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On the splitting of primes in an arithmetic progression, II

by

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1. Introduction. Let k be a number field and suppose $p \in \mathbf{Q}$ is tamely ramified in k: $p = P_1^{e_1} P_2^{e_2} \dots P_r^{e_r}$, $p \nmid e_i$. In this paper we show that there exists a set of rational primes with positive density in an arithmetic progression whose splitting in k depends on the ramification indices and residue class degrees of the P_i's. This is an extension of the result in [1].

2. Some preliminary results

Lemma 1. *Let k be a number field and suppose K is the narrow class field of the normal closure $\bar{k}$. Let P be a prime in K and suppose $I = I(P|P \cap \mathbf{Q})$ is the inertia group of P over $\mathbf{Q}$. If Q is any prime unramified in K such that $\left[\frac{K/\mathbf{Q}}{Q}\right] \in I$, then $q = Q \cap \mathbf{Q}$ splits into positive principal prime divisors.* (This means that the prime ideals have generators whose images under all real embeddings of k are positive.)

This is proved for (Hilbert) class field in [4] (also in [2]).

This easily carries over to narrow class-fields.

Theorem A. *Let k be a normal number field in which a prime p ramifies with ramification index $e_p = p^r e'_p$, $p \nmid e'_p$. Let a be a primitive root modulo p^l. Then there is a t_0, $0 \leqslant t_0 \leqslant r$, with the following property: The set of primes $q \equiv a \pmod{p^l}$ which have degree $e'_p p^{t_0}$ and which split into positive principal prime ideals in k has positive density.*

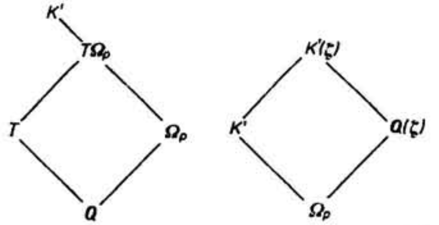
Proof. Let P be a prime ideal lying over p in the narrow class-field K of k. Let $I = I(P|p)$ be the inertia group of P over p and T the fixed field of the inertia group. Let V_1 be as usual,

$$V_1 = \{\sigma \in \mathrm{Gal}(K/\mathbf{Q}) |\ \sigma(\alpha) \equiv \alpha \pmod{P^2}\}.$$

Then V_1 is a normal subgroup of I and I/V_1 is cyclic. Let K' be the fixed field of V_1. Since V_1 is the p-Sylow subgroup of I, K'/T is a cyclic extension of degree e'_p. Let ζ denote a primitive p^l-th root of unity. Since T and $\mathbf{Q}(\zeta)$ are linearly disjoint,

$$\mathrm{Gal}(T(\zeta)/T) \cong \mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q}).$$

Let τ_a denote the automorphism such that $\tau_a(\zeta) = \zeta^a$, $\tau_a \in \mathrm{Gal}(\boldsymbol{Q}(\zeta)/\boldsymbol{Q})$. Since τ_a generates $\mathrm{Gal}(\boldsymbol{Q}(\zeta)/\boldsymbol{Q})$, $\tau_a|\Omega_p$ generates $\mathrm{Gal}(\Omega_p/\boldsymbol{Q})$ $(\Omega_p = K' \cap \boldsymbol{Q}(\zeta))$.



Now $T \cap \Omega_p = \boldsymbol{Q}$. Let σ_1 be an embedding of $T\Omega_p$ which restricts to identity on T and equals $\tau_a|\Omega_p$ on Ω_p. Then σ_1 generates $\mathrm{Gal}(T\Omega_p/T)$. There is a generator σ_p of $\mathrm{Gal}(K'/T)$ which restricts to σ_1. Now choose an embedding σ of $K'(\zeta)$ which restricts to σ_p on K' and to τ_a on $\boldsymbol{Q}(\zeta)$. Then $\sigma \in \mathrm{Gal}(K'(\zeta)/\boldsymbol{Q})$. The set of primes Q in T which have degree one over $\boldsymbol{Q}$ has density one. So the set (say) A of degree 1 primes in T which have σ as Frobenius automorphism has positive density. Now we have to prove that the primes $Q \in A$ are such that $q = Q \cap \boldsymbol{Q}$ satisfy the condition $q \equiv a \pmod{p^l}$. This follows from the fact that the Frobenius automorphism of q in $\boldsymbol{Q}(\zeta)$ is τ_a. Now suppose U is a prime in K' lying over $Q \in A$. Then $\left[\frac{K/Q}{U}\right]$ restricts to σ_p. So the order of the Frobenius automorphism $O\left(\left[\frac{K/Q}{U}\right]\right)$ (= degree of U over $Q \cap \boldsymbol{Q}$) is divisible by e_p. But since $Q \cap \boldsymbol{Q}$ splits completely into positive principal prime factors in k and Q has degree 1 over $\boldsymbol{Q}$, the degree of $q = Q \cap \boldsymbol{Q}$ is $O\left(\left[\frac{K/Q}{U}\right]\right) = e'_p p^t$, $0 \leqslant t \leqslant r$. Since there are only finitely many values of t and q's have positive density, there is a t_0 for which the corresponding q's have positive density.

