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On strong uniform distribution

by

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1. Introduction. For any real number x, let $\langle x\rangle = x-[x]$, where $[x]$ denotes the largest integer not greater than x. In this paper in answer to a question of R. C. Baker [3] we use ergodic theory to prove the following theorem.

THEOREM 1. *For a finite set of pairwise coprime integers $p_1, \ldots, p_r$, exceeding one, let $(m_k)_{k=1}^{\infty}$ denote the sequence of integers they generate multiplicatively, once ordered by size. Then given any function $f \in L^1([0, 1))$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\langle m_k x\rangle) = \int_0^1 f(t)\,dt \quad a.e. \tag{1}$$

Here and henceforth in this paper it is to be understood that pointwise convergence is always with respect to Lebesgue or Haar measure (depending on context), unless otherwise stated. Theorem 1, with the stronger assumption that f is bounded and measurable, instead of just being integrable, is due to J. M. Marstrand [11, Cor. 6.3]. Theorem 1 and all the other results in this paper are consequences of the following more general ergodic theorem which is itself a special case of a theorem due to T. Bewley [4].

THEOREM 2. *For a probability space (X, β, λ), suppose the finite set of measurable commuting maps $T_i\colon X \to X$ $(i = 1, 2, \ldots, r)$ are all measure preserving, that is, $\lambda(T_i^{-1}B) = \lambda(B)$ $(i = 1, \ldots, r)$, for each set $B\in\beta$. Suppose also that at least one, T_e (say), of the maps is ergodic, that is, for any $B\in\beta$, if $T_e^{-1}B = B$, then $\lambda(B)$ is either zero or one. For a collection of finite subsets $(N_t)_{t=0}^{\infty}$ of the set of r-tuples of integers $\mathbf{Z}^r$, assume the following are true:*

(i) *$N_{t_1} \subset N_{t_2}$ if $t_1 \leqslant t_2$;*

(ii) *For each $h = (h_1, \ldots, h_r)\in \mathbf{Z}^r$ and each t let*

$$h+N_t = \{(h_1+s_1, \ldots, h_r+s_r)\colon (s_1, \ldots, s_r)\in N_t\},$$

then for all $h\in \mathbf{Z}^r$,

$$\lim_{t\to\infty} \frac{\#\{(h+N_t)\Delta N_t\}}{\# N_t} = 0;$$

(iii) If $N_t - N_t = \{x-y\colon x, y \in N_t\}$, then there exists a positive constant K (possibly dependent on $(N_t)_{t=0}^{\infty}$ and r, but not dependent on t) such that

$$\#\{N_t - N_t\} \leqslant K \# N_t;$$

(iv) If for some positive integer t, $(s_1, \ldots, s_r) \in N_t$, with s_i negative for some i, this implies T_i is invertible off a set of measure zero.

Then if s represents $(s_1, \ldots, s_r) \in \mathbf{Z}^r$, for any function $f \in L^1(X, \beta, \lambda)$,

$$\lim_{t\to\infty} \frac{1}{\# N_t} \sum_{s \in N_t} f(T_1^{s_1} \ldots T_r^{s_r} x) = \int_X f\, d\lambda, \tag{2}$$

λ almost everywhere.

The term strong uniform distribution refers to a collection of results related to a well-known conjecture due to A. Khinchin [9] which said that if $S \subset [0, 1)$ has indicator function I_S and positive Lebesgue measure $|S|$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} I_S(\langle kx \rangle) = |S| \quad \text{a.e.}$$

Khinchin's conjecture was eventually disproved by J. M. Marstrand [11, Cor. 3.3 & Thm. 5.1], however, not before it had been shown by D. A. Raikov [13] that if p is an integer greater than one, then for any function $f \in L^1([0, 1))$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\langle p^k x \rangle) = \int_0^1 f(t)\, dt \quad \text{a.e.}$$

