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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

The metrical theory of complex continued fractions

by

HITOSHI NAKADA (Yokohama)

In the previous paper [2], the author constructed a transformation $\bar{T}$ which is associated with A. Schmidt's complex continued fractions over the Gaussian field. As a result, we found some metrical properties of these complex continued fractions. Schmidt had defined three sequences of convergents $\{p_{l,n}/q_{l,n}\}$, $l = 0$, 1 and ∞, and we established in [2] the law of large numbers for the number of solutions to the diophantine inequality $|z-(p/q)| < c/|q|^2$, $c > 0$, for these convergents. In the present paper, it is shown that the rate of the growth of $|q_{l,n}|$ is exponential for almost all z, and its explicit rate is given. Furthermore, the rate of the convergence of $p_{l,n}/q_{l,n}$ to z is determined.

MAIN THEOREM. *For almost all complex numbers z and any $l = 0$, 1 or ∞, we have*

(i) $$\lim_{n\to\infty} (\log|q_{l,n}|)/n = E/\pi,$$

(ii) $$\lim_{n\to\infty} (\log|z-(p_{l,n}/q_{l,n})|)/n = -2E/\pi,$$

where

$$E = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^2}.$$

We note that the method of the proof is quite different from the well-known method used by Billingsley [1] to prove Lévy's result for the case of simple continued fractions. In [2], we use the ergodic theorem and here, we combine this with the Borel–Cantelli lemma in an interesting way. This arises naturally since the quantities must be compared. The probability space $(\bar{X}, \bar{\mu})$ includes complex numbers as a set of measure 0. However, $\bar{T}$ on $\bar{X}$ induces a kind of contractive and expansive structure. This structure helps us to see that the set of measure one, for which our property holds, includes a set of relatively measure one of the complex numbers.

In Section 1, we recall some fundamental definitions and properties from [2]. In Section 2, we show some essential properties of $q_{l,n}$ and $p_{l,n}$ and in Section 3 we give the proof of the Main Theorem.

1. Let C and C^* be two complex planes. For a 2×2 matrix $A=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$ with Gaussian integer coefficients and $|\det A|=1$, we consider g_A, the linear fractional transformation defined by

$$g_A(z)=\frac{az+b}{cz+d}$$

on $(C\cup\{\infty\})\cup(C^*\cup\{\infty\})$, where g_A acts separately on each plane if $\det A=\pm 1$, and interchanges the planes if $\det A=\pm i$. (Here and henceforth, we use the same symbol ∞ for the two points of infinity associated with the two planes.)

We put

$$T_+=\{z=x_1+x_2 i:\ x_2\geqslant 0\}\cup\{\infty\}\subset C,$$
$$T_+^*=\{z=x_1+x_2 i:\ 0\leqslant x_1\leqslant 1,\ x_2\geqslant\sqrt{x_1-x_1^2}\}\cup\{\infty\}\subset C^*,$$
$$T_-=\{z=x_1-x_2 i:\ 0\leqslant x_1\leqslant 1,\ x_2\geqslant\sqrt{x_1-x_1^2}\}\cup\{\infty\}\subset C,$$
$$T_-^*=\{z=x_1-x_2 i:\ x_2\geqslant 0\}\cup\{\infty\}\subset C^*$$

and define partitions

$$\{\mathcal{V}_l,\ \mathcal{E}_l\ (l=1,2,3),\ \mathcal{C},\ \{\infty\}\}$$

of T_+ (Fig. 1(i)) and

$$\{\mathcal{V}_l^*\ (l=1,2,3),\ \mathcal{C}^*,\ \{\infty\}\}$$

of T_+^* (Fig. 1(ii)). The transformation $\bar{T}$ on $\bar{X}=(T_-\times T_+)\cup(T_-^*\times T_+^*)$ is defined by

$$\bar{T}(z_1,z_2)=(g_A(z_1),g_A(z_2))$$

with

$$(1)\qquad A=\begin{cases} V_l^{-1} & \text{if } z_2\in\mathcal{V}_l\cup\mathcal{V}_l^*,\\ E_l^{-1} & \text{if } z_2\in\mathcal{E}_l,\\ C^{-1} & \text{if } z_2\in\mathcal{C}\cup\mathcal{C}^*,\\ I & \text{if } z_2=\infty;\end{cases}$$