Remark 1. If k_0 is a field such that k is its normal closure, then the primes of positive density in the above theorem have principal prime ideal factors in k_0. This follows from Lemma 1.

Main theorem

THEOREM 1. *Suppose p is an odd rational prime which is tamely ramified in a number field k: $p = P_1^{e_1} P_2^{e_2} \ldots P_s^{e_s}$, $p \nmid e_i$, where P_i $(i = 1, \ldots, s)$ are prime ideals of k of residue class degree f_i. Then there exists an infinite set of rational primes q of positive density in the arithmetic progression $a \pmod{p^l}$ (a being a primitive root $\bmod p^l$) which split in the following manner:*

$$q = \prod_{i=1}^{s} \prod_{j=1}^{f_i} Q_{ij}$$

where each Q_{ij} is of degree e_i and is a positive principal prime ideal.

Proof. Since p is tamely ramified in k it is tamely ramified in $\bar{k}$. It is tamely ramified in K too. Therefore, with notation as in the previous lemma, we see that $V_1 = \{1\}$ and $I(P|p)$ is cyclic and $K = K'$. Let Q be a prime in A and $\mathfrak{M}$ a prime in K lying over Q. Then the Frobenius automorphism of $\mathfrak{M}$ with respect to Q is the generator of the inertia group $I(P|p)$. Consider $\mathfrak{M} \cap \bar{k} = \mathfrak{B}$. Then the Frobenius automorphism of $\mathfrak{B}$ over $\mathfrak{B} \cap \boldsymbol{Q}$ is the generator of the inertia group $I(\mathfrak{P}|\mathfrak{P} \cap \boldsymbol{Q})$, where $\mathfrak{P} = P \cap \bar{k}$. Fix a $q = Q \cap \boldsymbol{Q}$, $Q \in A$. Then the decomposition group of $\mathfrak{B}$ over q = Inertia group of $\mathfrak{P}$ over p, i.e., $D(\mathfrak{B}|\mathfrak{B} \cap \boldsymbol{Q}) = I(\mathfrak{P}|p)$. Let $f(\mathfrak{a}|\boldsymbol{Q})$ denote the residue class degree of a prime ideal $\mathfrak{a}$ in k over $\boldsymbol{Q}$. Then

$$f(\mathfrak{B} \cap k|\boldsymbol{Q}) = \left|\frac{D(\mathfrak{B}|q)}{D(\mathfrak{B}|\mathfrak{B} \cap k)}\right| = \left|\frac{I(\mathfrak{P}|p)}{I(\mathfrak{P}|\mathfrak{P} \cap k)}\right|$$

since

$$D(\mathfrak{B}|\mathfrak{B} \cap k) = D(\mathfrak{B}|q) \cap \mathrm{Gal}(\bar{k}/k) = I(\mathfrak{P}|p) \cap \mathrm{Gal}(\bar{k}/k) = I(\mathfrak{P}|\mathfrak{P} \cap k).$$

Here, if $\mathfrak{P} \cap k = P_1$, then $f(\mathfrak{B} \cap k|q) = e_1$. Consider now $\sigma(\mathfrak{B})$. Since

$$D(\sigma(\mathfrak{B})|q) = \sigma D(\mathfrak{B}|q)\sigma^{-1} = \sigma I(\mathfrak{P}|p)\sigma^{-1} = I(\sigma(\mathfrak{P})|p),$$

if $\sigma(\mathfrak{P}) \cap k$ is P_i, then

$$f(\sigma(\mathfrak{B}) \cap k|q) = e_i.$$

Let

$$H = \mathrm{Gal}(\bar{k}/k) \quad \text{and} \quad \Phi = \Phi(\mathfrak{B}|q) = \left[\frac{\bar{k}/\boldsymbol{Q}}{\mathfrak{B}}\right].$$

