Eisenstein's theorem on power series expansions
of algebraic functions

by

WOLFGANG M. SCHMIDT* (Boulder, Col.)

1. Introduction. A well-known theorem of Eisenstein asserts that if a formal
series
\begin{equation}
y = a_0 + a_1 X + a_2 X^2 + \ldots
\end{equation}
satisfies an equation \( F(X, y) = 0 \) where \( F \) is a nonzero polynomial with
algebraic coefficients, then \( a_0, a_1, \ldots \) lie in an algebraic number field, and there
are natural numbers \( a_0, a \) such that
\begin{equation}
a_0 a^j a_j \quad (j = 0, 1, \ldots)
\end{equation}
are algebraic integers. It is our purpose to make this more explicit.

In the special case when the polynomial \( F(X, Y) \) lies in \( \mathbb{Z}[X, Y] \) and has
no multiple factors, our results will imply that we may take \( a, a_0 \) with
\[ a < c_1(N) H^\mathbb{H}, \quad a_0 = a^N, \]
where \( N \) is the total degree, and \( H \) is the maximum modulus of the coefficients
of \( F \). The only quantitative version of Eisenstein's Theorem that I could find in
the literature** is due to Coates [2, Lemma 3], and implies a value
\[ a = a_0 < c_2(N) H^3 \]
with \( c_2(N) = (2N)^{6N^2} \).

\( F(X, Y) \) may be regarded as a polynomial in \( Y \) whose coefficients are
polynomials in \( X \). As such it has a discriminant \( D(X) \) which is a polynomial in
\( X \). We will suppose throughout that \( D(X) \neq 0 \), i.e., that \( F(X, Y) \) when
regarded as a polynomial in \( Y \) has no multiple factors. We will assume that \( F \) is
of degree \( m > 0 \) in \( X \) and of degree \( n > 0 \) in \( Y \), and that the coefficients of \( F \) lie
in an algebraic number field \( k \) of degree \( d \). It is well known and easily seen that

* Supported in part by NSF grant DMS-8603093.
Amer. Math. Soc. 280 (1983), 637–657 obtain a bound similar to Coates'.
if $y$ as above satisfies $F(X, y) = 0$, then the coefficients $a_0, a_1, \ldots$ generate a field $K$ over $k$ of degree $[K : k] \leq n$. Thus $K$ has degree $\delta = [K : \mathcal{O}] \leq nd$.

Let $a_i \to a_i^0$ ($i = 1, \ldots, \delta$) be the isomorphic embeddings of $K$ into $C$. It is known that there are positive reals $A_0, A$ such that

\begin{equation}
|z|^0_i \leq A_0 A^j \quad (1 \leq i \leq \delta; \ j = 0, 1, \ldots). \tag{1.3}
\end{equation}

This, together with the assertion on (1.2), implies that $y$ is a $G$-function as defined by Siegel [7].

By an absolute value of $k$ we will always understand an absolute value which is normalized so that it extends either the standard absolute value or a $p$-adic absolute value of $\mathcal{O}$. Given such an absolute value $|\_|_w$ of $k$, let $n_w$ be its local degree. Let $M(k)$ be a set of symbols $v$, such that with every $v \in M(k)$ there is associated an absolute value $|\_|_w$ of $k$, and moreover every absolute value $|\_|_w$ of $k$ is obtained for precisely $n_w$ elements of $M(k)$. In other words, $M(k)$ is the set of absolute values with multiplicities, so that a given $|\_|_w$ occurs $n_w$ times. With this convention, we have the product formula

\[ \prod_{v \in M(k)} \left| a_v \right| = 1 \quad \text{for } a \in k, \quad a \neq 0. \]

We will write $v|\infty$ if $v$ extends the Archimedean absolute value of $\mathcal{O}$, i.e., when $v$ is Archimedean. There are precisely $d$ such $v \in M(k)$. We will write $v|p$ if $v$ extends the $p$-adic absolute value of $\mathcal{O}$. Given a prime $p$, there are precisely $d$ such $v \in M(k)$.

We set

\[ M(k) = M_{\infty}(k) \cup M_1(k) \cup M_2(k), \]

where $M_{\infty}(k)$ consists of $v$ with $v|\infty$, where $M_1(k)$ consists of $v$ with $v|p$ where $p > n$, and $M_2(k)$ consists of $v$ with $v|p$ where $p \leq n$.

Now let $P$ be a polynomial in one or several variables and with coefficients in $k$. Given $v \in M(k)$, let $|P|_v$ be the maximum of $|a_v|$ over all the coefficients $a$ of $P$. We define the field height $H_k(P)$ of $P$ by

\[ H_k(P) = \prod_{v \in M(k)} |P|_v, \]

and the absolute height by $H(P) = H_k(P)^{1/d}$. (Warning: sometimes, e.g. in [6], a different height is used.) We define $M(K)$ in complete analogy with $M(k)$.

When $v \in M(k)$, we $M(K)$ and the restriction of $|\_|_w$ to $k$ is $|\_|_w$. We write $w|v$.

Given $v \in M(k)$, there are precisely $[K : k]$ elements $w \in M(K)$ with $w|v$.

**THEOREM 1.** Let $F, y$ be as above. There are real numbers $A_0 \geq 1$, defined for $v \in M(k)$ and with $A_0 = 1$ for all but finitely many $v$, such that

\begin{equation}
\left| a_v \right|_w \leq A_0^{a_0^i + j} \quad (j = 0, 1, \ldots) \tag{1.4}
\end{equation}

for every $v \in M(K)$, $w \in M(K)$ with $w|v$, and such that

\begin{equation}
\prod_{v \in M(k)} A_v \leq (m+1)(n+1)\sqrt{a}^{a_0^i + 1}H(F)^{2nd} = C, \tag{1.5}
\end{equation}

say, and

\begin{equation}
\prod_{v \in M(k)} A_v < (16m)^{1+\varepsilon}H(F)^{2nd} \tag{1.6}
\end{equation}

It is likely that the bound in (1.6) is weak and should be replaced by a bound similar to (1.5). In order to obtain Eisenstein’s Theorem we need a variation on Theorem 1. For $v \in M_{\infty}(k)$, let $G_v$ be the group $R^+$ of positive reals under multiplication. For $v \in M_1(k)$, let $G_v = G_v \cup M_2(k) = M(k), M_{\infty}(k)$, and $G_v = G_v \cup M_2(k)$.