Raikov's theorem was later shown by F. Riesz [14] to be a consequence of the pointwise ergodic theorem due to G. Birkhoff [6, p. 11]. The sequences $(p^k)_{k=1}^{\infty}$ and $(k)_{k=1}^{\infty}$ are both multiplicatively generated by integers. In light of Marstrand's disproof of Khinchin's conjecture and Theorem 1, a natural question which now arises is whether there exists a sequence of strictly increasing integers $(m_k^*)_{k=1}^{\infty}$, generated by a countable set of distinct integers, such that (1) remains true with $(m_k)_{k=1}^{\infty}$ replaced by $(m_k^*)_{k=1}^{\infty}$. More ambitiously, can we give necessary and sufficient conditions for a set of generators to have this property. These questions seem quite hard however at the present time. The proof of Theorem 1 in light of Theorem 2 may be viewed as an extension of F. Riesz's above-mentioned observation.

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2. Derivation of results. Without assuming that one of the maps T_i $(i = 1, 2, \ldots, r)$ is ergodic, the statement that the left-hand side of (2) converges λ almost everywhere to a limit $f_*(x)$ (say), in $L^1(X, \beta, \lambda)$, such that $\int_X f_*\, d\lambda = \int_X f\, d\lambda$ and $f_*(T_i x) = f_*(x)$ λ almost everywhere $(i = 1, 2, \ldots, r)$, is a special case of Theorem 3 in [4]. See also [5] and [16] for earlier versions of this theorem. To derive Theorem 2 we need only observe that $f_*(x)$ being T_e invariant forces it to be constant λ almost everywhere [17, p. 28].

We now turn to the derivation of the rest of the results from Theorem 2. Let G denote a compact Abelian group whose topology has a countable base. We say a map $E\colon G \to G$ is an *epimorphism* if it is continuous, $E(G) = G$ and given elements $g_1, g_2 \in G$, $E(g_1 + g_2) = E(g_1) + E(g_2)$. Here "+" denotes the group operation on G. We call maps $R_g\colon G \to G$, defined for $x \in G$ by $R_g(x) = x + g$, for some $g \in G$, *rotations*. All epimorphisms and rotations preserve Haar measure on G [17, p. 21] and an epimorphism or rotation A is ergodic if and only if there exists an element $x \in G$ such that $\{A^n(x)\colon n \in \mathbf{Z}_+\}$ is dense in G [17, p. 31]. Here and henceforth in this paper for any subset V of $\mathbf{R}^r$, the set of r-tuples of real numbers, let V_+ denote the subset of V consisting of elements with non-negative coordinates. Theorem 2 has the following corollary.

COROLLARY 3. *Let A_i $(i = 1, 2, \ldots, r)$ denote a finite set of commuting maps which are either all epimorphisms or all rotations of a compact, connected, Abelian group G whose topology has a countable base. Assume for one of the maps, A_e (say), and some $g_* \in G$ that $\{A_e^n(g_*)\colon n \in \mathbf{Z}_+\}$ is dense in G. Suppose $(N_t)_{t=0}^{\infty}$ denotes a class of finite subsets of $\mathbf{Z}^r$ satisfying* (i), (ii), (iii) *and* (iv) *of Theorem 2. Corresponding to each point $s = (s_1, \ldots, s_r) \in N_t$, for some $t \in \mathbf{Z}_+$, we associate the map $A_1^{s_1} \ldots A_r^{s_r}$, of G. Then for any function $f \in L^1(G)$ we have*

$$\lim_{t\to\infty} \frac{1}{\# N_t} \sum_{s \in N_t} f(A_1^{s_1} \ldots A_r^{s_r}(g)) = \int_G f(h)\, dh \quad \text{a.e.}$$

We now turn to the special case where $G = \mathbf{T}^m$, the m-dimensional torus $(1 \leqslant m \leqslant \infty)$. By $\mathbf{T}^{\infty}$ we mean the product of a countable number of copies of $\mathbf{T}$, given the Tikhonov product topology. For $y = (y_k)_{k=1}^m \in \mathbf{R}^m$, we use the abbreviation $\langle y \rangle$ to denote $(\langle y_k \rangle)_{k=1}^m$. In the case where all the maps A_i $(i = 1, \ldots, r)$ are epimorphisms, Corollary 3 has the following special case.

