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Large values of certain number-theoretic error terms

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1. Introduction. This paper is primarily concerned with the investigation of large values of certain number-theoretic error terms. The error terms in question essentially admit asymptotic expansions in terms of (generalized) Bessel functions, but before we give the general definition, it seems appropriate to give first some examples. Typical ones are $\Delta(x)$, the error term in the classical divisor problem, and $P(x)$, the error term in the circle problem. They are defined as

$$\Delta(x) = \sum_{n\leqslant x}{}' d(n) - x(\log x + 2\gamma - 1) - 1/4, \qquad P(x) = \sum_{n\leqslant x}{}' r(n) - \pi x + 1,$$

where $d(n)$ is the number of divisors of n, γ is Euler's constant, and $r(n)$ is the number of representations of n as a sum of two integer squares. In general, the sum $\sum'_{n\leqslant x} f(n)$ means that if x is an integer, then the last term in the sum is $f(x)/2$. By a classical formula of G. F. Voronoi [34] (see also Ch. 3 of [22])

$$\Delta(x) = \frac{1}{\pi}\sum_{n=1}^{\infty} d(n) n^{-1} f_0(\pi^2 x n), \tag{1.1}$$

$$f_0(x) = -x^{1/2}\left(Y_1(4\sqrt{x}) + \frac{2}{\pi}K_1(4\sqrt{x})\right),$$

where Y_ν is the ordinary Bessel function of the second kind and K_ν is the modified Bessel function (see G. N. Watson [35] or Ch. 3 of [22] for definition and properties of the Bessel functions).

The sums in the definition of $\Delta(x)$ and $P(x)$ are special cases of the general sum

$$F_\varrho(x) = \frac{1}{\Gamma(\varrho+1)}\sum_{n\leqslant x}{}' f(n)(x-n)^\varrho, \tag{1.2}$$

which may be defined for any arithmetical function $f(n)$ and $\varrho \geqslant 0$ (it is also defined for $\varrho < 0$ if x is not an integer). If $f(n) = r(n)$ in (1.2), then G. H. Hardy [17], [18] proved that, for $\varrho \geqslant 0$,

$$F_\varrho(x) = \frac{\pi x^{1+\varrho}}{\Gamma(\varrho+2)} - \frac{x^\varrho}{\Gamma(\varrho+1)} + \frac{1}{\pi^\varrho}\sum_{n=1}^{\infty} r(n)\left(\frac{x}{n}\right)^{(1+\varrho)/2} J_{1+\varrho}(2\pi\sqrt{nx}), \tag{1.3}$$

where J_ν is the ordinary Bessel function of order ν. Chandrasekharan and Narasimhan [6] showed that (1.3) also holds for $-1/2 < \varrho < 0$ if x is not an integer. The series of Bessel functions in (1.3) is absolutely convergent for $\varrho > 1/2$, since

$$(1.4) \qquad J_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos(x - \pi\nu/2 - \pi/4) + O(x^{-3/2}) \qquad (x \to \infty),$$

while for $-1/2 < \varrho \leqslant 1/2$ it converges only conditionally. Since

$$F_0(x) = \sum_{n \leqslant x}{}' r(n),$$

it follows that (1.3) gives an explicit formula for $P(x)$ in terms of Bessel functions (or an asymptotic expansion in terms of cosines, if one uses (1.4)).

Another classical expansion of similar type is

$$(1.5) \qquad \sum_{n \leqslant x}{}' \tau(n) = \sum_{n=1}^{\infty} \tau(n)(x/n)^6 J_{12}(4\pi\sqrt{nx}),$$

where $\tau(n)$ is the well-known multiplicative function of S. Ramanujan (see T. M. Apostol [1] and G. H. Hardy [19]), defined by

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\{(1-x)(1-x^2)(1-x^3)\ldots\}^{24} \qquad (|x| < 1).$$

Note that on the right-hand side of (1.5) there is no main term, but only the oscillatory Bessel-function terms. This reflects the oscillatory character of $\tau(n)$, which is perhaps the best known example of a larger class of arithmetical functions. This is the class of functions $a(n)$, which are the Fourier coefficients of a cusp form of weight $\varkappa = 2n$ ($\geqslant 12$) for the full modular group (see T. M. Apostol [1]). If $\varphi(s)$ is the Dirichlet series (zeta-function) attached to $a(n)$, that is, if

$$\varphi(s) = \sum_{n=1}^{\infty} a(n)n^{-s} \qquad (\sigma = \operatorname{Re} s > (\varkappa+1)/2),$$

then E. Hecke showed that $\varphi(s)$ satisfies a simple functional equation. This is

$$(1.6) \qquad (2\pi)^{-s}\Gamma(s)\varphi(s) = (-1)^{\varkappa/2}(2\pi)^{-(\varkappa-s)}\Gamma(\varkappa-s)\varphi(\varkappa-s),$$

which plays a fundamental role in the study of $\varphi(s)$ (in the case of $\tau(n)$ one has $\varkappa = 12$).