**THEOREM 2.** Let $F, y$ be as above. There are numbers $B_0 \in G_v$ for each $v \in M(k)$, having $B_v \geq 1$, and $B_v = 1$ for all but finitely many $v$, such that

\begin{equation}
\left| a_v \right|_w \leq B_0^{a_0^i + j} \quad (j = 0, 1, \ldots) \tag{1.7}
\end{equation}

for every $v \in M(k)$, $w \in M(K)$ with $w|v$, and such that

\begin{equation}
\prod_{v \in M(k)} B_v < (2^{14} m n^2) H(F)^{4n+mn+d} = C_1, \quad \text{say}. \tag{1.8}
\end{equation}

It is an immediate consequence of Theorem 1 that

\[ |\alpha|^0_i \leq C_0^{\alpha_0^i + j} \quad (1 \leq i \leq \delta; \ j = 0, 1, \ldots) \]

so that (1.3) holds with $A = C, A_0 = C_0$. On the other hand, for $v \in M_{\infty}(k)$ let $\mathfrak{P}_v$ be the prime ideal in the ring of integers in $k$ consisting of $x$ with $|x|_w < 1$. If $v|p^m$ then $(p^m) = \mathfrak{P}_v \mathfrak{P}_v^2 \ldots \mathfrak{P}_v^m$ for prime ideals $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_s$ and exponents $e_v, e_2, \ldots, e_s$. The value group $G_v$ is generated by $p_1^{e_1}, \ldots, p_s^{e_s}$. The ideal $\mathfrak{P}_v$ generates an ideal in the ring of integers of $k$ which we will also denote by $\mathfrak{P}_v$, and every $a \in \mathfrak{P}_v, a \in \mathfrak{P}_v$ has $|a|_w \leq p_0^{a_0^i e_1}$. The ideal $\mathfrak{P}_v$ is the ideal

\[ \mathfrak{P}_v = \prod_{v \in M_{\infty}(k)} \mathfrak{P}_v. \]

where $\mathfrak{P}_0(k)$ (in contrast to $\mathfrak{P}_0(k)$) indexes every absolute value just once. Then by (1.7),

\begin{equation}
\mathfrak{P}_v^{\alpha_0^i e_0} = \mathfrak{P}_v^j \quad (j = 0, 1, \ldots) \tag{1.9}
\end{equation}

are integral ideals in the ring of integers of $k$. Moreover, $\mathfrak{P}_v$ has some norm $N(\mathfrak{P}_v) = p_0^{a_0^i}$ with $e_0 f_0 \leq d$, and

\[ N(\alpha) = \prod_{v \in M_{\infty}(k)} N(\mathfrak{P}_v)^{e_0} = \prod_{v \in M_{\infty}(k)} p_0^{a_0^i e_0} = \prod_{v \in M_{\infty}(k)} B_v = C_1. \]

Setting $a = N(\alpha)$, we obtain the following quantitative version of Eisenstein’s Theorem.
THEOREM 3. There is an ideal \( \mathfrak{a} \) in the ring of integers of \( k \) with \( \mathcal{N}(\mathfrak{a}) \leq C_1 \) such that the ideals (1.9) are integral. There is a natural number \( a \) with \( a \leq C_1 \) such that

\[
d^{n+1}x_j \quad (j = 0, 1, \ldots)
\]

are algebraic integers.

We remark that some precision is lost in going from the ideal \( \mathfrak{a} \) to the natural number \( a \). In fact, the formulation in Theorem 1 may be best. For example, if \( k = K = Q \), and if 2 occurs in the denominator of \( x_j \) to the exponent \([j/2]\) where \([ \cdot ]\) denotes integer parts, then in Theorem 1 we may take \( A_2 = 2^{[j/2]} \), but in Theorem 2 we have to take \( B_2 = 2 \), and in Theorem 3 we need to take a divisible by 2.

The quantitative version of Eisenstein's Theorem due to Coates [2, Lemma 3], has \( a_0 = a \leq c_2(n, m)H(F)^3 \) and \( c_2 = (4n^2d)^{2m} \).

The proofs of Theorems 1 and 2 will distinguish between elements \( v \) in \( M_\omega(k), M_\lambda(k), M_\lambda(k) \). The argument for \( v \in M_\omega(k) \), will follow classical lines. For \( v \in M_\omega(k) \), a result of Dwork and Robba [3] on \( p \)-adic radii of convergence will be crucial. A conjectured variation (see Section 2, Lemma 1) of this result for \( v \in M_\lambda(k) \) would lead to a great simplification and to better bounds. Since such a variation has not been proved, in order to deal with \( v \in M_\lambda(k) \), we have to derive a linear differential equation satisfied by \( y \), and to use a paper of Clark [1] on \( p \)-adic convergence of solutions of such differential equations. I am grateful to Professor Dwork for drawing my attention to this work of Clark.

Eisenstein in [4] apparently supposes that the discriminant \( D(X) \) does not vanish at \( x = 0 \). Under this assumption, his theorem becomes considerably easier, and our bounds could be much improved.

2. Quantities \( \varrho \) and \( \sigma \). For \( v \in M_\omega(k) \), let \( C_v \) be the algebraic closure of the completion of \( K \) under \( \vline\cdot\vline_v \). Thus \( C_v \cong C \). For \( v \in M_\lambda(k) \), let \( C_v \) be the completion of the algebraic closure of the completion of \( K \) under \( \vline\cdot\vline_v \). There is a natural extension of \( \vline\cdot\vline_v \) to \( C_v \). Similarly define \( C_v \) for \( v \in M(k) \), and extend \( \vline\cdot\vline_v \) to \( C_v \).

For \( w \in M(k) \), let \( \varrho_w \) be the \( w \)-adic radius of convergence of \( y \). Thus \( \varrho_w \) is the supremum of the numbers \( q \) such that the series for \( y \) converges \( w \)-adically for every \( x \in C_w \) with \( \vline x \vline_w \leq q \). We will see in the course of our investigation that \( \varrho_w > 0 \) for each \( w \).

Let \( D(X) \) be the discriminant of \( F(X, Y) \) when considered as a polynomial with coefficients in \( k[X] \). Then \( D(X) \in k[X] \), and \( D(X) \neq 0 \) by hypothesis. Write

\[
F(X, Y) = A_n(X)Y^n + \cdots + A_0(X),
\]

so that \( A_n(X) \neq 0 \). Put

\[
R(X) = A_n(X)D(X);
\]  

then \( \deg R(X) \leq (2n-1)m \). Write

\[
R(X) = X^qR^*(X)
\]

where \( R^* \) is a polynomial with \( R^*(0) \neq 0 \).