THEOREM 4. *Let P_i $(i = 1, \ldots, r)$ denote a finite set of commuting integer entry $m \times m$ matrices $(1 \leqslant m < \infty)$ such that $|\det(P_i)|$ is greater than one for each i and at least one of which has no roots of unity among its eigenvalues. Suppose $(M_k)_{k=1}^{\infty}$ denotes the set of $m \times m$ matrices,*

$$\{P_1^{s_1} \ldots P_r^{s_r}\colon (s_1, \ldots, s_r) \in \mathbf{Z}_+^r\},$$

given any order such that $k < k'$ only if $|\det(M_k)| \leqslant |\det(M_{k'})|$. For each positive integer k, let

$$a_k = \#\{(s_1, \ldots, s_r) \in \mathbf{Z}_+^r\colon M_k = P_1^{s_1} \ldots P_r^{s_r}\}$$

and let

$$D_N = a_1 + \ldots + a_N \quad (N = 1, 2, \ldots).$$

Then, for any function $f \in L^1(\mathbf{T}^m)$,

$$\lim_{N\to\infty} \frac{1}{D_N} \sum_{k=1}^{N} a_k f(\langle M_k x \rangle) = \int_{\mathbf{T}} m f(t)\, dt \quad \text{a.e.}$$

When $|\det(P_i)|$ $(i=1,\ldots,r)$ are pairwise coprime then $a_k=1$ for all positive integers k and so $D_N=N$ for all positive integers N. In the special case when $m=1$ and $P_i=p_i$ $(i=1,\ldots,r)$, for pairwise coprime integers p_i all greater than one, Theorem 4 reduces to Theorem 1.

To derive Theorem 4 from Corollary 3, we need the following two lemmas, the first of which is a variation of one due to M. M. Day [7] only used to prove the second. We begin with some notation. For sets A and B contained in $\boldsymbol{R}^r$, a point x in $\boldsymbol{R}^r$ and scalar δ we denote $\{a+b\colon a\in A,\ b\in B\}$ by $A+B$, $\{x+a\colon a\in A\}$ by $x+A$ and $\{\delta a\colon a\in A\}$ by δA. Also let $|A|$ denote the r-dimensional Lebesgue measure of A and A^c its complement in $\boldsymbol{R}^r$.

LEMMA 5. *Suppose C denotes a convex set in $\boldsymbol{R}^r$ which contains an r-dimensional ball $B(x,\varrho)$, with centre x and radius ϱ. Then for all $\gamma>0$*

$$C+B(0,\gamma)\subset x+\left(\frac{\varrho+\gamma}{\varrho}\right)(-x+C).$$

Proof. By translation we can assume $x=0$ without loss of generality. Suppose $y\in B(0,\gamma)$. Then there exists $y'\in B(0,\varrho)$ such that $y=\gamma\varrho^{-1}y'$. For any $y''\in C$,

$$y+y''=\varrho^{-1}(\varrho+\gamma)\big(\gamma(\varrho+\gamma)^{-1}y'+\varrho(\varrho+\gamma)^{-1}y''\big)=\varrho^{-1}(\varrho+\gamma)y_0,$$

for some $y_0\in C$, using the convexity of C. ∎

LEMMA 6. *Suppose for each $t\in\boldsymbol{Z}_+$, $N_t=N_t^*\cap\boldsymbol{Z}^r$, where N_t^* denotes a bounded convex subset of $\boldsymbol{R}^r$. Suppose that there exists a positive constant K (possibly dependent on r and $(N_t)_{t=0}^{\infty}$, but not on t), an unbounded increasing sequence of real numbers $(\varrho_t)_{t=0}^{\infty}$ and a sequence of points $(x_t)_{t=0}^{\infty}$ of $\boldsymbol{R}^r$ such that for each t,*

$$B(x_t,\varrho_t)\subset N_t^*\subset B(x_t,K\varrho_t). \tag{3}$$

Then $(N_t)_{t=0}^{\infty}$ satisfies conditions (ii) *and* (iii) *of Theorem* 2.