In the general case one has (see Chandrasekharan–Narasimhan [6] or M. Jutila [27])

$$(1.7) \qquad A_\varrho(x) = \frac{1}{\Gamma(\varrho+1)} \sum_{n \leqslant x}{}' a(n)(x-n)^\varrho$$
$$= (-1)^{\varkappa/2}(2\pi)^{-\varrho} \sum_{n=1}^{\infty} a(n)\left(\frac{x}{n}\right)^{(\varkappa+\varrho)/2} J_{\varkappa+\varrho}(4\pi\sqrt{nx})$$

for $\varrho > -1/2$ (for $-1/2 < \varrho < 0$ x may not be an integer, and the series is absolutely convergent for $\varrho > 1/2$ by (1.4)). For $\varrho = 0$ and $a(n) = \tau(n)$ (1.7) reduces to (1.5).

Asymptotic expansions of the type (1.7) are only special cases of the more general situation where $a(n)$ is replaced by Fourier coefficients of a so-called automorphic representation, such as the Maass wave forms. Another possibility is to insert besides $a(n)$ a suitable exponential factor (see M. Jutila [27]). However, in order to avoid technicalities, we shall not go that far in our investigations.

Before we pass to the general case of number-theoretic error terms which we shall investigate, we finally point out another important example. This is the function $E(T)$, defined by

$$(1.8) \qquad E(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt - T\big(\log(T/(2\pi)) + 2\gamma - 1\big).$$

From a classical formula of F. V. Atkinson [2] (see also Ch. 15 of [22]) it easily follows that $E(T) = O(T^{1/2}\log T)$, so that $E(T)$ may be considered as the error term in the mean square formula for the Riemann zeta-function on the critical line $\sigma = 1/2$. Atkinson was the first to point out certain analogies between $2\pi\Delta(T/(2\pi))$ and $E(T)$, which were later investigated in more detail by M. Jutila [26]. However, unlike the previous examples, the function $E(T)$ does not arise directly in connection with a particular arithmetical function. Its connection with the divisor function $d(n)$ is only indirect.

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2. The functional equation. We are now going to give the definition of the general number-theoretic error term (due to Chandrasekharan–Narasimhan [7]), which includes all the previous examples except $E(T)$. The identities (1.1), (1.3), (1.5) and (1.7) all involve arithmetical functions whose generating Dirichlet series satisfy a certain type of functional equation, analogous to (1.6). In the case of $d(n)$, generated by $\zeta^2(s)$, such a functional equation follows simply by squaring the classical equation

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)$$

for the Riemann zeta-function $\zeta(s)$. K. Chandrasekharan and R. Narasimhan [6], [7], [8], were the first to establish identities (asymptotic expansions) for arithmetical functions associated with Dirichlet series satisfying a general functional equation. As in the simplest case of $\zeta(s)$, the functional equation that they considered involved gamma factors. Their work was continued by other researchers, including B. C. Berndt [3], [4], [5] and J. L. Hafner [11]. Our number-theoretic error terms will be the error terms in the asymptotic formula for the summatory functions of arithmetical functions whose Dirichlet series satisfy a functional equation with multiple gamma factors. The definition of this type of functional equation, given first by Chandrasekharan–Narasimhan [7], is as follows:

Let $\{f(n)\}$ and $\{g(n)\}$ be two sequences of complex numbers not identically zero, and let λ_n, μ_n be two strictly increasing sequences of positive numbers tending to infinity. Suppose the series

$$\varphi(s) = \sum_{n=1}^{\infty} f(n)\lambda_n^{-s}, \qquad \psi(s) = \sum_{n=1}^{\infty} g(n)\mu_n^{-s}$$

converge in some half-plane and have abscissae of absolute convergence σ_a^* and σ_a, respectively. For each $\nu = 1, 2, \ldots, N$ suppose that $\alpha_\nu > 0$ and β_ν is complex, and let

$$\Delta(s) = \prod_{\nu=1}^{N} \Gamma(\alpha_\nu s + \beta_\nu).$$

If r is real, then φ and ψ satisfy the functional equation

$$\Delta(s)\varphi(s) = \Delta(r-s)\psi(r-s) \tag{2.1}$$

if there exists in the s-plane a domain D that is the exterior of a compact set S and on which there exists a holomorphic function $\chi(s)$ ($s = \sigma + it$, σ and t real) such that