Now let \( v \in M(k) \), and \( k_v \) the completion of \( k \) under \( \vline\cdot\vline_v \), so that \( k_v \cong C_v \). In \( k_v[X] \) we have a factorization \( R^*(X) = R_1(X) \cdots R_l(X) \) into irreducible factors. Say \( R_i(X) = (X - \beta_{1i}) \cdots (X - \beta_{li}) \in C_v \). Then it is well known that \( \vline \beta_{1i} \vline_v = \vline \beta_{2i} \vline_v = \cdots = \vline \beta_{li} \vline_v = \nu_i \), say \( i = 1, \ldots, l \). Thus the set of \( \nu \)-adic absolute values of the roots of \( R^* \) is \( \{ \nu_1, \ldots, \nu_l \} \). Set

\[
\sigma_v = \min(1, \nu_1, \ldots, \nu_l).
\]

Suppose now that \( E \) is an algebraic extension of \( k \) in which \( R^*(X) \) factors into linear factors, say \( R^*(X) = c(X - \beta_1) \cdots (X - \beta_l) \), and let \( \vline\cdot\vline_w \) be an extension of \( \vline\cdot\vline_v \) to \( E \). Then, since \( E \) can be embedded into \( C_v \), the set of absolute values \( \vline \beta_i \vline_w \) (\( i = 1, \ldots, l \)) is the same as \( \{ \nu_1, \ldots, \nu_l \} \). In other words, this set is independent of \( E \) and of \( w \). Thus

\[
\sigma_w = \min(1, \nu_1, \ldots, \nu_l) = \vline \beta_i \vline_w.
\]

LEMMA 1. Suppose \( w | v \) with \( w \in M(k), v \in M_\omega(k) \cup M_\lambda(k) \). Then \( \varrho_w \geq \sigma_v \).

Proof. The case when \( v \in M_\omega(k) \) is classical. We may suppose that \( K \) is embedded in \( C \) and that \( \vline\cdot\vline_v, \vline\cdot\vline_w \) are ordinary absolute values. The equation \( F(X, y) = 0 \) has \( n \) Puiseux series solutions \( y_1(x), \ldots, y_n(x) \) at \( x = 0 \). Since \( A_0(x) \neq 0 \) and \( F_0(x, y) = 0 \) for every \( x \in C \) with \( 0 < \vline x \vline_v < \sigma_v \), \( F(x, y) = 0 \), each of the series \( y_1(x), \ldots, y_n(x) \) can be continued analytically to \( 0 < \vline x \vline_v < \sigma_v \). Since there can be no more than \( n \) formal Puiseux series solutions to \( F(x, y) = 0 \), the given series \( y \) of (1.1) is among \( y_1(x), \ldots, y_n(x) \), hence is analytic in \( |x| < \sigma_v \). Therefore its radius of convergence is \( \geq \sigma_v \).

The case when \( v \in M_\lambda(k) \) is due to Dwork and Robba [3]. Again, at each \( x_0 \in C_v \) with \( 0 < \vline x_0 \vline_v < \sigma_v \), the equation \( F(x, y) = 0 \) has \( n \) distinct locally analytic solutions \( y_1(x), \ldots, y_n(x) \). Pick \( \zeta \in C_w \) with \( \vline \zeta \vline_w = \sigma_v \), and set \( G(X, Y) = F(\zeta X, Y) \). Then at each \( x_0 \in C_w \) with \( 0 < \vline x_0 \vline_v < \sigma_v \), the equation \( G(x, y) = 0 \) has \( n \) distinct locally analytic solutions. By Dwork and Robba's Theorem 2.1, \( y(x) \) is convergent for \( |x| < \sigma_v \), so that \( y(x) \) itself is convergent for \( |x| < \sigma_v \). Thus \( \varrho_w \geq \sigma_v \).

I conjecture that when \( v \in M_2(k) \) and everything else is as above, then \( y(x) \) is convergent for \( |x| < c(n) \), where \( c(n) \) depends on \( n \) only.

In what follows, write

\[
A_n(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0X^0
\]

with \( a_n \neq 0, a_0 \neq 0 \). Theorems 1, 2 are invariant under multiplication of \( F \) by a nonzero element of \( k \). We therefore may, and will, suppose in the sequel that

\[
a_n = 1.
\]
Lemma 2. (a) Suppose that \( w \mid v \) where \( w \in M(K) \) and \( v \in M_\infty(k) \). Then
\[
|\sigma_j|_w \leq 2n|F|_e(2/\sigma_j)^{m+i} \quad (j = 0, 1, \ldots).
\]
(b) Suppose that \( w \mid v \) with \( w \in M(K) \) and \( v \in M_\infty(k) \). Suppose that \( \tau_w > 0 \), where
\[
\tau_w = \min(\sigma_w, \varrho_w).
\]
Then
\[
|\sigma_j|_w \leq |F|_e(1/\tau_w)^{m+i} \quad (j = 0, 1, \ldots).
\]

Proof. (a) We may suppose that \( K \) is embedded in \( C \) and that \( |\cdot|_w, |\cdot|_e \) are the ordinary absolute values. We factor
\[
A_n(z) = a_0 z^n + \cdots + a_n z = 0, \quad (n = 1, 2, \ldots).
\]
(When \( r = u \), we have \( A_n(x) = a_0 x^n + \cdots + a_n z = 0 \).) Let \( z \) be complex with \( |z| = |z|/2 \). Then \( |y(z)/z| \leq |y(z)|/2 \) (i = 1, 2, \ldots, \( n \)).

