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Ω-results for sums of Fourier coefficients of cusp forms

by

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1. Introduction. Let f be a normalized Hecke eigenform of weight k for the full modular group which is a cusp form and let $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$ be its Fourier expansion at the cusp $i\infty$.

Hardy [5] and Rankin [9] showed

$$a(n) = \Omega(n^{(k-1)/2})$$

and

$$\limsup_{n\to\infty} \frac{|a(n)|}{n^{(k-1)/2}} = +\infty$$

respectively.

R. Balasubramanian and M. Ram Murty [1] proved:

$$a(n) = \Omega\big(n^{(k-1)/2} \exp(c(\log n)^{1/k-\varepsilon})\big).$$

Later, for an arbitrary cusp form, which is not necessarily an eigenfunction, Ram Murty [8] proved:

$$a(n) = \Omega\left(n^{(k-1)/2} \exp\left(\frac{c\log n}{\log\log n}\right)\right),$$

which is best possible in view of Deligne's result.

In the same paper [8], Ram Murty conjectured that, if $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$ is an arbitrary cusp form of weight k for the full modular group and $a(n) \in \mathbf{R}$, then

$$\sum_{p\le x} a(p)\, p^{-(k-1)/2} = \Omega_{\pm}\left(\frac{x^{1/2}\log\log\log x}{\log x}\right)$$

and proved that for a normalized eigenform $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$ of weight k for the full modular group, this is true, provided

$$L_f(s) = \prod_p \left(1 - \frac{e^{i\theta(p)}}{p^s}\right)^{-1} \left(1 - \frac{e^{-i\theta(p)}}{p^s}\right)^{-1}$$

has no real zero in $1/2 \le s \le 1$.

Here $\theta(p)$ is given by

$$a(p) = 2p^{(k-1)/2}\cos(\theta(p)).$$

From the work of Deligne [2], we know that $\theta(p)$ is real, which gives

$$|a(p)| \leqslant 2p^{(k-1)/2}.$$

We prove:

THEOREM 1. *If $F(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi i n z}$ ($c(n) \in \mathbf{R}$) is a cusp form of integral weight k for $\Gamma_0(N)$ (for some integer $N \geqslant 1$) with real character, $F(z)$ is an eigenfunction of the Hecke operators and it does not vanish on $\{iy \mid 0 < y < \infty\}$, then*

$$\sum_{p \leqslant x} c(p)p^{-(k-1)/2}\log p = \Omega_{\pm}(x^{1/2}\log\log\log x).$$

EXAMPLES. Recently, Dummit, Kisilevsky and Mckay [4] have characterized the products of η-function whose Fourier coefficients are multiplicative. They are:

$$\prod_{i=1}^{t} \eta(n_i z), \quad \text{with} \quad \sum_{i=1}^{t} n_i = 24,$$

where the corresponding partitions of 24 are given by:

$$(24),\ (8^3),\ (23,1),\ (22,2),\ (21,3),\ (20,4),\ (18,6),\ (16,8),$$
$$(12^2),\ (15,5,3,1),\ (14,7,2,1),\ (12,6,4,2),\ (11^2,1^2),\ (10^2,2^2),$$
$$(9^2,3^2),\ (8^2,4^2),\ (6^4),\ (8^2,4,2,1^2),\ (7^3,1^3),\ (6^3,2^3),$$
$$(4^6),\ (6^2,3^2,2^2,1^2),\ (5^4,1^4),\ (4^4,2^4),\ (3^6,1^6),$$
$$(3^8),\ (4^4,2^2,1^4),\ (2^{12}),\ (2^8,1^8),\ (1^{24}),$$

where (8^3) stands for the partition (8, 8, 8) and so on.

If $(n_1, \ldots, n_t)$ is one of the above partitions, then $\varphi(z) = \prod_{i=1}^{t} \eta(n_i z)$ is a cusp form of weight k for $\Gamma_0(N)$ with real character, where $k = t/2$ and $N = (\min n_i)(\max n_i)$.

For weight $\geqslant 2$, these functions are eigenfunctions of the Hecke operators. Also, since $\eta^{24}(z) = \Delta(z)$ does not vanish on the upper half plane, $\varphi(z)$ does not vanish there.

