\nu > T} \left( \eta^2 \right)^{-1} \right) = O \left( \frac{\log T}{\eta T} \right).
\]

Therefore,

\[
\left(3.3\right) \quad \frac{1}{2\pi w-\eta} \sum_{\nu > 0} G(u) \, du = -2S(w) + O(1) + O \left( \frac{\log T}{\eta T} \right),
\]

where

\[
S(w) = \sum_{0 < \gamma < \epsilon} \frac{\sin \eta \gamma \sin \gamma \gamma}{\gamma}
\]

\[
\psi_0(x) = \Omega_\pm \left( x^{1/2} \log \log \log x \right)
\]
Now, we utilize the theorem of Dirichlet (see Titchmarsh’s *Theory of Functions*): Given \( \theta_1, \ldots, \theta_N, N \) real numbers, \( q > 1, \tau > 0 \), the interval \( [\tau, \tau q^N] \) contains a \( U \in \mathbb{Z} \) such that \( \|U \theta_i\| < 1/q, 1 \leq i \leq N \).

Now, applying this to \( \theta_j = \gamma_j/(2\pi), 1 \leq j \leq N(T) \), for \( \tau = q^{N(T)} \) (\( q \) will be chosen later) we obtain:

There is \( U \in \mathbb{Z}, q^{N(T)} \leq U \leq q^{2N(T)} \), such that

\[
\|UY/2\pi\| < 1/q.
\]

Therefore, for all real \( v \),

\[
|S(U + v) - S(v)| \leq \frac{2\pi}{q} \sum_{j < \tau T} \frac{1}{q^j}
\]

by the mean value theorem and (3.4).

Therefore, by Corollary 2.2

\[
|S(U + v) - S(v)| = O\left(\frac{\log^2 T}{q}\right).
\]

Let \( 0 < \eta < 1/2 \). Setting \( w = U + \eta \), \( w = 2\eta \) in (3.3) and subtracting the corresponding expressions, we have by the above results

\[
\frac{1}{2\eta} \int_{-\eta}^{\eta} \left[ G(U + \eta + y) - G(2\eta + y) \right] dy = O\left(\frac{\log^2 T}{q}\right) + O(1) + O\left(\frac{\log T}{\eta T}\right).
\]

Choosing \( q = \log^2 T, \eta = (\log T)/T \) gives

\[
\frac{1}{2\eta} \int_{-\eta}^{\eta} \left[ G(U + \eta + y) - G(2\eta + y) \right] dy = O(1).
\]

Since \( y \in [-\eta, \eta] \), we have \( 2\eta + y = (2+\theta)\eta \) where \( |\theta| \leq 1 \). As \( \eta \rightarrow 0 \),

\[
G(2\eta + y) = 2\log(1-e^{-\eta-y/2}) + O(1) = 2\log(\eta + y/2) + O(1) = 2\log \eta + O(1).
\]

Therefore,

\[
\frac{1}{2\eta} \int_{-\eta}^{\eta} G(2\eta + y) dy \leq 2\log \eta + O(1).
\]

Hence, from (3.5),

\[
\frac{1}{2\eta} \int_{-\eta}^{\eta} G(U + \eta + y) dy \leq 2\log \eta + O(1).
\]

Now,

\[
\frac{1}{2\eta} \int_{-\eta}^{\eta} G(U + \eta + y) dy \leq 2\log \eta + O(1)
\]

implies that there exists \( u \in [U + \eta, U + 3\eta] \), such that \( G(u) \leq 2\log \eta + O(1) \).

Again \( -\log \eta \sim \log T \) and \( \log \log u = \log \log U + O(1) \). But (by Lemma 2)

\[
\log \log U = \log N(T) + O(\log \log q) = \log T + O(\log \log q) + O(\log T).
\]

Hence,

\[
\liminf_{u \rightarrow \infty} \frac{G(u)}{\log \log u} \leq -2,
\]

that is, \( G(u) = \Omega_-(\log \log u) \).

A similar analysis with \( -\log \eta \) yields

\[
\limsup_{u \rightarrow \infty} \frac{G(u)}{\log \log u} \geq 2.
\]

Hence (3.2) is true.

By Remark 2.1,

\[
\Psi_1(x) = \Omega_{\pm}(\sqrt[4]{\log \log x}).
\]

Since

\[
\Psi_1(x) = \sum_{p \leq x} y(p) + \sum_{p \leq x} Y(p^2) + \cdots + \sum_{p \leq x} Y(p^m), \text{ where } m = \left[ \frac{\log x}{\log 2} \right],
\]

it follows that

\[
\sum_{p \leq x} Y(p^3) + \cdots + \sum_{p \leq x} Y(p^m) \leq \left( \frac{\log x}{\log 2} \right) \left( \sum_{p \leq x} \log p \right) = O\left(\sqrt{x} \log x \right)
\]

and

\[
\sum_{p \leq x} Y(p^3) + \cdots + \sum_{p \leq x} Y(p^m) \leq \left( \frac{\log x}{\log 2} \right) \left( \sum_{p \leq x} \log p \right) = O\left(\sqrt{x} \log x \right)
\]

with \( m \) as above, we get

\[
\sum_{p \leq x} Y(p) \leq \sum_{p \leq x} a(p) \log p = \Omega_{\pm}(\sqrt[4]{\log \log x})
\]

which proves Theorem 1.

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References


Mean value estimates for exponential sums with applications to $L$-functions

by

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1. Introduction

1.1. In our previous paper [J3], we studied the mean square of the exponential sum

$$S(M, M'; v, y) = \sum_{M} d(m) g(m, v, y) e(f(m, v, y))$$

with respect to $v$ running over an interval $[0, V]$ and $y$ running over a well-spaced system of real numbers. Here $d(m)$ is the usual divisor function, $e(z) = e^{2\pi iz}$, and the functions $f$ and $g$ are supposed to satisfy certain conditions. The main result, a general mean value theorem, was applied to the fourth moment of $\zeta(1/2 + it)$ over a system of short intervals. In this way, we reproved a theorem of H. Iwaniec [Iw], which was in fact our principal motivation.

Our object in this paper is to generalize Iwaniec's theorem to $L$-functions. To this end, we need a mean value estimate for exponential sums

$$S(M, M'; v, y, a) = \sum_{M} \chi(m) d(m) g(m, v, y) e(f(m, v, y))$$

involving Dirichlet characters. If $\chi$ is a primitive character (mod $D$), then the sum $S_z$ can be written in terms of the Gaussian sum

$$\tau_z = \sum_{a=1}^{D} \chi(a) e(a/D)$$

and the exponential sum

$$S(M, M'; v, y, a) = \sum_{M} d(m) g(m, v, y) e(f(m, v, y) + ma)$$

as follows:

$$S_z = (\tau_z)^{-1} \sum_{a=1}^{D} \chi(a) S(M, M'; v, y, a/D).$$