$$V_1=\begin{bmatrix}1 & i\\ 0 & 1\end{bmatrix},\quad V_2=\begin{bmatrix}1 & 0\\ -i & 1\end{bmatrix},\quad V_3=\begin{bmatrix}1-i & i\\ -i & 1+i\end{bmatrix},$$

$$E_1=\begin{bmatrix}1 & 0\\ 1-i & i\end{bmatrix},\quad E_2=\begin{bmatrix}1 & -1+i\\ 0 & i\end{bmatrix},\quad E_3=\begin{bmatrix}i & 0\\ 0 & 1\end{bmatrix},$$

$$C=\begin{bmatrix}1 & -1+i\\ 1-i & i\end{bmatrix},\quad I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

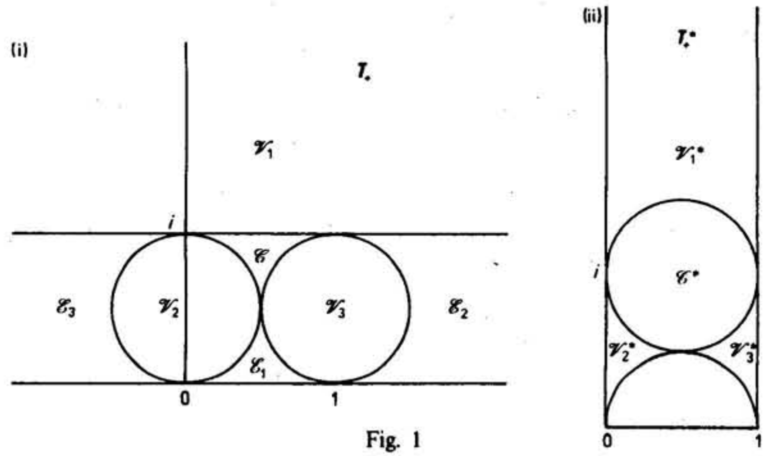

Fig. 1

We also define T on $X=T_+\cup T_+^*$ (which is just a projected version of $\bar{T}$) by

$$T(z_2)=g_A(z_2)$$

for $z_2\in X$ with A in (1).

THEOREM 0 ([2]). *The dynamical system $(\bar{X},\bar{T})$ is ergodic with respect to the invariant measure $\bar{\mu}$ defined by*

$$d\bar{\mu}=\frac{2dx_1\,dx_2\,dx_1'\,dx_2'}{\pi^2|z_1-z_2|^4},$$

with $z_1=x_1+x_2 i$ and $z_2=x_1'+x_2' i$.

For $(z_1,z_2)\in\bar{X}$, we define for $n\geqslant 0$

$$t_n=V_l,\quad E_l,\quad C\quad\text{or}\quad I$$

according to whether the second coordinate of $\bar{T}^n(z_1,z_2)$ is in $\mathcal{V}_l\cup\mathcal{V}_l^*$, $\mathcal{E}_l$, $\mathcal{C}\cup\mathcal{C}^*$ or $\{\infty\}$, respectively; for $n<0$, similarly we define

$$t_n=V_l^{-1},\quad E_l^{-1},\quad C^{-1}\quad\text{or}\quad I.$$

Thus we get a sequence of matrices $\dots t_{-2}t_{-1}t_0t_1t_2\dots$

It is easy to see that $\bar{T}$ corresponds to the shift operator on a subset of the set of sequences of matrices. We may identify a complex number $z\in Z$ with $(\infty,z)\in\bar{X}$ (in this sense, X is regarded as a set of measure 0 in $\bar{X}$). We note that for any (z_1,z_2) and $(z_1',z_2)\in\bar{X}$ with $T^n z_2\neq 0$, 1 or ∞ for $n\geqslant 0$, the distance between the first coordinates of $\bar{T}^n(z_1,z_2)$ and $\bar{T}^n(z_1',z_2)$ tends to 0 as n goes to $+\infty$. We will see in Section 3 that this decay rate is exponential (Lemma 3 and the Main Theorem).