2. Some lemmas. Let $S_k(N, \chi)$ denote the space of cusp forms of weight k for $\Gamma_0(N)$ with a real character χ. Then the map

$$f \mapsto f\left|\begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}_k\right. := N^{k/2}(Nz)^{-k}f(-1/(Nz))$$

is an isomorphism of the vector space $S_k(N, \chi)$.

Defining

$$f^+ = \frac{1}{2}\left(f + i^k f\left|\begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}_k\right.\right), \quad f^- = \frac{1}{2}\left(f - i^k f\left|\begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}_k\right.\right),$$

we see that $f = f^+ + f^-$, where

$$(2.1) \qquad f\left|\begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}_k\right. = i^{-k}f^+, \quad f\left|\begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}_k\right. = -i^{-k}f^-.$$

Let $F(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi i n z}$ be as in Theorem 1. Then

$$|c(p)| \leqslant 2p^{(k-1)/2}$$

for primes p not dividing N. (Deligne [2] for $k \geqslant 2$, Deligne and Serre [3] for $k = 1$.)

Let

$$F^+(z) = \sum_{n=1}^{\infty} c_1(n)e^{2\pi i n z}, \quad F^-(z) = \sum_{n=1}^{\infty} c_2(n)e^{2\pi i n z}$$

be the Fourier expansions of F^+ and F^- respectively.

Let

$$L(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}, \quad L_1(s) = \sum_{n=1}^{\infty} \frac{c_1(n)}{n^s}, \quad L_2(s) = \sum_{n=1}^{\infty} \frac{c_2(n)}{n^s}$$

($\mathrm{Re}\, s > k/2 + 1/2$) be the Dirichlet series corresponding to $F(z)$, $F^+(z)$, and $F^-(z)$ respectively.

Using (2.1), by standard methods (see e.g. Koblitz [7], p. 140) one gets functional equations for $L_1(s)$ and $L_2(s)$, and hence the following lemma:

LEMMA 1. *If*

$$V(s) = (\sqrt{N}/(2\pi))^s\Gamma(s)[L_1(s) + L_2(s)],$$
$$V^*(s) = (\sqrt{N}/(2\pi))^s\Gamma(s)[L_1(s) - L_2(s)],$$

then we have

$$V(s) = V^*(k-s).$$

Also, $(L_1(s) + L_2(s))$ and $(L_1(s) - L_2(s))$ have analytic continuation to the whole complex plane as entire functions.

Now, we write

$$\tilde{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \tilde{L}_j(s) = \sum_{n=1}^{\infty} \frac{a_j(n)}{n^s},$$

where $a(n) = c(n)n^{-(k-1)/2}$ and $a_j(n) = c_j(n)n^{-(k-1)/2}$, $j = 1, 2$.

Therefore, writing

$$\Lambda(s) = (\sqrt{N}/(2\pi))^{s+k/2-1/2}\Gamma(s+k/2-1/2)[\tilde{L}_1(s)+\tilde{L}_2(s)],$$

$$\Lambda^*(s) = (\sqrt{N}/(2\pi))^{s+k/2-1/2}\Gamma(s+k/2-1/2)[\tilde{L}_1(s)-\tilde{L}_2(s)],$$

we have

$$\Lambda(s) = \Lambda^*(1-s). \tag{2.2}$$

Now,

$$L(s) = \prod_p (1-c(p)p^{-s}+\chi(p)p^{k-1-2s})^{-1} = \prod_p (1-\beta_p p^{-s})^{-1}(1-\chi(p)\bar{\beta}_p p^{-s})^{-1},$$

where $\bar{\beta}_p$ is the complex conjugate of β_p. Therefore, we have

$$\tilde{L}(s) = L(s+k/2-1/2) = \prod_p (1-\gamma_p p^{-s})^{-1}(1-\chi(p)\bar{\gamma}_p p^{-s})^{-1}, \tag{2.3}$$

where

$$\gamma_p = \beta_p p^{-k/2+1/2}, \quad \gamma_p+\chi(p)\bar{\gamma}_p = a(p), \quad |\gamma_p| = |\bar{\gamma}_p| = 1.$$

From (2.3), we have

$$\begin{aligned} -\frac{\tilde{L}'}{\tilde{L}}(s) &= \sum_p [(\gamma_p p^{-s}\log p+\gamma_p^2 p^{-2s}\log p+\ldots) \\ &\quad +(\chi(p)\bar{\gamma}_p p^{-s}\log p+\chi^2(p)\bar{\gamma}_p^2 p^{-2s}\log p+\ldots)] \\ &= \sum_{n=1}^{\infty} Y(n)n^{-s}. \end{aligned} \tag{2.4}$$