On the other hand, since \( |z| = |z|/2 < 1/2 \),
\[
|A_n(z)| < 2|F|_e \quad (i = 0, 1, \ldots, n).
\]
Since \( y(z) \) satisfies \( F(z, w(z)) = A_n(z) w(z)^n + \cdots + A_0(z) w \), we have
\[
|y(z)| < 2n|F|_e(2/\sigma_j)^{m+i}.
\]

By Cauchy's formula, the coefficient \( \alpha_j \) in the expansion of \( y(z) \) is given by
\[
\alpha_j = \frac{1}{2\pi i} \int_C \frac{y(z)}{z^{j+1}} \, dz,
\]
where \( C \) is, say, the circle \( |z| = |z|/2 \). On this circle, \( |y(z)/z| \leq 2n|F|_e(2/\sigma_j)^{m+i+1} \). The path of integration has length \( 2\pi(\pi/2) \). We obtain
\[
|\alpha_j|_w = |\alpha_j| \leq 2n|F|_e(2/\sigma_j)^{m+i+1} \quad (j = 0, 1, \ldots)
\]
(b) Let \( q \) be a number in \( 0 < q < \tau_w \) of the type \( q = p^t \) where \( v \mid p \) and \( t \in Q \). Since \( q < \tau_w \), the series for \( y(z) \) is \( w \)-adically convergent for \( z \in C_w \) with \( |z| = q \). In \( C_w \) we again have a factorization (2.7), and \( |y(z)|_w \leq \sigma_w \leq q \) \( i = 1, 2, \ldots, \). Therefore \( z \) with \( |z| = q \) has
\[
|A_n(z)|_w = |a_0 z^n|_w \quad |a_n z^n|_w = q^n |a_n z^n|_w = q^n \geq q^n.
\]
On the other hand, since \( |z| = q < 1 \),
\[
|A_n(z)|_w \leq |F|_e \quad (i = 0, 1, \ldots, n).
\]

The quotients have \( |A_n(z)/A_{n-1}(z)|_w \leq |F|_e q^{-m} \quad (i = 0, 1, \ldots, n) \). Note that the right side here is \( > 1 \) since \( q < 1 \), and since \( |F|_e \geq 1 \) by (2.6). Since \( y(z) \) satisfies \( A_n(z) y(z)^n + \cdots + A_0(z) = 0 \), we have
\[
|y(z)|_w \leq |F|_e q^{-m}.
\]

Pick \( c \in C_w \) with \( |c|_w = q \), and set
\[
y(X) = y(c + x) = a_0 + a_1 x + a_2 x^2 + \cdots = b_0 + b_1 x + b_2 x^2 + \cdots,
\]
saying that \( y(z) \leq |F|_e q^{-m} \) for every \( z \in C_w \), having \( |z| = 1 \). Furthermore, since \( y(z) \) is \( w \)-adically convergent for such \( z \), \( |b_j|_w \to 0 \) as \( j \to \infty \). Set
\[
B = \max_{j \geq 1} |b_j|_w,
\]
and when \( B \neq 0 \) pick \( t \) such that \( |b_j|_w < B \) for \( j > t \). There is a \( z \in C_w \) with \( |z| = 1 \) and
\[
|b_0 + b_1 z + \cdots + b_t z^t|_w = B;
\]
then also \( |y(z)|_w = B \). This implies that \( B \leq |F|_e q^{-m} \), so that \( |b_j|_w \leq |F|_e q^{-m} \), i.e., \( |b_j|_w \leq |F|_e q^{-m} \), and therefore
\[
|y(z)|_w \leq |F|_e q^{-m} |z|_w = B.
\]
Since this is true for every \( q < \tau_w \) of the type specified above, assertion (b) follows.

3. On \( R(X) \) and its roots. Given \( e \in E \) where \( E \) is an algebraic number field, define its field height to be
\[
h_E(e) = \prod_{w \in M(E)} \max(1, |e|_w),
\]
and its absolute height by \( h(e) = h_E(e)^{1/e} \), where \( e = [E:Q] \).

Lemma 3. Suppose \( P(X) = q(X - e_1) \cdots (X - e_t) \) with \( q, e_1, \ldots, e_t \) in \( E \). Then
\[
h(e_1) \cdots h(e_t) \leq (t + 1) H(P).
\]

Proof. For \( w \in M(E) \), put
\[
\mathcal{M}_w = |e|_w \max(1, |e|_w) \cdots \max(1, |e|_w).
\]
When \( w \in M_0(E) \), so that \( |e|_w \) is non-Archimedean, Gauss' Lemma yields
\[
\mathcal{M}_w = |P|_w.
\]
Now suppose that \( w \) is Archimedean. After embedding \( E \) in a suitable way into \( C \), we may suppose that \( |e|_w \) is the standard absolute value of \( C \). Then \( \mathcal{M}_w \) is the Mahler height of \( P \), and therefore
\[
\mathcal{M}_w = \exp \frac{1}{2} \log |P(e^{2\pi i})| \, dx
\]
We obtain
$$H_k(R) = \prod_{w \in M(k)} |R| |(m+1)(n+1)\sqrt{n}^{2n-1}dH_k(F)^{2n-1},$$
where $d = [k:Q]$. The lemma follows.

**Lemma 5.**
$$\prod_{\nu \in M(k)} \sigma_{\nu} \geq (m+1)(n+1)\sqrt{n}^{2n-1}dH(F)^{2n-1}.$$  

**Proof.** Let $E$ be an extension in which $R^+(X)$ splits into linear factors, say
$$R^+(X) = c(X - \beta_1) \cdots (X - \beta_s).$$
For $\nu \in M(k)$ and any extension $|\cdot|_\nu$ of $|\cdot|_k$ to $E$ we have (2.4). Thus if $q = [E:k]$, then
$$\sigma_{\nu} \geq \min_{w \in M(E)} \frac{1}{|\beta|_w}, \ldots, \frac{1}{|\beta|_w}.$$
Therefore
$$\prod_{\nu \in M(k)} \max_{w \in M(E)} \frac{1}{|\beta|_w} \leq h(e_1) \cdots h(e_s) \leq (m+1)(n+1)H(R) \leq 2nmH(R).$$

The lemma now follows upon extracting $q$th roots, in view of (3.1) and of
$$2nm \leq (m+1)(n+1)\sqrt{n}.$$
\[ \prod_{v \in \mathcal{M}_\omega(k) \cup M_1(k)} A_v \leq (4n)^d H(F) \prod_{v \in \mathcal{M}(k)} \sigma_v^{-1} \]
\[ \leq (m+1)(n+1)\sqrt{n}^{2n+1} d H(F)^2 = C, \]
so that (1.5) is true.