Proof. We consider first the proof of (ii). Suppose $h=(h_1,\ldots,h_r)\in\boldsymbol{R}^r$ and let $\|h\|=(h_1^2+\ldots+h_r^2)^{1/2}$. Also suppose for a set $S\subset\boldsymbol{R}^r$, with interior S^0 and closure $\bar{S}$, that $\partial(S)=\bar{S}\setminus S^0$. Finally, for positive v, let

$$\partial_v(S)=\{y\in\boldsymbol{R}^r\colon \inf_{x\in\partial(S)}\|y-x\|\leqslant v\}.$$

The idea of the proof of (ii) is to compare $\#N_t$ with $|N_t^*|$. First observe that

$$(h+N_t)\Delta N_t\subset\partial_{\|h\|}(N_t^*)\cap\boldsymbol{Z}^r.$$

Hence to prove (ii) it is sufficient to show that

$$\lim_{t\to\infty}\frac{\#\{\partial_{\|h\|}(N_t^*)\cap\boldsymbol{Z}^r\}}{\#N_t}=0.$$

Now

$$\partial_v(N_t^*)\cap(N_t^*)^c=\big(N_t^*+B(0,v)\big)\setminus N_t^*,$$

which, by Lemma 5 and (3), is contained in

$$\{x_t+\varrho_t^{-1}(\varrho_t+v)(-x_t+N_t^*)\}\setminus N_t^*.$$

This implies that

$$\frac{|\partial_v(N_t^*)\cap(N_t^*)^c|}{|N_t^*|}\leqslant\varrho_t^{-r}(\varrho_t+v)^r-1. \tag{4}$$

In addition

$$\frac{|\partial_v(N_t^*)\cap N_t^*|}{|N_t^*|}=1-\frac{|N_t^*\cap(\partial_v(N_t^*))^c|}{|N_t^*|} \tag{5}$$

and as

$$\{N_t^*\cap(\partial_v(N_t^*))^c+B(0,v)\}=N_t^*,$$

we know, by Lemma 5, that

$$\varrho_t^{-r}(\varrho_t-v)^r\leqslant\frac{|N_t^*\cap(\partial_v(N_t^*))^c|}{|N_t^*|}.$$

Thus the right-hand side of (5) is

$$\leqslant 1-\varrho_t^{-r}(\varrho_t-v)^r. \tag{6}$$

Together (4) and (6) now give

$$\frac{|\partial_v(N_t^*)|}{|N_t^*|}\leqslant\varrho_t^{-r}\{(\varrho_t+v)^r-(\varrho_t-v)^r\}. \tag{7}$$

Comparing number of lattice points with volume

$$\#\{\partial_v(N_t^*)\cap\boldsymbol{Z}^r\}\leqslant|\partial_{v+\sqrt{r}}(N_t^*)|, \tag{8}$$

and using Lemma 5, similarly we have

$$|N_t^*|\leqslant\varrho_t^r(\varrho_t-r^{1/2})^{-r}\#N_t. \tag{9}$$

Thus combining (8) and (9) gives

$$\frac{\#\{\partial_{\|h\|}(N_t^*)\cap\boldsymbol{Z}^r\}}{\#N_t}\leqslant\varrho_t^r(\varrho_t-r^{1/2})^{-r}\frac{|\partial_{\|h\|+\sqrt{r}}(N_t^*)|}{|N_t^*|},$$

which by (7) is

$$\leqslant(\varrho_t-r^{1/2})^{-r}\{(\varrho_t+\|h\|+r^{1/2})^r-(\varrho_t-\|h\|-r^{1/2})^r\}.$$

By hypothesis $(\varrho_t)_{t=0}^{\infty}$ is unbounded and increasing so (ii) is proved.