Consider the orbits of the cosets of H in G under the action of Φ:

$$\{Hg_1, Hg_1\Phi, \ldots, Hg_1\Phi^{m_1-1}\},$$
$$\{Hg_2, Hg_2\Phi, \ldots, Hg_2\Phi^{m_2-1}\}, \ldots, \{Hg_n, \ldots, Hg_n\Phi^{m_n-1}\}.$$

We know that q splits as $q = \mathfrak{B}_1 \ldots \mathfrak{B}_n$ in k where $\mathfrak{B}_i = g_i(\mathfrak{B}) \cap k$ and $m_i = f(\mathfrak{B}_i|q)$ (cf. [3], Theorem 33). Now, choose $\sigma_1, \sigma_2, \ldots, \sigma_s$ such that $\sigma_i(\mathfrak{P}) \cap k = P_i$. For a fixed i, let $\sigma_i(\mathfrak{P}) = \hat{\mathfrak{P}}$ and $\sigma_i(\mathfrak{B}) = \hat{\mathfrak{B}}$. We know that $f(\hat{\mathfrak{B}} \cap k|\boldsymbol{Q}) = e_i$. Let us consider

$$\frac{D(\hat{\mathfrak{P}}|p)}{D(\hat{\mathfrak{B}}|q)D(\hat{\mathfrak{P}}|\hat{\mathfrak{P}} \cap k)}.$$

Since

$$\left|\frac{D(\hat{\mathfrak{P}}|p)}{D(\hat{\mathfrak{B}}|q)}\right| = \left|\frac{D(\hat{\mathfrak{P}}|p)}{D(\hat{\mathfrak{B}}|q)D(\hat{\mathfrak{P}}|\hat{\mathfrak{P}} \cap k)}\right| \left|\frac{D(\hat{\mathfrak{B}}|q)D(\hat{\mathfrak{P}}|\hat{\mathfrak{P}} \cap k)}{D(\hat{\mathfrak{B}}|q)}\right|,$$

$$\frac{D(\hat{\mathfrak{B}}|q)D(\hat{\mathfrak{P}}|\hat{\mathfrak{P}} \cap k)}{D(\hat{\mathfrak{B}}|q)} \cong \frac{D(\hat{\mathfrak{P}}|\hat{\mathfrak{P}} \cap k)}{D(\hat{\mathfrak{B}}|q) \cap D(\hat{\mathfrak{P}}|\hat{\mathfrak{P}} \cap k)} = \frac{D(\hat{\mathfrak{P}}|\hat{\mathfrak{P}} \cap k)}{D(\hat{\mathfrak{B}}|\hat{\mathfrak{B}} \cap k)},$$

it follows that

$$\left|\frac{D(\hat{\mathfrak{P}}|p)}{D(\hat{\mathfrak{B}}|q)D(\hat{\mathfrak{P}}|\hat{\mathfrak{P}} \cap k)}\right| = \left|\frac{D(\hat{\mathfrak{P}}|p)}{D(\hat{\mathfrak{B}}|q)}\right| \div \left|\frac{D(\hat{\mathfrak{P}}|\hat{\mathfrak{P}} \cap k)}{D(\hat{\mathfrak{B}}|\hat{\mathfrak{B}} \cap k)}\right| = f(\hat{\mathfrak{P}} \cap k|q) = f_i.$$

Let $\tau_1, \ldots, \tau_{f_i}$ be a set of coset representatives of

$$\frac{D(\hat{\mathfrak{P}}|p)}{D(\hat{\mathfrak{B}}|q)D(\hat{\mathfrak{P}}|\hat{\mathfrak{P}} \cap k)}.$$

Consider the f_i orbits

$$\{H\tau_1\sigma_i, H\tau_1\sigma_i\Phi, \dots, H\tau_1\sigma_i\Phi^{e_i-1}\},$$
$$\{H\tau_2\sigma_i, H\tau_2\sigma_i\Phi, \dots, H\tau_2\sigma_i\Phi^{e_i-1}\}, \dots, \{H\tau_{f_i}\sigma_i, \dots, H\tau_{f_i}\sigma_i\Phi^{e_i-1}\}.$$