Encouraged by this, let us do the part of Theorem 2 concerned with
\( v \in \mathcal{M}_\omega(k) \cup M_1(k). \) Set
\[ B_v = A_v = 2n |F|/\sigma_v \quad \text{for} \quad v \in \mathcal{M}_\omega(k). \]

When \( v \in M_1(k), \) we observe that \( |F|/\sigma_v \in G_v, \) but not necessarily \( \sigma_v \in G_v. \) Each \( \beta_i \)
in (2.4) generates a field over \( k \) of degree \( \leq dR \leq 2nm, \) and therefore for each
\( i \) there is an \( e_i < 2nm \) with \( \beta_i \sigma_v \in G_v. \) Therefore there is some \( e_v \) in \( 1 \leq e_v < 2nm \)
with \( \sigma_v^{e_v} \in G_v. \) Put
\[ B_v = |F|/\sigma_v^{e_v} \quad \text{for} \quad v \in M_1(k). \]

Then (1.7) is certainly true, and
\[ (1.3) \quad \prod_{v \in \mathcal{M}_\omega(k) \cup M_1(k)} B_v \leq \prod_{v \in \mathcal{M}_\omega(k) \cup M_1(k)} A_v^{2nm} = C^{2nm}. \]

4. A differential equation. It remains for us to deal with \( v \in M_2(k). \) For this
case we have to put in a lot of extra effort, but on the other hand, our auxiliary
theorem on differential equations may be of independent interest.

Our solution \( y \) of \( F(X, y) = 0 \) generates a function field \( \mathcal{X} \) over the field
of rational functions \( k(X); \) and \( \mathcal{X} : k(X) \leq n. \) It is well known (see also our
arguments below) that all the derivatives \( y, y', y'', \ldots \) lie in \( \mathcal{X}. \) Now \( y, y', \ldots, y^n \)
must be linearly dependent over \( k(X), \) so that \( y \) satisfies an \( n \)th order linear
differential equation with coefficients in \( k(X), \) and in fact with coefficients in the
polynomial ring \( k[X]. \) We will make this more precise.

Let \( \mathcal{L} \) be a linear differential operator,
\[ \mathcal{L} = L_n(X) \frac{d^n}{dX^n} + \ldots + L_1(X) \frac{d}{dX} + L_0(X) \]
with coefficients \( L_i(X) \in k[X] \) \( (i = 0, \ldots, n). \) We define
\[ \deg \mathcal{L} = \max(\deg L_n, \ldots, \deg L_0). \]

We further define the height by
\[ H_\mathcal{L}(\mathcal{L}) = \prod_{v \in \mathcal{M}(k)} |\mathcal{L}|_v, \]
where
\[ |\mathcal{L}|_v = \max(|L_n|_v, \ldots, |L_0|_v). \]
The absolute height is \( H(\mathcal{L}) = H_\mathcal{L}(\mathcal{L})^{1/d}. \)

Theorem 4. Let \( F, y \) be as above. Then \( y \) satisfies a nontrivial \( n \)-th order linear

differential equation \( \mathcal{L} y = 0, \)
\[ (4.1) \quad \deg \mathcal{L} \leq 2n^2 m, \]
\[ (4.2) \quad H(\mathcal{L}) \leq (16n)^{2n} H(F)^{2n}. \]

We have to begin with a series of lemmas.

Lemma 6. We have
\[ (4.3) \quad A_1(X) y^j = B_{j,0}(X) + B_{j,1}(X) y + \ldots + B_{j,n-1}(X) y^{n-1} \quad (j = 1, 2, \ldots) \]
with certain polynomials \( B_{j,0} \in k[X] \) satisfying
\[ (4.4) \quad \deg B_{j,0} \leq j m, \]
\[ (4.5) \quad |B_{j,0}| \leq (2m+2)|F| \quad \text{when} \quad v \in \mathcal{M}_\omega(k), \]
\[ (4.6) \quad |B_{j,0}| \leq |F| \quad \text{when} \quad v \in M_1(k). \]

Proof. When \( j < n-1 \) we set \( B_{j,0} = A_{n,0} , \) and \( B_{j,0} = 0 \) for \( l \neq j. \) Then (4.3),
(4.4), (4.6) hold. In order to prove (4.5), it is enough to observe the fact, which
will be used repeatedly, that if \( S(X), T(X) \) are polynomials with \( \deg S = s, \)
\( \deg T = t, \) then
\[ (4.7) \quad |S T|_v \leq (1 + \min(s, t)) |S|_v |T|_v. \]
Thus for \( j < n-1, \) (4.5) holds in the strengthened form that
\[ |B_{j,0}| \leq (m+1)^{j+1} |F|_v. \]

Suppose now that the assertion is true for some \( j > n-1. \) Then
\[ A_{n+1,0} y^{n+1} = A_{n,0} B_{j,0} + A_{n+1,1} B_{j,0} y + \ldots + A_{n+1,B_{j,n-2}} y^{n+1} + A_{n+1,B_{j,n-1}} y^n \]
and
\[ A_{n,0} B_{j,n-1} y^{n} = A_{n,0} B_{j,n-1} + A_{n,0} B_{j,n-1} y - \ldots - A_{n,0} B_{j,n-1} y^{n-1}. \]
Therefore (4.3) holds for \( j+1 \) with
\[ B_{j+1,0} = -A_{n,0} B_{j,n-1}, \]
\[ B_{j+1,1} = A_{n,0} B_{j,n-1} - A_{n,0} B_{j,n-1} \quad (0 < i < n). \]

Now (4.4), (4.5), (4.6) follow by induction, where for (4.5) we use the observation
(4.7).

In what follows, denote the partial derivatives of \( F(X, Y) \) by \( F_X, F_Y, F_{XX}, F_{XY}, \ldots \) Further \( G_{i} = G_{i}(X, Y) \) will be a polynomial parametrized by
\( t = 1, 2, \ldots, \) and with partial derivatives \( G_{iX}, G_{iY}. \)

Lemma 7. For \( t = 1, 2, \ldots \) we have
\[ (4.8) \quad F_X(X, y)^{t-1} y^0 = G_{t}(X, y) \]
where \( G_1(X, Y) \in k[X, Y] \) has
\[
\deg_x G_i \leq (2m-1)t - m,
\]
\[
\deg_y G_i \leq (2n-2)t + 2 - n,
\]
\[
|G_{i,v} \leq (20(m+1)^3 n^t) t |F|^{2t-1} \quad \text{when } v \in M_v(k),
\]
\[
|G_{i,v} \leq |F|^{2t-1} \quad \text{when } v \in M_0(k).
\]

Proof. Differentiating \( F(X, y) = 0 \) we obtain \( F_x + F_y y' = 0 \), so that (4.8) is true for \( t = 1 \) with \( G_1 = -F_x \). Then also (4.9), (4.10) hold for \( t = 1 \). When \( v \) is Archimedean, \( |F_{x,v}| \leq |F| \), so that (4.11) is certainly true for \( t = 1 \). Similarly, we have (4.11).