Now consider (iii). By (3)

$$B(x_t,\varrho_t)\subset N_t^*\subset B(x_t,K\varrho_t),$$

for some $x_t \in \boldsymbol{R}^r$ and real number ϱ_t. Let D denote the largest r-dimensional cube with sides parallel to the coordinate axes of $\boldsymbol{R}^r$, which is contained in $B(x_t, \varrho_t)$ and has vertices in N_t. We can cover $B(x_t, K\varrho_t) \cap \boldsymbol{Z}^r$ and hence N_t by a finite number of translates

$$h_1+(D\cap \boldsymbol{Z}^r), \quad h_2+(D\cap \boldsymbol{Z}^r), \quad \ldots, \quad h_q+(D\cap \boldsymbol{Z}^r) \text{ of } D\cap \boldsymbol{Z}^r$$

$(h_1, \ldots, h_q \in \boldsymbol{Z}^r)$, where q depends on K and $(N_t)_{t=0}^{\infty}$, but not on t. This means that

$$N_t - N_t \subset \bigcup_{i,j=1}^{q} \{(h_i-h_j)+(D\cap \boldsymbol{Z}^r)-(D\cap \boldsymbol{Z}^r)\}.$$

Hence as $\#\{(D\cap \boldsymbol{Z}^r)-(D\cap \boldsymbol{Z}^r)\} \leqslant 2^r \#(D\cap \boldsymbol{Z}^r)$,

$$\#\{N_t - N_t\} \leqslant q^2 2^r \# N_t,$$

proving (iii) and Lemma 6. ∎

We are now in a position to complete the proof of Theorem 4. To an arbitrary $m\times m$ $(1 \leqslant m < \infty)$, integer entry, non-singular matrix $(P_{i,j})$ (say), there corresponds an epimorphism η of $\boldsymbol{T}^m$ such that if $x = (x_1, \ldots, x_m) \in \boldsymbol{T}^m$, $\eta(x) = (x_1^*, \ldots, x_m^*)$ is given by

$$x_j^* = \langle \sum_{i=1}^{m} P_{i,j} x_i \rangle \quad (j = 1, 2, \ldots, m)$$

[15]. Observe first, for a finite set P_i $(i = 1, \ldots, r)$ of commuting integer entry $m\times m$ matrices, with corresponding epimorphisms η_i $(i = 1, \ldots, r)$, that for each $x \in \boldsymbol{T}^m$ and each $s = (s_1, \ldots, s_r) \in \boldsymbol{Z}_+^r$,

$$\eta_1^{s_1} \ldots \eta_r^{s_r}(x) = \langle P_1^{s_1} \ldots P_r^{s_r} x \rangle.$$

Note also that any epimorphism η is ergodic, or equivalently it possesses an $x \in \boldsymbol{T}^m$ such that $\{\eta^s(x) \colon s \in \boldsymbol{Z}_+\}$ is dense in $\boldsymbol{T}^m$, if and only if its matrix has no roots of unity among its eigenvalues [15]. Now let

$$N_t = \{(s_1, \ldots, s_r) \in \boldsymbol{Z}_+^r \colon |\det(P_1^{s_1} \ldots P_r^{s_r})| \leqslant t\}.$$

Then we note, upon taking logarithms, that $N_t = N_t^* \cap \boldsymbol{Z}^r$, where N_t^* is the r-dimensional tetrahedron in $\boldsymbol{R}^r$ bounded by the non-negative parts of the coordinate axes and the hyperplane

$$s_1 \log|\det(P_1)| + \ldots + s_r \log|\det(P_r)| = \log t.$$

Clearly, for each positive number t, N_t^* is bounded, convex and satisfies (3) so by using Lemma 6, Corollary 3 gives Theorem 4. ∎

For $\alpha = (\alpha_k)_{k=1}^{m} \in \boldsymbol{R}^m$ $(1 \leqslant m \leqslant \infty)$, the map R_α of $\boldsymbol{T}^m$ determined for $x \in \boldsymbol{T}^m$ by

$$R_\alpha(x) = \langle x+\alpha \rangle$$

is obviously Lebesgue measure preserving.