Here,

$$Y(n) = \begin{cases} (\gamma_p^m+\chi^m(p)\bar{\gamma}_p^m)\log p & \text{if } n = p^m\ (m > 1), \text{ for some prime } p, \\ 0 & \text{otherwise.} \end{cases}$$

We note that

$$Y(p) = (\gamma_p+\chi(p)\bar{\gamma}_p)\log p = a(p)\log p.$$

Now, the following lemma follows by standard methods (see e.g. Ingham [6], pp. 68–70).

LEMMA 2. *For $T > 0$, let $N(T)$ denote the number of zeros of $\tilde{L}(s)$ in the rectangle $0 \leqslant \sigma \leqslant 1$, $0 \leqslant t \leqslant T$. Then, as $T \to \infty$,*

$$N(T) = T\log T+(\log(\sqrt{N}/(2\pi))-1)T+O(\log T).$$

The following are easy consequences of Lemma 2.

COROLLARY 2.1. *If h is a fixed positive number, then*

$$N(T+h)-N(T) = O(\log T).$$

COROLLARY 2.2. *If $\varrho = \beta+\gamma i$, $0 \leqslant \beta \leqslant 1$ are zeros of $\tilde{L}(s)$ in the critical strip, then*

$$\sum_{0\leqslant\gamma\leqslant T}\frac{1}{\gamma} = O(\log^2 T), \qquad \sum_{\gamma > T}\frac{1}{\gamma^2} = O\left(\frac{\log T}{T}\right).$$

DEFINITIONS. We define

$$\Psi_1(x) = \sum_{n\leqslant x} Y(n)$$

where $Y(n)$ is defined in (2.4) and

$$\Psi_0(x) = (\Psi_1(x+0)+\Psi_1(x-0))/2.$$

Remark 2.1. $\Psi_0(x)$ differs from $\Psi_1(x)$ only when x is a prime-power p^m, the difference then being $\frac{1}{2}(\gamma_p^m+\chi^m(p)\bar{\gamma}_p^m)\log p$.

Now, by standard methods (see Ingham [6], Theorems 26–29) one gets the explicit formula

$$\Psi_0(x) = -\sum_{\varrho}\frac{x^{\varrho}}{\varrho}-\frac{\tilde{L}'}{\tilde{L}}(0)-2\log\left(1-\frac{1}{\sqrt{x}}\right). \tag{2.5}$$

We have $F(iy) \neq 0$ for all $y > 0$. Hence, the equation

$$(-2\pi i)^{-s}\Gamma(s)L(s) = \int_0^{i\infty} F(z)z^{s-1}\,dz$$

implies that $L(s) \neq 0$ for $s > 0$ and hence the following lemma:

LEMMA 3. *$\tilde{L}(s)$ has no zeros on the part of the real axis given by $s > (-k+1)/2$.*

LEMMA 4. *If θ denotes the upper bound of the real parts of the complex zeros of $\tilde{L}(s)$, then $\Psi_1(x) = \Omega_{\pm}(x^{\theta-\delta})$ for any fixed positive number δ.*

By Abel's identity,

$$-\frac{\tilde{L}'}{\tilde{L}}(s) = s\int_1^{\infty}\frac{\Psi_1(x)}{x^{s+1}}\,dx \quad (s > 1).$$

Writing

$$c(x) = (\Psi_1(x)-x^{\alpha})/x,$$

for some $0 < \alpha < \theta$,

$$\begin{aligned} \int_1^{\infty}\frac{c(x)}{x^s}\,dx &= -\left(\frac{1}{s}\right)\frac{\tilde{L}'}{\tilde{L}}(s)-\frac{1}{s-\alpha} \quad (s > 1) \\ &= f(s), \text{ say.} \end{aligned} \tag{2.6}$$

If σ_0 is the abscissa of convergence of the Dirichlet integral in (2.6), then $\sigma_0 \geqslant \theta$. Also, $f(s)$ has no singularities on the stretch $s > \alpha$, since $\tilde{L}(s)$ is regular and has no zeros on the positive real axis by Lemma 3.

Since $\sigma_0 \geqslant \theta > \alpha$, $s = \sigma_0$ is not a singularity of $f(s)$.