Now define ∞, 0 and 1 convergents $p_{\infty,n}/q_{\infty,n}$, $p_{0,n}/q_{0,n}$ and $p_{1,n}/q_{1,n}$, $n \geq 1$, by

$$\begin{bmatrix} p_{\infty,n} \\ q_{\infty,n} \end{bmatrix} = (t_0 t_1 \dots t_{n-1}) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} p_{0,n} \\ q_{0,n} \end{bmatrix} = (t_0 \dots t_{n-1}) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$(2) \qquad \begin{bmatrix} p_{1,n} \\ q_{1,n} \end{bmatrix} = (t_0 \dots t_{n-1}) \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

that is, $p_{l,n}/q_{l,n} = (t_0 \dots t_{n-1})(l)$ for $l = \infty, 0$ and 1. (Here, we do not distinguish 0,1 in C from those in C^*.) More generally, it is possible to define l-convergents for any Gaussian integer l in the same way. The particular advantage of the choice $l = \infty, 0$ and 1 is shown by Theorem 2.5 of Schmidt [4]. Though $t_0 \dots t_{n-1}$ and $p_{l,n}/q_{l,n}$ depend on z, we do not bother mentioning this unless it is not clear from the context. We always assume that n is a positive integer.

2. From the definition (2), we have

$$(3) \qquad t_0 \cdot t_1 \dots t_{n-1} = \begin{bmatrix} p_{\infty,n} & p_{0,n} \\ q_{\infty,n} & q_{0,n} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p_{1,n} \\ q_{1,n} \end{bmatrix} = \begin{bmatrix} p_{\infty,n} + p_{0,n} \\ q_{\infty,n} + q_{0,n} \end{bmatrix}.$$

It is easy to see the following:

LEMMA 1. (i) *If* $t_n = V_1, E_2$ *or* E_3, *then*

$$p_{\infty,n+1} = p_{\infty,n} \text{ or } ip_{\infty,n}, \quad q_{\infty,n+1} = q_{\infty,n} \text{ or } iq_{\infty,n}.$$

(ii) *If* $t_n = V_2, E_3$ *or* E_1, *then*

$$p_{0,n+1} = p_{0,n} \text{ or } ip_{0,n}, \quad q_{0,n+1} = q_{0,n} \text{ or } iq_{0,n}.$$

(iii) *If* $t_n = V_3, E_1$ *or* E_2, *then*

$$p_{1,n+1} = p_{1,n} \text{ or } ip_{1,n}, \quad q_{1,n+1} = q_{1,n} \text{ or } iq_{1,n}.$$

Moreover, we see the following:

LEMMA 2. (i) *If* $T^n(z) \in T_+$ (*or* $T^n(z) \in T_+^*$), *then* $-q_{0,n}/q_{\infty,n}$ *and* $-p_{0,n}/p_{\infty,n} \in T_-$ (*or* T_-^*, *respectively*).

(ii) *If* $T^{n-1}(z) \in T_+^*$ *and* $t_{n-1} = C$, *then* $-q_{0,n}/q_{\infty,n}$ *and* $-p_{0,n}/p_{\infty,n} \in \{z: |z-(1/2-i)| \leq 1/4\} \subset T_-$, *see Fig.* 2.

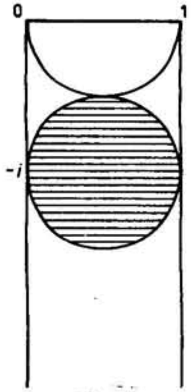

Fig. 2

Proof. Since $-q_{0,n}/q_{\infty,n} = (t_0 \dots t_{n-1})^{-1}(\infty)$ and $-p_{0,n}/p_{\infty,n} = (t_0 \dots \dots t_{n-1})^{-1}(0)$ by (3), we have