LEMMA 7 [5, p. 97]. *For a compact, connected, Abelian topological group G, and an element $g_0 \in G$, the following are equivalent:*

(a) *The map $R_{g_0}(x) = x+g_0$ is ergodic with respect to Haar measure;*

(b) *For any non-trivial character χ of G, $\chi(g_0) \neq 1$.*

When G is $\boldsymbol{T}^m$ $(1 \leqslant m \leqslant \infty)$, an arbitrary character evaluated at $x = (x_k)_{k=1}^{m} \in \boldsymbol{T}^m$ is of the form

$$\chi(x) = \exp(2\pi i \sum_{k=1}^{m} m_k x_k).$$

Here m_k is an integer and non-zero for only finitely many positive integers k [8, p. 373]. From Lemma 7, we know that for a sequence of real numbers $\alpha = (\alpha_k)_{k=1}^{m}$ the map R_α is ergodic if and only if for each finite subset $\{k_1, \ldots, k_b\}$ of $\boldsymbol{Z}_+ \cap [1, m]$, $\alpha_{k_1}, \ldots, \alpha_{k_b}$ and 1 are linearly independent over the rationals. This is the case for example when $\alpha_k = \theta^k$ $(k \in \boldsymbol{Z}_+ \cap [1, m])$ for some transcendental number θ. In light of Corollary 3 we have

COROLLARY 8. *Let $\gamma_i = (\gamma_{i,k})_{k=1}^{m}$ $(1 \leqslant i \leqslant r;\ 1 \leqslant m \leqslant \infty)$ be a finite number of sequences of real numbers. Suppose that for at least one i, for all finite subsets $\{k_1, \ldots, k_b\}$ of $\boldsymbol{Z}_+ \cap [1, m]$, $\gamma_{i,k_1}, \ldots, \gamma_{i,k_b}$ and 1 are linearly independent over the rational numbers. Further, let $(N_t)_{t=0}^{\infty}$ be a collection of subsets of $\boldsymbol{Z}^r$ which satisfy* (i), (ii) *and* (iii) *of Theorem 2. If we now associate to each point $x \in \boldsymbol{T}^m$, and each point $s = (s_1, \ldots, s_r) \in \boldsymbol{Z}^r$, the point*

$$\langle s_1\gamma_1 + \ldots + s_r\gamma_r + x \rangle \in \boldsymbol{T}^m,$$

then for any function $f \in L^1(\boldsymbol{T}^m)$,

$$\lim_{t\to\infty} \frac{1}{\# N_t} \sum_{s \in N_t} f(\langle s_1\gamma_1 + \ldots + s_r\gamma_r + x \rangle) = \int_{\boldsymbol{T}} mf(t)\,dt \quad a.e.$$

Let $C_n(x)$ $(n \in \boldsymbol{Z}_+)$ denote the nth Chebyshev polynomial, that is, $C_n(x) = \cos n\theta$, where $x = \cos\theta$. It was shown by R. L. Adler and T. J. Rivlin [2] that on $[-1, 1]$, with respect to the measure ξ defined for Lebesgue measurable sets $A \subset [-1, 1]$ by

$$\xi(A) = \frac{2}{\pi} \int_A \frac{dt}{\sqrt{1-t^2}}, \tag{10}$$

$C_n(x)$ is ergodic and measure preserving. Using the fact that

$$C_{nm}(x) = C_n(C_m(x)) \quad (n, m \in \boldsymbol{Z}_+),$$

which follows from the definition of $C_n(x)$, Theorem 2 gives

COROLLARY 9. *For any finite set of integers $p_1, \ldots, p_r$, all greater than one, let $(m_k)_{k=1}^{\infty}$ denote the sequence of integers they generate multiplicatively, once* *ordered by size. Also let*

$$a_k = \#\{(s_1, \ldots, s_r) \in Z^r_+ : \ m_k = p_1^{s_1} \ldots p_r^{s_r}\} \quad (k = 1, 2, \ldots),$$

$$D_N = a_1 + \ldots + a_N \quad (N = 1, 2, \ldots).$$

Then for any function $f \in L^1([-1, 1], \xi)$, with ξ and $C_n(x)$ $(n = 1, 2, \ldots)$ defined by (10), we have

$$\lim_{N\to\infty} \frac{1}{D_N} \sum_{k=1}^{N} a_k f(C_{m_k}(x)) = \int_{-1}^{1} f\, d\xi \quad a.e.$$