We now proceed by induction on \( t \). Differentiating (4.8) we obtain
\[
(2t-1) F_x^{2t-2}(F_{xy} + F_{yy} y') y^{2t} + F_x^{2t-1} y^{2t+1} = G_{ix} + G_{iy} y'.
\]
We multiply by \( F_y^{2t} \) and note \( F_x^{2t} y^{2t} = G_i \) and \( F_y y' = -F_x \) to obtain
\[
(2t-1) G_i(F_x F_{xy} - F_x F_{yy} y') + F_x^{2t} y^{2t+1} = (F_y G_{ix} - F_x G_{iy}) F_y.
\]
Thus (4.8) holds for \( t + 1 \) with
\[
G_{i+1} = (2t-1) \left| \begin{array}{ccc} F_x & F_y & F_x F_y \\ F_{xy} & F_{yy} & F_x F_{xy} \\ 0 & 0 & F_y \\ \end{array} \right| G_{i,v} - F_y G_{ix} F_{iy}.
\]
Therefore
\[
\deg_x G_{i+1} \leq 2m - 1 + \deg_x G_i,
\]
\[
\deg_y G_{i+1} \leq 2n - 2 + \deg_y G_i,
\]
so that the truth of (4.9), (4.10) for \( t \) implies it for \( t + 1 \).

In what follows, we will use the fact that if \( S(X, Y) \), \( T(X, Y) \) are in \( k[X, Y] \) with \( \deg_x S = s_x \), \( \deg_y S = s_y \), \( \deg_x T = t_x \), \( \deg_y T = t_y \), then
\[
|ST|_v \leq (1 + \min(s_x, t_x))(1 + \min(s_y, t_y)) |S|_v |T|_v.
\]
(Of course, much more is true when \( v \in M_0(k) \).) We have
\[
|F_x|_v \leq |F|_v, \quad |F_y|_v \leq n |F|_v, \quad |F_{xy}|_v \leq mn |F|_v, \quad |F_{xyy}|_v \leq n^2 |F|_v.
\]
and
\[
|F_x F_{xy}|_v \leq m(n-1) |F|_v, \quad |F_{xyy}|_v \leq m^2 n^3 |F|_v, \quad |F_{xyy}|_v \leq m^2 n |F|_v,
\]
so that
\[
|F_x F_{xy} - F_x F_{xyy}|_v \leq 2m^2 n^3 |F|_v.
\]

(Here and below, the \( \leq \) may often be replaced by \( < \) if \( F \neq 0 \).) We further obtain
\[
|F_x F_{xy} - F_x F_{xyy}|_v \leq (2m)(2n-1) |F_x F_{xy} - F_x F_{xyy}|_v |G_{i,v} |_v \leq 8m^3 n^t |F|^{2t} |G|_v.
\]

On the other hand,
\[
|F_x G_{ix}|_v \leq m(n+1) |F|_v |G_{ix}|_v \leq m(n+1) n |F|_v |G_{ix}|_v \leq 2m^2 n^3 t |F|_v |G|_v,
\]
\[
|F_x G_{iy}|_v \leq (m+1) n |F|_v |G_{iy}|_v \leq 2(m+1) n^2 t |F|_v |G|_v,
\]
so that
\[
|F_x G_{ix} - F_x G_{iy}|_v \leq (m+1) n^2 t |F|_v |G|_v.
\]
We further obtain
\[
|F_y (F_x G_{ix} - F_x G_{iy})|_v \leq (m+1) n^2 t |F|_v |F_x G_{ix} - F_x G_{iy}|_v \leq 4(m+1) n^2 t |F|_v |G|_v.
\]
Combining our estimates, we see that (4.13) yields
\[
|G_{i+1}|_v \leq \left[ (2t-1) - 8m^3 n^t + 4(m+1) n^t \right] |F|^{2t} |G|_v \leq 20(m+1)^3 n^t t |F|^{2t} |G|_v.
\]
Now (4.11) follows by induction on \( t \). The proof of (4.13) is similar and simpler.

**Lemma 8.** For \( 0 \leq t \leq n \) we have
\[
F_x(X, y) \geq 2^{n-1} y^{2^n} = N_t(X, y)
\]
where \( N_t(X, Y) \in k[X, Y] \) with
\[
\deg_x N_t \leq 2nm - m - t,
\]
\[
\deg_y N_t \leq 2n^2 - 3n + 2,
\]
\[
|N_{i,v}|_v \leq (20(m+1)^3 n^t)^{n-t} |F|^{2^{n-1}} \quad \text{when } v \in M_0(k),
\]
\[
|N_{i,v}|_v \leq |F|^{2^{n-1}} \quad \text{when } v \in M_0(k).
\]
Proof. When \( t = 0 \) we set \( N_0 = Y \cdot F_0^2 \). Then (4.16), (4.17), (4.19) are clear. On the other hand,
\[
|F_0|_\nu \leq n|F|_\nu \quad \text{and} \quad |F_0'|_\nu \leq ((m+1)n)^{-1} |F|_\nu
\]
by (4.14) and induction on \( j \) for \( j = 1, 2, \ldots \), so that
\[
|F_j|_\nu \leq ((m+1)n)^{-1} |F|_\nu \quad (j = 1, 2, \ldots).
\]
With \( j = 2n-1 \) we obtain (4.18) for \( t = 0 \).

When \( 0 < t \leq n \), we take
\[
N_t(X, Y) = G_t(X, Y) \cdot F_t(X, Y)^{2n^2 - 2t}.
\]
In view of (4.9), (4.10) we obtain
\[
\deg_X N_t \leq ((2m-1)t - m + (2n-2)t) = 2mn - m - t,
\]
\[
\deg_Y N_t \leq (2n-2)t + 2 - n + (2n-2)t(n-1) = 2n^2 - 3n + 2,
\]

i.e., (4.16), (4.17). Using (4.21), (4.14), (4.16), (4.17), (4.11), (4.20), we get
\[
|N_t|_\nu \leq (2mn - 2n + t - 3n) |G_t|_\nu |F_t|^{2n^2 - 2t} |F_t|^{2n^2 - t}.
\]

Since \( t \leq n \), we may conclude that
\[
|N_t|_\nu \leq (2mn)(2n^2)(2m+1)^{n^2} |F_t|^{2n^2 - 2t} \leq (2m+1)^{n^2} |F_t|^{2n^2 - t} + 1,
\]

i.e., (4.18). The proof of (4.19) is similar and simpler.