$$\left.\begin{array}{l} \bar{T}^n(\infty, z) = (-q_{0,n}/q_{\infty,n}, T^n(z)) \\ \bar{T}^n(0, z) = (-p_{0,n}/p_{\infty,n}, T^n(z)) \end{array}\right\} \in \bar{X},$$

whenever $z \in X$. This implies the assertion (i). The assertion (ii) follows from the fact that

$$g_{C^{-1}}(T_-^*) = \{z: |z-(1/2-i)| \leq 1/4\} \subset C.$$

Let $S = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. It is easy to see that

$$(4) \qquad S^2 = S^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad S(1) = S^{-1}(0) = \infty \quad \text{and} \quad S(T_+^*) = T_+^*.$$

Furthermore it is possible to show that

$$t_n(S(z)) = St_n(z)S^{-1}.$$

Thus we have

$$\begin{bmatrix} p_{\infty,n}(S(z)) & p_{0,n}(S(z)) \\ q_{\infty,n}(S(z)) & q_{0,n}(S(z)) \end{bmatrix} = S \begin{bmatrix} p_{\infty,n}(z) & p_{0,n}(z) \\ q_{\infty,n}(z) & q_{0,n}(z) \end{bmatrix} S^{-1}$$

and

$$(5) \qquad \begin{array}{l} p_{\infty,n}(S(z)) = q_{1,n}(z), \\ p_{0,n}(S(z)) = -q_{\infty,n}(z), \\ p_{1,n}(S(z)) = q_{0,n}(z). \end{array}$$

THEOREM 1. (i) *If either* $T^n(z) \notin C^*$ *or* $t_n \in C$, *then*

$$|q_{l,n+1}| \geq |q_{l,n}| \quad \text{and} \quad |p_{l,n+1}| \geq |p_{l,n}| \quad \text{for } l = \infty, 0, \text{ and } 1.$$

(ii) *If* $|q_{l,n+1}| < |q_{l,n}|$ (*or* $|p_{l,n+1}| < |p_{l,n}|$) *for some* $l = \infty, 0$ *or* 1, *then* $t_n = C$, $T^n(z) \in C^*$ *and*

$$|q_{l,n}| < \sqrt{2}\,|q_{l,n+1}|, \quad |p_{l,n}| < \sqrt{2}\,|p_{l,n+1}|.$$

In addition, if $p_{l,n+1}/q_{l,n+1} = \dots = p_{l,n+k-1}/q_{l,n+k-1} \neq p_{l,n+k}/q_{l,n+k}$, *then*

$$|q_{l,n+k}| > |q_{l,n}| \quad \text{and} \quad |p_{l,n+k}| > |p_{l,n}|.$$

Proof. From the relation (5), we only need to show the assertion for $q_{\infty,n}$, $p_{\infty,n}$ and $q_{0,n}$. First we consider the case of $l = \infty$. We put $t_n = V_2$, then we have by (2) and (3)

$$\begin{bmatrix} p_{\infty,n+1} \\ q_{\infty,n+1} \end{bmatrix} = \begin{bmatrix} p_{\infty,n} & p_{0,n} \\ q_{\infty,n} & q_{0,n} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{\infty,n} - ip_{0,n} \\ q_{\infty,n} - iq_{0,n} \end{bmatrix}.$$

So we get

$$\frac{q_{\infty,n+1}}{q_{\infty,n}} = 1+i\left(-\frac{q_{0,n}}{q_{\infty,n}}\right) \quad \text{and} \quad \frac{p_{\infty,n+1}}{p_{\infty,n}} = 1+i\left(-\frac{p_{0,n}}{p_{\infty,n}}\right).$$