In the case $r = 1$, with $N_t = Z \cap [1, t]$, Theorem 2 gives Birkhoff's ergodic theorem for an ergodic transformation. This has a number of well-known implications for metric number theory, some of them a consequence of the following theorem of R. L. Adler [1] (see also [6, pp. 169&290]).

THEOREM 10. *Suppose that for each positive integer w,*

$$[0, 1) = \bigcup I^w_{n_1, \ldots, n_w},$$

where the $I^w_{n_1,\ldots,n_w}$ are disjoint intervals such that

$$\bigcup_{n_w} I^w_{n_1,\ldots,n_w} = I^{w-1}_{n_1,\ldots,n_{w-1}} \quad (w = 2, 3, \ldots),$$

indexed by w-tuples $n_1, \ldots, n_w$ which belong to an indexing subset J^w of Z^w_+. Suppose also that φ: $[0, 1) \to [0, 1)$ is monotone and twice continuously differentiable on $I^1_{n_1}$ $(n_1 \in J^1)$, that $\varphi(I^1_{n_1}) = [0, 1)$ $(n_1 \in J^1)$ and that

$$\varphi(I^w_{n_1,\ldots,n_w}) = I^{w-1}_{n_1,\ldots,n_{w-1}} \quad (w = 2, 3, \ldots).$$

Further, suppose that there exists a natural number s such that if φ^a denotes the a-fold composition of φ $(a = 1, 2, \ldots)$, then

$$\inf_{n_1 \in J^1} \inf_{x \in I^1_{n_1}} \left| \frac{d\varphi^s(x)}{dx} \right| > 1.$$

Finally suppose that

$$\sup_{n_1 \in J^1} \sup_{x_1, x_2 \in I^1_{n_1}} \frac{|(d^2\varphi/dx^2)(x_1)|}{[(d\varphi/dx)(x_2)]^2} < +\infty.$$

Then there exists a measure μ on $[0, 1)$, absolutely continuous with respect to Lebesgue measure, which is preserved by φ and with respect to which φ is ergodic. (See [15] for a generalization of Theorem 10.)

Suppose now that we are given the strictly increasing sequences of positive integers $(d_k)_{k=1}^{\infty}$ and $(e_k)_{k=1}^{\infty}$ such that for some $F > 0$,

(11) $$e_k \geqslant F d_{k-1} \quad (k = 2, 3, \ldots).$$

Then, as is readily checked, the subsets $(N_t)_{t=0}^{\infty}$ of Z_+ given by

$$N_t = [1, t] \cap Z \cap \bigcup_{k=1}^{\infty} [d_k, d_k + e_k],$$

satisfy (i), (ii), (iii) and (iv) of Theorem 2 (needing to use (11) only to verify (iii)). Let $(a_k)_{k=1}^{\infty}$ be the set $\bigcup_{k=1}^{\infty} [d_k, d_k + e_k] \cap Z$ once ordered into a sequence by size. Theorems 2 and 10 now give

COROLLARY 11. *If φ: $[0, 1) \to [0, 1)$ and μ are as in Theorem 10 and $(a_k)_{k=1}^{\infty}$ is as just above, then for any function $f \in L^1([0, 1), \mu)$,*

(12) $$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\varphi^{a_k}(x)) = \int_0^1 f(t)\, d\mu \quad a.e.$$

Corollary 11 is of interest because, if in addition to (11), $e_k = o(d_k)$, then as is easy to verify, $(a_k)_{k=1}^{\infty}$ has zero density, that is,

$$\lim_{N\to\infty} \frac{\#\{(a_k)_{k=1}^{\infty} \cap [1, N]\}}{N} = 0.$$

This means that you cannot hope to derive Corollary 11 from Birkhoff's theorem by any direct manipulation.