Lemma 9. For \( 0 \leq t \leq n \) we have
\[
A_t(X)^{2n^2 - 3n^4} F_r(X, y)^{2n^2 - 2t} y^j = Q_t(X, y)
\]
where
\[
\deg_X Q_t \leq 2n^2 m,
\]
\[
\deg_Y Q_t \leq n - 1,
\]
\[
|Q_t|_\nu \leq 2^{2n^2 + 3} (2m + 2)^{2n^2 + 7} m^{5n^2 + 11} |F_t|^{2n^2 - t} + 1 \quad \text{when} \ t \in M_\nu(k),
\]
\[
|Q_t|_\nu \leq |F_t|^{2n^2 - t} + 1 \quad \text{when} \ t \in M_0(k).
\]

Proof. Write
\[
N_t(X, Y) = N_0(X) + N_{i1}(X) Y + \cdots + N_{it}(X) Y^t
\]
where \( s = 2n^2 - 3n^2 + 2 \). Since by Lemma 6,
\[
A_t y^j = B_{j0} + B_{j1} y + \cdots + B_{jn-1} y^{n-1} = B_j(X, y),
\]
say, (4.22) will hold with
\[
Q_t(X, Y) = \sum_{j=0}^{n} A_t(X)^{2n^2 - 3n^4} F_r(X, y)^{2n^2 - 2t} y^j
\]

Then (4.24) is clear. On the other hand, by (4.16), (4.4), a typical summand in (4.27) has
\[
\deg_X \leq (s-j)m + (2nm - m - t) + jm \leq (s+2n-1)m \leq 2n^2 m,
\]
so that (4.23) holds.

We have
\[
A_t^{-1} \leq (m+1)^{-1} |F_t|^{2n^2 - t} + 1
\]

with \( q = (20m + 1)^{3n^2} \).

by (4.18),
\[
|B_j|_\nu \leq (2m + 2)^{2n^2} |F_t|^{2n^2 - t} + 1
\]

by (4.5). A typical summand on the right hand side of (4.27) has \( \nu \)-adic absolute value
\[
\leq (2m + 2)^{(2n^2) \nu} \leq q(2m + 2)^{2n^2} |F_t|^{2n^2 - t} + 1
\]

\[
\leq 6n^2 m^2 (2m + 2)^{(2n^2) \nu} + 1 |F_t|^{2n^2 - t} + 1
\]

\[
\leq 2^n^{2n^2 - 2} (2n + 2)^{2n^2 - 2} |F_t|^{2n^2 - t} + 1
\]

\[
= 2^n^{2n^2 - 2} (2n + 2)^{2n^2 - 7} |F_t|^{2n^2 - t} + 1.
\]

After multiplying by the number \( s+1 \leq 2n^2 \) of summands in (4.27), we obtain
(4.25). The proof of (4.26) is similar and simpler.

Proof of Theorem 4. We may suppose that some coefficient of \( F \) is 1, so that
\[
|F|_\nu \geq 1
\]
for each \( \nu \). Since the discriminant \( D(X) \neq 0 \) by hypothesis, \( y \) with \( F(X, y) = 0 \) has \( F_r(X, y) \neq 0 \).

Write
\[
Q_t(X, Y) = Q_{t0}(X) + Q_{t1}(X) Y + \cdots + Q_{tn-1}(X) Y^{n-1}.
\]

In view of Lemma 9, it will suffice to choose the coefficients \( L_{ij} \), \( i, j = 0, \ldots, n-1 \), of the desired linear differential operator \( L' \) with
\[
\sum_{j=0}^{n} Q_{tj} L_{ij} = 0 \quad (j = 0, \ldots, n-1).
\]
This is a system of \( n \) linear equations in the \( n+1 \) unknowns \( L_0, \ldots, L_n \). Say this system has rank \( r \), and the equations with \( j = 0, \ldots, r-1 \) are independent, say the submatrix \( (Q_{ij})_r \) with \( 0 \leq i, j \leq r-1 \) is nonsingular. We set 
\[ L_{r+1} = \ldots = L_n = 0 \] 
There are no such when \( r = n \), and for \( L_0, \ldots, L_r \) we take certain obvious determinants of order \( r \) from \( (Q_{ij})_r \) with \( 0 \leq t \leq r, 0 \leq j \leq r-1 \).

Then the \( L_i \) will be in \( k[X] \), and (4.23) yields
\[
\deg L_i(X) \leq 2n^2 mr \leq 2n^2 m \quad (i = 0, \ldots, n),
\]
and therefore (4.1). The determinant for \( L_i \) has \( r! \) summands, and each summand is (up to sign) a product of \( r \) factors \( Q_{ij} \). Therefore by (4.14), (4.23), each summand has \( v \)-adic norm
\[
\leq (2n^2 m + 1)^{-1} \max_{0 \leq t \leq r} |Q_{ij}|^t,
\]
and by (4.25) this is
\[
< (2n^2 m)^{r-1} 2^{2a+3b} (2m+2)^{2a+3b} 2^{2a+3b} n^{2a+3b} 2^{2a+3b} |F|^2 a^3,
\]
in view of (4.28). Since the determinant for each \( L_i \) has \( r! \) summands, we obtain
\[
|L_i|_v < 2^{2a+3b} 2^{2a+3b} 2^{2a+3b} n^{2a+3b} 2^{2a+3b} |F|^2 a^3,
\]
This holds for \( v \in M(k) \), whether Archimedean or not. For \( v \in M_0(k) \), i.e., non-Archimedean, we obtain in a similar manner that
\[
|L_i|_v \leq |F|^2 a^3.
\]
Theorem 4 follows.