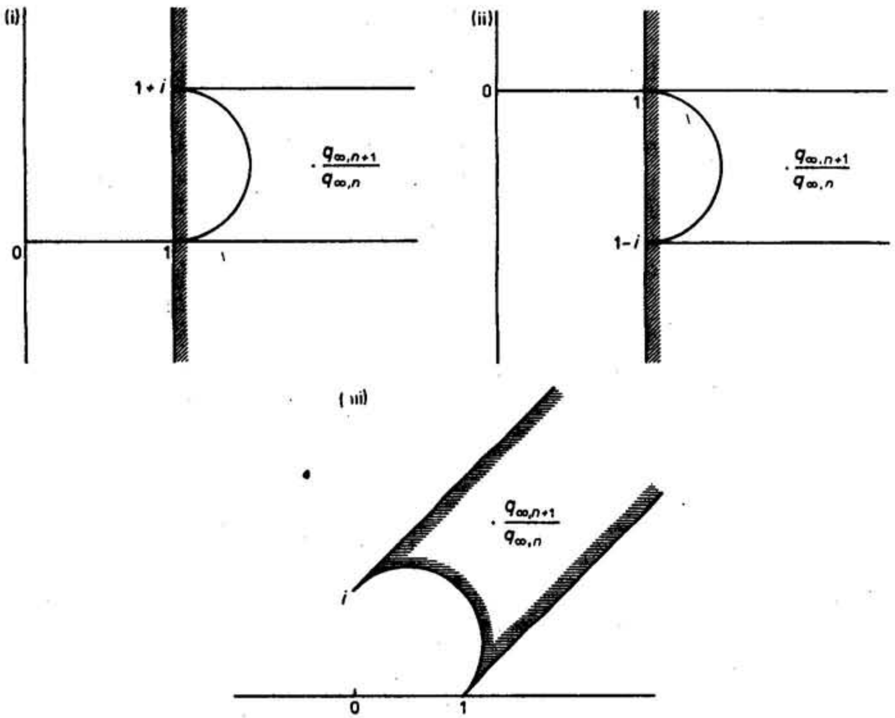

Fig. 3

From Lemma 2(i), it turns out that

$$1+i\left(-\frac{q_{0,n}}{q_{\infty,n}}\right) \quad \text{and} \quad 1+i\left(-\frac{p_{0,n}}{p_{\infty,n}}\right) \in \{z = x+iy:\ x \geqslant 1\},$$

see Fig. 3(i). Thus we see

$$(6) \qquad \left|\frac{q_{\infty,n+1}}{q_{\infty,n}}\right| > 1 \quad \text{and} \quad \left|\frac{p_{\infty,n+1}}{p_{\infty,n}}\right| \geqslant 1.$$

(Equality holds if and only if $p_{0,n} = 0$.) If $t_n = V_3$, then we have

$$\begin{bmatrix} p_{\infty,n+1} \\ q_{\infty,n+1} \end{bmatrix} = \begin{bmatrix} (1-i)p_{\infty,n} - ip_{0,n} \\ (1-i)q_{\infty,n} - iq_{0,n} \end{bmatrix}$$

and

$$\frac{q_{\infty,n+1}}{q_{\infty,n}} = (1-i)+i\left(-\frac{q_{0,n}}{q_{\infty,n}}\right), \quad \frac{p_{\infty,n+1}}{p_{\infty,n}} = (1-i)+i\left(-\frac{p_{0,n}}{p_{\infty,n}}\right),$$

Thus we have (6) again, see Fig. 3(ii). If $t_n = E_1$ or C, then

$$\frac{q_{\infty,n+1}}{q_{\infty,n}} = 1+(-1+i)\left(-\frac{q_{0,n}}{q_{\infty,n}}\right), \quad \frac{p_{\infty,n+1}}{p_{\infty,n}} = 1+(-1+i)\left(-\frac{p_{0,n}}{p_{\infty,n}}\right).$$

Then we have (6) when $T^n(z) \notin T_+^*$, see Fig. 3 (iii). Now if $T^n(z) \in T_+^*$ and $t_n = C$, then $|q_{\infty,n+1}| < |q_{\infty,n}|$ when