A famous example of a map φ that satisfies the conditions of Theorem 10 is the continued fraction transformation, defined for $x \in (0, 1)$ by

$$\varphi(x) = \varphi_\sigma(x) = \langle 1/x \rangle.$$

This is known to preserve the invariant measure σ defined for Lebesgue measurable sets $A \subset [0, 1)$ by

$$\sigma(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x},$$

with respect to which φ_σ is ergodic. Using Corollary 11 we obtain a variant of classical results about arithmetic and geometric means of the partial quotients of the continued fraction expansion of a real number due to Khinchin which we quote from [6, pp. 165–175].

COROLLARY 12. *Let $c_k(x)$ $(k = 1, 2, \ldots)$ denote the k-th partial quotient of the continued fraction expansion of the real number $x \in (0, 1)$, that is, $c_1(x) = [1/x]$ and $c_k(x) = c_{k-1}(\varphi_\sigma(x))$ $(k = 2, 3, \ldots)$. Let $(a_k)_{k=1}^{\infty}$ be the sequence of natural numbers defined just before Corollary 11. Then*

(13) $$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} c_{a_k}(x) = \infty \quad a.e.,$$

(14) $$\lim_{N\to\infty} \left(\prod_{k=1}^{N} c_{a_k}(x)\right)^{1/N} = \prod_{j=1}^{\infty} \left(1 + \frac{1}{(j+1)^2 - 1}\right)^{\log j/\log 2} \quad a.e.$$

Proof. Let $f(x) = c_1(x)$, that is, $f(x) = k$ for $x \in ((k+1)^{-1}, k^{-1})$, $(k = 1, 2, \ldots)$. Then if $f_M(x) = \min(f(x), M)$ for positive M, we have

$$\frac{1}{N}\sum_{k=1}^{N} c_{a_k}(x) = \frac{1}{N}\sum_{k=1}^{N} f(\varphi_\sigma^{a_k}(x)) \geqslant \frac{1}{N}\sum_{k=1}^{N} f_M(\varphi_\sigma^{a_k}(x)),$$

the right-hand side of which tends almost everywhere to

$$\frac{1}{\log 2}\int_0^1 f_M(t)\frac{dt}{1+t},$$

by Corollary 11. Now letting $M \to \infty$ proves (13) because f is not integrable.

To see (14) choose $f(x) = \log c_1(x)$, because then Corollary 12 gives

$$\frac{1}{\log 2}\int_0^1 \frac{f(x)}{1+x}dx = \sum_{k=1}^{\infty}\frac{\log k}{\log 2}\log\left(1+\frac{1}{(k+1)^2-1}\right). \blacksquare$$

The proof of Corollary 12 is modelled on [6, p. 175].

COMMENT. In the statement of Theorem 2, we can replace Z^r by Z^r_+. This is another special case of Theorem 3 of [4]. In this context (iv) is vacuously satisfied and a version of Lemma 7 can be proved in which condition (3) used to verify (iii) can be replaced by the condition that for each t, if $(s_1, \ldots, s_r) \in N_t$ and $0 \leqslant s_i^* \leqslant s_i$ $(i = 1, \ldots, r)$, then $s^* = (s_1^*, \ldots, s_r^*) \in N_t$ if $s^* \in Z^r$. This condition implies $N_t - N_t \subset N_t$ in Z^r_+ and is far less restrictive than (3). For instance, choose $(N_t)_{t=0}^{\infty}$ such that $N_t = N_t^* \cap Z^r_+$ for nested r-dimensional rectangles $(N_t^*)_{t=0}^{\infty}$ in R^r_+, with one corner at the origin and with edges parallel to the coordinate axes, whose lengths tend to infinity with t, at different rates. Then even if $(N_t)_{t=0}^{\infty}$ is not assumed to satisfy (3), it satisfies (i), (ii) and (iii) in Z^r_+.

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