We claim that they are distinct. Suppose $H\tau_u\sigma_i = H\tau_v\sigma_i\Phi^j$. Then

$$\tau_u\sigma_i = h\tau_v\sigma_i\Phi^j$$

which means

$$\tau_u\sigma_i\Phi^{-j}\sigma_i^{-1}\tau_v^{-1} = \tau_u\tau_v^{-1}(\tau_v\sigma_i\Phi^{-j}\sigma_i^{-1}\tau_v^{-1}) = h.$$

But $\tau_v\sigma_i\Phi^{-j}\sigma_i^{-1}\tau_v^{-1} \in D(\mathfrak{B}|q)$ since $D(\mathfrak{B}|q)$ is normal in $D(\mathfrak{P}|p)$. So τ_u and τ_v are in the same coset, contradicting our choice of τ_i's. Notice that $\tau_u\sigma_i\Phi^j(\mathfrak{P}) = \sigma_i(\mathfrak{P}) = \mathfrak{P}$. So the primes corresponding to these orbits will have degree e_i.

To complete the proof we have to show that any coset $H\sigma$ is of the form $H\tau_k\sigma_i\Phi^j$ for some i.

Suppose $\sigma(\mathfrak{P}) \cap k = P_i = \sigma_i(\mathfrak{P}) \cap k$. Then there exists $h \in \mathrm{Gal}(\bar{k}/k)$ such that

$$h\sigma(\mathfrak{P}) = \sigma_i(\mathfrak{P}), \quad \text{i.e.} \quad \sigma_i^{-1}h\sigma(\mathfrak{P}) = \mathfrak{P}.$$

Therefore

$$\sigma_i^{-1}h\sigma = \tau, \quad \tau \in D(\mathfrak{P}|p),$$

which means

$$h\sigma = \sigma_i\tau = \sigma_i\tau\sigma_i^{-1}\sigma_i.$$

Now $\sigma_i\tau\sigma_i^{-1} \in D(\mathfrak{P}|p)$. Therefore there is an h' such that

$$\sigma_i\tau\sigma_i^{-1} = h'\sigma_i\Phi^w\sigma_i^{-1}\tau_k$$

which means $h\sigma = h'\sigma_i\Phi^w\sigma_i^{-1}\tau_k\sigma_i = h'\tau_k\sigma_i\Phi^s\sigma_i^{-1}\sigma_i$ (since $D(\mathfrak{B}|q)$ is normal in $D(\mathfrak{P}|p)) = h'\tau_k\sigma_i\Phi^s$. Hence

$$H\sigma = H\tau_k\sigma_i\Phi^s.$$

This completes the proof.

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Polynomials with high multiplicity

by

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0. Introduction. Let S be a non-empty finite subset of C^n. Following Waldschmidt (see [W2], § 1.3e)) we define $\omega_M(S)$ as the minimum degree of an algebraic hypersurface having a singularity of order $\geqslant M$ at any point of S. We are looking for inequalities between $\omega_1(S)$ and $\omega_M(S)$, $M > 1$. Trivially, we have

$$\frac{1}{M}\omega_M(S) \leqslant \omega_1(S). \tag{1}$$

In the opposite sense, using powerful tools from complex analysis, Waldschmidt proved

$$\frac{1}{n}\omega_1(S) \leqslant \frac{1}{M}\omega_M(S) \tag{2}$$

(see [W2], § 7.5b)). The last inequality follows from Bombieri–Skoda's existence theorem, which in turn derives from some L^2-estimates and from existence theorems for the operator $\bar{\partial}$, due to Hörmander.

Weaker results of the following kind:

$$\frac{1}{c_n}\omega_1(S) \leqslant \frac{1}{M}\omega_M(S) \tag{2'}$$

where c_n is some constant greater than n, were obtained by Masser and Wüstholz independently (see [M] and [Wu]).

More recently, using deep arguments from projective geometry, Esnault and Viehweg (see [E–V]) have obtained the following improvement of (2):

$$\frac{\omega_1(S)+1}{n} \leqslant \frac{1}{M}\omega_M(S) \quad \text{for } n > 1.$$

A conjecture of J. P. Demailly asserts that one should have

$$\frac{\omega_1(S)+n-1}{n} \leqslant \frac{1}{M}\omega_M(S) \quad \text{for } n \geqslant 1.$$

In this paper we give some results of the type (2′) in the ring $\boldsymbol{Z}[x_1, \dots, x_n]$ with explicit bounds for the height of the polynomials.