$$-q_{0,n}/q_{\infty,n} \in \{z:\ |z-(1/2+i\cdot 1/2)| < 1/\sqrt{2}\},$$

see Fig. 4. In this case, it is easy to see that

$$|q_{\infty,n+1}/q_{\infty,n}| \geqslant 1/\sqrt{2}.$$

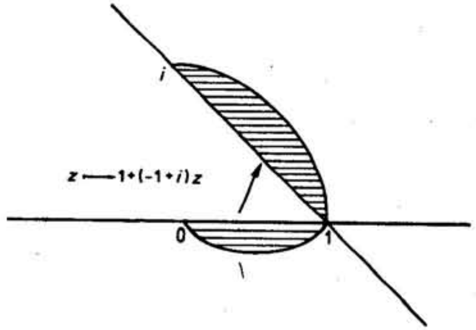

Fig. 4

Furthermore, we see by Lemma 2 (ii) that

$$-q_{0,n+1}/q_{\infty,n+1} \in \{z:\ |z-(1/2-i)| \leqslant 1/4\}.$$

So if $t_{n+1} \neq V_1$, E_2 or E_3, in addition, then we have

$$|q_{\infty,n+2}/q_{\infty,n+1}| > \sqrt{2}.$$

On the other hand, if $t_{n+1} = V_1$, E_2 or E_3, then we also have the same inequality for $|q_{\infty,n+k}/q_{\infty,n+k-1}|$ with k as in the assumption of the theorem. The same holds for $p_{\infty,n}$.

Next, let $l = 0$ and suppose $t_n = V_1$. We see that

$$\begin{bmatrix} p_{0,n+1} \\ q_{0,n+1} \end{bmatrix} = \begin{bmatrix} ip_{\infty,n} + p_{0,n} \\ iq_{\infty,n} + q_{0,n} \end{bmatrix}$$

and

$$q_{0,n+1}/q_{0,n} = 1 - i(-q_{\infty,n}/q_{0,n}).$$

This corresponds to the case of Fig. 3 (ii) and we have

$$|q_{0,n+1}/q_{0,n}| > 1.$$

In the same way, if $t_n = V_3$, then

$$q_{0,n+1}/q_{0,n} = (1+i) - i(-q_{\infty,n}/q_{0,n})$$

and this is the case of Fig. 3 (i), and if $t_n = E_2$ or C, then we have

$$q_{0,n+1}/q_{0,n} = i + (1-i)(-q_{\infty,n}/q_{0,n}).$$

Similarly to the case of $l = \infty$, we get the assertion of the theorem.

3. In this section, we discuss the metrical theory of the convergents. We put $(w(z, n), z(n)) = \bar{T}^n(w, z)$ for $(w, z) \in \bar{X}$.

LEMMA 3. *For any complex number z ($\neq 0, 1$) with (w_1, z) and $(w_2, z) \in \bar{X}$, we have*

$$|w_1(z, n) - w_2(z, n)| = O(|q_{\infty,n}|^{-2})$$

as n tends to $+\infty$.

Proof. From (3), we have

$$w_1(z, n) = (q_{0,n} w_1 - p_{0,n})/(-q_{\infty,n} w_1 + p_{\infty,n}),$$
$$w_2(z, n) = (q_{0,n} w_2 - p_{0,n})/(-q_{\infty,n} w_2 + p_{\infty,n}).$$

Here we suppose that w_1 and w_2 are not both equal to ∞ (this is not essential). So we see that

$$|w_1(z, n) - w_2(z, n)| = \frac{|w_1 - w_2|}{|q_{\infty,n} w_1 - p_{\infty,n}|\,|q_{\infty,n} w_2 - p_{\infty,n}|}$$
$$= \frac{|w_1 - w_2|}{|q_{\infty,n}|^2 \,|w_1 - (p_{\infty,n}/q_{\infty,n})|\,|w_2 - (p_{\infty,n}/q_{\infty,n})|}.$$

Since $\{p_{\infty,n}/q_{\infty,n}\}$ converges to z, we get the assertion of the lemma.

LEMMA 4. *If we fix $\hat{w} = \infty$, then we have for any $\varepsilon > 0$,*

$$\#\{n: |\hat{w}(z, n)| > e^{n\varepsilon}\} < +\infty,$$
$$\#\{n: |\hat{w}(z, n)| < e^{-n\varepsilon}\} < +\infty,$$
$$\#\{n: |\hat{w}(z, n) - 1| < e^{-n\varepsilon}\} < +\infty$$

for a.a. $z \in X$.