5. Application of Clark's Theorem. Let \( \mathcal{L} \) be the differential operator of Theorem 4 and \( l = \deg \mathcal{L} \), so that
\[
l \leq 2n^2 m.
\]
Let \( r \) be least such that each \( X^{-1} L_i(X) \) \( (i = 0, \ldots, n) \) is a polynomial. After multiplying our differential operator by \( X^r \), and denoting the new differential operator and its coefficients again by \( \mathcal{L} \), \( L_{n+i} \), \( L_i \), we have
\[
L_i(X) = \sum_{j=0}^{r+i} \lambda_{ij} X^{i+j} \quad (i = 0, \ldots, n),
\]
where the \( \lambda_{ij} \) lie in \( k \), and \( \lambda_{0i}, \lambda_{1i}, \ldots, \lambda_{ri} \) are not all \( 0 \). \( \mathcal{L} \) applied to \( X^s \) is
\[
\mathcal{L}(X^s) = \sum_{j=0}^{s+i} \Phi_j(s+j) X^{i+j} \quad (s = 0, 1, \ldots),
\]
where \( \Phi_j (j = 0, \ldots, l+i) \) is the polynomial with
\[
\Phi_j(s+j) = \sum_{i=0}^{s-j} \lambda_{ij} j(s-1) \ldots (s-i+1).
\]
Then \( \Phi_0 \) is the classical indicial polynomial of \( \mathcal{L} \). We have
\[
\Phi_j(X) = \sum_{i=0}^{j} \lambda_{ij} (X-j)(X-j-1) \ldots (X-j-i+1),
\]
where the summand with \( i = 0 \) is to be interpreted as \( \lambda_{00} \). Therefore
\[
|\Phi_j|_v = (1+(j+1)+(j+1)(j+2)+\ldots+(j+1)(j+n))|\mathcal{L}|_v,
\]
and
\[
|\Phi_j|_v \leq (1+j+n)|\mathcal{L}|_v \quad \text{when} \quad v \in M_0(k).
\]
On the other hand,
\[
|\Phi_j|_v \leq |\mathcal{L}|_v \quad \text{when} \quad v \in M_0(k).
\]
Put
\[
\psi_v = \max_{j=0, \ldots, l+i} |\Phi_j|_v^{1/2};
\]
and in (5.7) below, the term on the right for \( j = 0 \) is to mean 1.

Lemma 10. We have
\[
\prod_{v \in M(k)} \psi_v \leq (n+2)^{2a} H_4(\mathcal{L}).
\]

Proof. When \( v \in M_0(k) \), we have from (5.4) (and the fact that \( (1+j+n)^{1/2} \leq n+2 \) when \( j \geq 1 \) that
\[
(5.7) \quad \psi_v \leq (n+2)^{2a} \max_{j=0, \ldots, l+i} (|\mathcal{L}|_v |\Phi_0|_v)^{1/2} \leq (n+2)^{2a} \max_{j=0, \ldots, l+i} (|\mathcal{L}|_v |\Phi_0|_v). \]

When \( v \in M_0(k) \), then
\[
\psi_v \leq \max_{j=0, \ldots, l+i} (|\mathcal{L}|_v |\Phi_0|_v).
\]
Therefore, using (5.4), (5.5) again,
\[
\prod_{v \in M(k)} \psi_v \leq (n+2)^{2a} \left( \prod_{v \in M(k)} \max_{j=0, \ldots, l+i} (|\mathcal{L}|_v |\Phi_0|_v) H_4(\Phi_0)^{-1} \right) \leq (n+2)^{2a} \prod_{v \in M(k)} |\mathcal{L}|_v = (n+2)^{2a} H_4(\mathcal{L}).
\]

When \( v \in M_0(k) \) and \( v \mid p \), put
\[
(5.8) \quad \omega_v = p^{-n(p-1)} \psi_v^{-1}.
\]

Lemma 11. Suppose \( v \in M_0(k) \), \( v \in M(K) \) and \( w \mid v \). Then \( \omega_w \geq \omega_v \).

Proof. This follows from Clark's Theorem 3 in [1]. Clark uses the order function rather than our absolute value \( |_v \), so that his value group is additive rather than multiplicative. His \( b_1(i) \) corresponds to our \( \psi_v^{-1} \), his \( b(i) \) corresponds to our \( \omega_v \). The zeros of our indicial polynomial \( \Phi_0 \) are algebraic, so that
they are non-Liouville as defined by Clark. His function \( w(\alpha) \) (see Definition 2 in [1]) has \( 0 \leq w(\alpha) \leq 1/(p-1) \). Since \( \Phi_0 \) is of degree \( \leq n \), Clark's \( w(\Phi_0) \) is \( n/(p-1) \), which explains the exponent in (5.8).

6. Conclusion. For \( v \mid w \) with \( w \in M(K), v \in M_0(k), \) the quantity \( r_v \) of Lemma 2 has \( r_v \geq \min(\sigma_v, \omega_v) \) by Lemma 11. When \( v \mid p \), let \( r_v^* \) be the largest number of the type \( \rho \) with \( \rho \in Z \) having \( r_v^* \leq \min(\sigma_v, \omega_v) \). Then \( r_v^* \in G_v \),

\[
\tau_v^* \geq p^{-1} \min(\sigma_v, \omega_v),
\]

and

\[
|\delta| \leq |F| (1/\tau_v^*)^{n+1}
\]

by Lemma 2.

For \( v \in M_2(k) \) we set

\[
A_v = B_v = |F|/\tau_v^*.
\]

Then \( B_v \in G_v \), and (1.4), (1.7) are true for \( w \mid v \). We have

\[
\prod_{v \in M_2(k)} A_v \leq H_v(F) \prod_{v \in M_1(k)} (p \max(\sigma_v^{-1}, \omega_v^{-1}))
\]

where \( p = p(v) \) with \( v \mid p \). We obtain

\[
H_v(F) \leq \prod_{v \in M_1(k)} (p^{1+n(2p-1)} \sigma_v^{-1} \psi_v)
\]

from (5.8), since \( \sigma_v < 1, \psi_v \geq 1 \). Given \( p \), there are \( \delta \) elements \( v \in M(k) \) with \( v \mid p \), so that

\[
\prod_{v \in M_1(k)} p^{1+n(2p-1)} \leq \prod_{p \leq \delta} p^{2nd(p-1)} \leq \delta^{2nd}.
\]

(We are very generous!) Combining this with Lemma 5 and 10, we get

\[
\prod_{v \in M_2(k)} A_v \leq (n^\delta(n+1)(n+1) \sqrt{n(n+2)})^{2nd} H(F)^{2nd}.
\]

With the help from Theorem 4 we finally obtain

\[
\prod_{v \in M_1(k)} A_v \leq (16m)^{11n+\delta} H(F)^{2n^2+2\delta},
\]

so that (1.6) holds. Theorem 1 is established.

For (1.8) and Theorem 2, we recall that \( B_v = A_v \) when \( v \in M_2(k) \), so that (3.3) in conjunction with (6.1) yields

\[
\prod_{v \in M_1(k)} B_v \leq (16m)^{11n^2} H(F)^{(2n^2+2\delta) n^2m},
\]

further by (1.5),

\[
\prod_{v \in M_1(k)} B_v \leq (16m)^{11n^2} (4mn \sqrt{n})^{n^2m} H(F)^{(2n^2+2n+4n^2m)d},
\]

and therefore (1.8).