Therefore, by Landau's theorem, we cannot have either $c(x) \geqslant 0$ or $c(x) \leqslant 0$ for sufficiently large x, which proves the lemma.

Remark 2.2. If $\theta > \frac{1}{2}$, Lemma 4 would imply

$$\Psi_1(x) = \Omega_{\pm}(x^{1/2}). \tag{2.7}$$

If $\theta = 1/2$, writing

$$c(x) = \frac{\Psi_1(x) - cx^{1/2}}{x} \quad \text{and} \quad f(s) = -\left(\frac{1}{s}\right)\frac{\tilde{L}'}{\tilde{L}}(s) + \frac{c}{s-1/2}$$

for some $c > 0$, we have

$$\int_1^{\infty} \frac{c(x)}{x^s}\,dx = f(s) \quad (\sigma > 1). \tag{2.8}$$

If $f(s)$ has no singularities on the real axis to the right of 1/2, Landau's theorem would imply that the abscissa of convergence of the integral in (2.8) is $\sigma_0 = 1/2$ and hence (2.8) is valid for $\sigma > 1/2$.

If possible, let $c(x) \geqslant 0$ for all $x \geqslant X$ (> 1). Then for $\sigma > 1/2$,

$$|f(\sigma + ti)| \leqslant \int_1^X \frac{|c(x)|}{x^\sigma}\,dx + \int_X^\infty \frac{c(x)}{x^\sigma}\,dx = \int_1^X \frac{|c(x)| - c(x)}{x^\sigma}\,dx + f(\sigma)$$

$$\leqslant 2\int_1^X \frac{|c(x)|}{x^{1/2}}\,dx + f(\sigma) = K + f(\sigma)$$

where K is independent of σ and t.

If $1/2 + \gamma_1 i$ is the zero with least positive γ, let $t = \gamma_1$ and then multiplying both sides by $\sigma - 1/2$ and making $\sigma \to 1/2 + 0$, we get from above

$$\frac{m_1}{|1/2 + \gamma_1 i|} \leqslant c,$$

where m_1 is the order of multiplicity of the zero $1/2 + \gamma_1 i$. But we culd have chosen $0 < c < m_1/|1/2 + \gamma_1 i|$ and that shows that the supposition $c(x) \geqslant 0$ for $x \geqslant X$ leads to a contradiction.

So $c(x) < 0$ for arbitrary large x. Similarly one can show that $c(x) > 0$ for arbitrary large x, i.e., (2.7) holds in the case $\theta = 1/2$, as well.

3. Proof of Theorem 1. We multiply the explicit formula (2.5) by $x^{-3/2}$, make the change of variable $x = e^u$ and integrate the resulting expression in u from $w - \eta$ to $w + \eta$ (where w, η are parameters to be chosen). This gives

$$\int_{w-\eta}^{w+\eta} e^{-u/2}\left(\Psi_0(e^u) + \frac{\tilde{L}'}{\tilde{L}}(0) + 2\log(1 - e^{-u/2})\right)du = -\sum_{\varrho} \int_{w-\eta}^{w+\eta} \frac{e^{u(\varrho - 1/2)}}{\varrho}\,du. \tag{3.1}$$

We put

$$G(u) = e^{-u/2}\left(\Psi_0(e^u) + \frac{\tilde{L}'}{\tilde{L}}(0) + 2\log(1 - e^{-u/2})\right).$$

Clearly,

$$G(u) = \Omega_{\pm}(\log\log u) \Leftrightarrow \Psi_0(x) = \Omega_{\pm}(x^{1/2}\log\log\log x).$$

If the Riemann hypothesis is false for $\tilde{L}$, i.e., $\theta > 1/2$, then from Lemma 4 (see Remark 2.1) a result stronger than

$$\Psi_0(x) = \Omega_{\pm}(x^{1/2}\log\log\log x) \tag{3.2}$$

is true.

So, we can assume the Riemann hypothesis.