Proof. For a fixed $\varepsilon > 0$, we put

$$A_n = \{(w, z) \in \bar{X}: |w| > e^{n\varepsilon}/2\}.$$

Since $\sum \bar{\mu}(A_n) < +\infty$,

(7) $$\#\{n: \bar{T}^n(w, z) \in A_n\} < +\infty$$

for a.a. $(w, z) \in \bar{X}$ by the Borel–Cantelli lemma. Hence for a.a. $z \in X$, there exists w such that (w, z) has the property (7). We choose such a point $(w, z) \in \bar{X}$. From Lemma 3, there exists a positive integer n_0 such that $n \geq n_0$ implies

$$|w(z, n) - \hat{w}(z, n)| < e^{\varepsilon}/2.$$

So $\bar{T}^n(z, w) \notin A_n$ implies $|\hat{w}(z, n)| < e^{n\varepsilon}$ whenever $n \geq n_0$. This shows

$$\#\{n: |\hat{w}(z, n)| > e^{n\varepsilon}\} < +\infty.$$

By using the same method, we see that

$$\#\{n: |\tilde{w}(z, n)| > e^{n\varepsilon}\} < +\infty$$

with $\tilde{w} = 1$. On the other hand, it is easy to see that

$$\big(S(\tilde{w}(S^{-1}(z), n)), ST^n S^{-1}(z)\big) = \bar{T}^n(\hat{w}, z)$$

and

$$\{w: |w| > e^{n\varepsilon}\} = S\{w: |w - 1| < e^{-n\varepsilon}\}.$$

Thus we have

$$\#\{n: |\hat{w}(z, n) - 1| < e^{-n\varepsilon}\} < +\infty$$

for a.a. z. Finally, by the equality

$$\big(\bar{w}(S(z), n), S^{-1} T^n S(z)\big) = \bar{T}^n(\hat{w}, z)$$

with $\bar{w} = 0$, it is possible to show that

$$\#\{n: |\hat{w}(z, n)| < e^{-n\varepsilon}\} < +\infty$$

for a.a. z. This completes the proof of the lemma.

LEMMA 5. *For a.a. $z \in X$ and any l and l' ($= \infty, 0$ or 1), we have*

$$\lim_{n\to\infty} (\log |q_{l,n}/q_{l',n}|)/n = 0.$$

Proof. From Lemma 4, it is easy to see that

(8) $$\lim_{n\to\infty} (\log |q_{0,n}/q_{\infty,n}|)/n = 0$$

for a.a. $z \in X$, since $\hat{w}(z, n) = -q_{0,n}/q_{\infty,n}$. Moreover, since

$$\log |q_{1,n}/q_{\infty,n}| = \log |1 + (q_{0,n}/q_{\infty,n})|,$$

we have

(9) $$\lim_{n\to\infty} (\log |q_{1,n}/q_{\infty,n}|)/n = 0$$

for a.a. $z \in X$, by Lemma 4, again. The rest of the assertion follows from (8) and (9).

Now we can prove the Main Theorem:

Proof of the Main Theorem. We put

$L_N = \#\{p/q$: there exists l $(=\infty, 0$ or $1)$, and n, $1 \leqslant n \leqslant N$, such that

$$p/q = p_{l,n}/q_{l,n} \text{ and } |z-(p/q)| < 1/(2|q|^2)\}.$$

From Theorem 2.5 of [4] and Theorem 1 of this paper, we have

$$\frac{L_N}{\log \bar{N}} < \frac{\#\{p/q\colon |q| \leqslant \bar{N}, |z-(p/q)| < 1/(2|q|^2), (p,q)=1\}+3}{\log \bar{N}},$$

$$\frac{L_N}{\log \underline{N}} > \frac{\#\{p/q\colon |q| \leqslant \underline{N}, |z-(p/q)| < 1/(2|q|^2), (p,q)=1\}-3}{\log \underline{N}},$$