Putting $\varrho = 1/2 + i\gamma$ in the formula (3.1) and integrating

$$\frac{1}{2\eta}\int_{w-\eta}^{w+\eta} G(u)\,du = -\sum_{\varrho} \frac{\sin\gamma\eta}{\gamma\eta}\frac{e^{i\gamma w}}{\varrho}.$$

Let $1/2 + \gamma_1 i$ be the first zero of $\tilde{L}(s)$ on the line 1/2 and let $T > \max\{e^2, \gamma_1\}$. Then

$$\sum_{\varrho} \frac{\sin\gamma\eta}{\gamma\eta}\frac{e^{i\gamma w}}{\varrho} = \sum_{|\gamma| \leqslant T} \frac{\sin\gamma\eta}{\gamma\eta}\frac{e^{i\gamma w}}{\varrho} + \sum_{|\gamma| \geqslant T} \frac{\sin\gamma\eta}{\gamma\eta}\frac{e^{i\gamma w}}{\varrho}.$$

Now,

$$\sum_{|\gamma| \leqslant T} \frac{\sin\gamma\eta}{\gamma\eta}\frac{e^{i\gamma w}}{\varrho} = \sum_{|\gamma| \leqslant T} \frac{\sin\gamma\eta}{\gamma\eta}\frac{e^{i\gamma w}}{i\gamma} + O\Big(\sum_{|\gamma| \leqslant T} \gamma^{-2}\Big)$$

$$= \sum_{|\gamma| \leqslant T} \frac{\sin\gamma\eta}{\gamma\eta}\frac{\cos\gamma w}{i\gamma} + \sum_{|\gamma| \leqslant T} \frac{\sin\gamma\eta}{\gamma\eta}\frac{\sin\gamma w}{\gamma} + O\Big(\sum_{|\gamma| \leqslant T} \gamma^{-2}\Big)$$

$$= \sum_{|\gamma| \leqslant T} \frac{\sin\gamma\eta}{\gamma\eta}\frac{\sin\gamma w}{\gamma} + O(1).$$

On the other hand, by Corollary 2.1

$$\sum_{|\gamma| \geqslant T} \frac{\sin\gamma\eta}{\gamma\eta}\frac{e^{i\gamma w}}{\varrho} = O\Big(\sum_{|\gamma| \geqslant T} (\eta\gamma^2)^{-1}\Big) = O\left(\frac{\log T}{\eta T}\right).$$

Therefore,

$$\frac{1}{2\eta}\int_{w-\eta}^{w+\eta} G(u)\,du = -2S(w) + O(1) + O\left(\frac{\log T}{\eta T}\right), \tag{3.3}$$

where

$$S(w) = \sum_{0 < \gamma \leqslant T} \frac{\sin\gamma\eta}{\gamma\eta}\frac{\sin\gamma w}{\gamma}.$$

Now, we utilize the theorem of Dirichlet (see Titchmarsh's *Theory of Functions*): Given $\theta_1, \ldots, \theta_N$, N real numbers, $q > 1$, $\tau > 0$, the interval $[\tau, \tau q^N]$ contains a $U \in \mathbf{Z}$ such that $\|U\theta_i\| < 1/q$, $1 \leqslant i \leqslant N$.

Now, applying this to $\theta_j = \gamma_j/(2\pi)$, $1 \leqslant j \leqslant N(T)$, for $\tau = q^{N(T)}$ (q will be chosen later) we obtain:

There is $U \in \mathbf{Z}$, $q^{N(T)} \leqslant U \leqslant q^{2N(T)}$, such that

$$\|U\gamma_j/2\pi\| < 1/q. \tag{3.4}$$

Therefore, for all real v,

$$|\pm S(U \pm v) - S(v)| \leqslant \frac{2\pi}{q} \sum_{\gamma < T} \frac{1}{\gamma}$$

by the mean value theorem and (3.4).

Therefore, by Corollary 2.2

$$|\pm S(U \pm v) - S(v)| = O\left(\frac{\log^2 T}{q}\right).$$

Let $0 < \eta < 1/2$. Setting, $w = U \pm 2\eta$, $w = 2\eta$ in (3.3) and subtracting the corresponding expressions, we have by the above results

$$\frac{1}{2\eta} \int_{-\eta}^{\eta} [\pm G(U \pm 2\eta + y) - G(2\eta + y)]\, dy = O\left(\frac{\log^2 T}{q}\right) + O(1) + O\left(\frac{\log T}{\eta T}\right).$$

Choosing $q = \log^2 T$, $\eta = (\log T)/T$ gives

$$\frac{1}{2\eta} \int_{-\eta}^{\eta} [\pm G(U \pm 2\eta + y) - G(2\eta + y)]\, dy = O(1). \tag{3.5}$$

Since $y \in [-\eta, \eta]$, we have $2\eta + y = (2+\theta)\eta$ where $|\theta| \leqslant 1$. As $\eta \to 0$,

$$G(2\eta + y) = 2\log(1 - e^{-\eta - y/2}) + O(1) = 2\log(\eta + y/2) + O(1) = 2\log \eta + O(1).$$

Therefore,

$$\frac{1}{2\eta} \int_{-\eta}^{\eta} G(2\eta + y)\, dy \leqslant 2\log \eta + O(1).$$

Hence, from (3.5),

$$\frac{1}{2\eta} \int_{-\eta}^{\eta} \pm G(U \pm 2\eta + y)\, dy \leqslant 2\log \eta + O(1).$$

Now,

$$\frac{1}{2\eta} \int_{-\eta}^{\eta} G(U + 2\eta + y)\, dy \leqslant 2\log \eta + O(1)$$

implies that there exists $u \in [U + \eta, U + 3\eta]$, such that $G(u) \leqslant 2\log \eta + O(1)$.

Again $-\log \eta \sim \log T$ and $\log\log u = \log\log U + O(1)$. But (by Lemma 2)

$$\log\log U = \log N(T) + O(\log\log q) = \log T + O(\log\log q) + O(\log\log T).$$

Hence,

$$\liminf_{u \to \infty} \frac{G(u)}{\log\log u} \leqslant -2,$$

that is, $G(u) = \Omega_-(\log\log u)$.

A similar analysis with $-G(u)$ yields

$$\limsup_{u \to \infty} \frac{G(u)}{\log\log u} \geqslant 2.$$

Hence (3.2) is true.

By Remark 2.1,

$$\Psi_1(x) = \Omega_\pm(x^{1/2} \log\log\log x).$$

Since

$$\Psi_1(x) = \sum_{p \leqslant x} Y(p) + \sum_{p^2 \leqslant x} Y(p^2) + \ldots + \sum_{p^m \leqslant x} Y(p^m), \quad \text{where } m = \left[\frac{\log x}{\log 2}\right],$$

$$\sum_{p^2 \leqslant x} Y(p^2) \leqslant 2 \sum_{p \leqslant \sqrt{x}} \log p = O(\sqrt{x})$$

and

$$\sum_{p^3 \leqslant x} Y(p^3) + \ldots + \sum_{p^m \leqslant x} Y(p^m) \leqslant \left[\frac{\log x}{\log 2}\right] (2 \sum_{p \leqslant x^{1/3}} \log p) = O(x^{1/3} \log x)$$

with m as above, we get

$$\sum_{p \leqslant x} Y(p) \leqslant \sum_{p \leqslant x} a(p) \log p = \Omega_\pm(x^{1/2} \log\log\log x)$$

which proves Theorem 1.

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Mean value estimates for exponential sums with applications to *L*-functions

by

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1. Introduction

1.1. In our previous paper [J3], we studied the mean square of the exponential sum

$$S(M, M'; v, y) = \sum_{M}^{M'} d(m) g(m, v, y) e(f(m, v, y))$$

with respect to v running over an interval $[0, V]$ and y running over a well-spaced system of real numbers. Here $d(m)$ is the usual divisor function, $e(\alpha) = e^{2\pi i\alpha}$, and the functions f and g are supposed to satisfy certain conditions. The main result, a general mean value theorem, was applied to the fourth moment of $\zeta(1/2+it)$ over a system of short intervals. In this way, we reproved a theorem of H. Iwaniec [Iw], which was in fact our principal motivation.

Our object in this paper is to generalize Iwaniec's theorem to *L*-functions. To this end, we need a mean value estimate for exponential sums

$$S_\chi(M, M'; v, y) = \sum_{M}^{M'} \chi(m) d(m) g(m, v, y) e(f(m, v, y)) \tag{1.1}$$

involving Dirichlet characters. If χ is a primitive character (mod D), then the sum S_χ can be written in terms of the Gaussian sum

$$\tau_\chi = \sum_{a=1}^{D} \chi(a) e(a/D)$$

and the exponential sum

$$S(M, M'; v, y, \alpha) = \sum_{M}^{M'} d(m) g(m, v, y) e(f(m, v, y) + m\alpha) \tag{1.2}$$

as follows:

$$S_\chi = (\tau_{\bar{\chi}})^{-1} \sum_{a=1}^{D} \bar{\chi}(a) S(M, M'; v, y, a/D).$$