where $\bar{N} = \max\{|q_{l,N}|; l=\infty, 0, 1\}$ and $\underline{N} = \min\{|q_{l,N}|; l=\infty, 0, 1\}$. From [2] and [3], we know

$$\lim_{N\to\infty} L_N/N = 3/4\pi$$

and

$$\lim_{Q\to\infty} \#\{p/q\colon |q| < Q, |z-(p/q)| < 1/(2|q|^2), (p,q)=1\}/\log Q$$

$$= \frac{\pi^2}{8\zeta_{-1}(2)} = \frac{\pi^2}{8\zeta(2)E}$$

for a.a. z. Thus, by Lemma 5, we have

$$\lim_{N\to\infty} (\log |q_{l,N}|)/N = (8\zeta(2)E/\pi^2)(3/4\pi) = E/\pi$$

for a.a. z.

To prove (ii), we note the following:

$$|z-(p_{\infty,n}/q_{\infty,n})| = 1/(|q_{\infty,n}|^2 |z(n)-\hat{w}(z,n)|).$$

Similarly to the proof of Lemma 4, we see that

$$\lim_{n\to\infty} (\log |z(n)-\hat{w}(z,n)|)/n = 0$$

for a.a. z. Moreover, by using the symmetry of 0, 1 and ∞ with respect to S, we get the desired result.

Finally, we compute the entropy of T with respect to μ. We denote by δ_n the Euclidean diameter of the circle $(t_0 \ldots t_{n-1})(T_+)$ (if there exists k, $0 \leqslant k \leqslant n-1$, such that $t_k \neq V_1, E_2$ or E_3, otherwise $\delta_n = \infty$). Let $\{n(m)\}$ be a subsequence of $\{n\}$ so that $T^{n(m)}(z) \in T_+$.

Proposition. *For a.a. $z \in X$, we have*

$$\lim_{m\to\infty} (\log \delta_{n(m)})/n(m) = -2E/\pi.$$

Remark. By the Shannon–McMillan–Breiman theorem, it turns out that the entropy $h(T, \mu)$ is equal to $4E/\pi$.

Proof. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|ad-bc| = 1$, then the radius of $g_A(T_+)$ is equal to

$$[\{c\bar{c}+d\bar{d}+(c+d)\overline{(c+d)}\}^2 - 2\{(c\bar{c})^2+(d\bar{d})^2+((c+d)\overline{(c+d)})^2\}]^{-1/2}$$
$$= -(c\bar{d}-\bar{c}d)^{-1}.$$

From this, we have

$$\delta_n = 1/|\mathrm{Im}\, q_{\infty,n}\bar{q}_{0,n}| = |q_{\infty,n}|^{-2} |\mathrm{Im}(-q_{0,n}/q_{\infty,n})|^{-1}.$$

If $T^n(z) \in T_+$, then for a.e. $z \in X$, we see that

$$-\varepsilon < (\log |\mathrm{Im}(-q_{0,n}/q_{\infty,n})|)/n < \varepsilon$$

for sufficiently large n (by using the Borel–Cantelli lemma). This shows the assertion of the Proposition.

Note. By using the same method, it is possible to get a similar result for the transformation $\hat{T}$ in [2]. Since the constant K' of Theorem 7.5 in [2] is equal to $((24/\sqrt{15})(\arccos(1/4)) - 2\pi)\pi/2$, we see the explicit value of the constant L of problem 3 in [5]:

$$L = E/((24/\sqrt{15})(\arccos(1/4)) - 2\pi).$$

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References

[1] P. Billingsley, *Ergodic Theory and Information*, J. Wiley, 1965.

[2] H. Nakada, *On ergodic theory of A. Schmidt's complex continued fractions over gaussian field*, Monatsh. Math. 105 (1988), 131–150.

[3] — *On metrical theory of diophantine approximation over imaginary quadratic field*, Acta Arith. 51 (1988), 393–403.

[4] A. L. Schmidt, *Diophantine approximation of complex numbers*, Acta Math. 134 (1975), 1–85.

[5] — *Ergodic theory for complex continued fractions*, Monatsh. Math. 93 (1983), 39–62.

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