(i)
$$\lim_{|t|\to\infty} \chi(\sigma+it) = 0$$

uniformly in every interval $-\infty < \sigma_1 \leqslant \sigma \leqslant \sigma_2 < \infty$, and

(ii)
$$\chi(s) = \begin{cases} \Delta(s)\varphi(s) & \text{for } \sigma > \sigma_a^*, \\ \Delta(r-s)\psi(r-s) & \text{for } \sigma < r-\sigma_a. \end{cases}$$

Thus with $\varphi(s) = \psi(s) = \pi^{-s}\zeta^2(s)$, $\Delta(s) = \Gamma^2(s/2)$, $r = 1$ we get the functional equation associated with $d(n)$. If

$$L(s) = \sum_{n=1}^{\infty} r(n)n^{-s} \qquad (\sigma > 1),$$

then the functional equation associated with $r(n)$ follows for $\varphi(s) = \psi(s) = \pi^{-s}L(s)$, $\Delta(s) = \Gamma(s)$, $r = 1$. The functional equation associated with $a(n)$ is (2.1) with $\psi(s) = (-1)^{\varkappa/2}\varphi(s)$, $\Delta(s) = \Gamma(s)$, $r = \varkappa$, where

$$\varphi(s) = (2\pi)^{-s}\sum_{n=1}^{\infty} a(n)n^{-s} \qquad (\sigma > (\varkappa+1)/2).$$

For $x > 0$ and ϱ a real number, the general summatory function of the arithmetical function $f(n)$ (of order ϱ) is defined as

$$F_\varrho(x) = \frac{1}{\Gamma(\varrho+1)} {\sum_{\lambda_n \leqslant x}}' f(n)(x-\lambda_n)^\varrho, \tag{2.2}$$

which reduces to (1.2) if $\lambda_n = n$, and where the dash in the sum indicates that the last term is to be halved if $\varrho = 0$ and $x = \lambda_n$ (if $\varrho < 0$ then (2.2) is defined only if x is not any λ_n). If henceforth $\alpha = \sum_{j=1}^{N} \alpha_j$, then for $\varrho > 2\alpha\sigma_a - \alpha r - 1/2$ (see Chandrasekharan–Narasimhan [7] and J. L. Hafner [11]) we have

$$F_\varrho(x) = Q_\varrho(x) + \sum_{n=1}^{\infty} g(n)\mu_n^{-r-\varrho} f_\varrho(x\mu_n). \tag{2.3}$$

In this formula $Q_\varrho(x)$ may be considered as a "main term" or "residual function". It is defined by the integral representation

$$Q_\varrho(x) = \frac{1}{2\pi i}\int_C \frac{\Gamma(s)\varphi(s)x^{\varrho+s}}{\Gamma(s+\varrho+1)}\,ds, \tag{2.4}$$

where C is a suitable rectangle containing all singularities of $\varphi(s)$, so that $Q_\varrho(x)$ can be evaluated fairly easily in case $\varphi(s)$ has only poles. The function $f_\varrho(x)$ is the so-called "generalized Bessel function", since in case when $\varrho \geqslant 0$ is an integer $f_\varrho(x)$ can be often expressed in terms of classical Bessel functions. Such is the case of the classical expansions (1.1) and (1.3). Explicitly one has (see J. L. Hafner [11])

$$f_\varrho(x) = \frac{1}{2\pi i}\int_{C_{a,b}} \frac{\Gamma(r-s)\Delta(s)x^{r+\varrho-s}}{\Gamma(r+\varrho-s+1)\Delta(r-s)}\,ds \qquad (\varrho > \min(2\alpha\sigma_a - \alpha r - 4, -1)), \tag{2.5}$$

where $C_{a,b}$ is the line $\sigma = a > 0$, $a = \min(\sigma_a - 2/\alpha, r/2 - 1/(2\alpha))$, suitably indented to the right to contain all the singularities of the integrand in the right half-plane. In [11] Hafner proves that, for $\varrho > \min(2\alpha\sigma_a - \alpha r - 4, -1)$,

$$\frac{d}{dx} f_{\varrho+1}(x) = f_\varrho(x) \tag{2.6}$$

and that for any fixed integer $m \geqslant 0$

$$f_\varrho(x) = \sum_{j=0}^{m} e_j(\varrho)x^{\theta_\varrho - j/(2\alpha)}\cos(hx^{1/(2\alpha)} + k_j(\varrho)) + O(x^{\theta_\varrho - (m+1)/(2\alpha)}) + O(x^{r+\varrho-b}) \tag{2.7}$$

for $b > 0$ a large constant and

$$\theta_\varrho = \frac{r}{2} - \frac{1}{4\alpha} + \varrho\left(1 - \frac{1}{2\alpha}\right). \tag{2.8}$$

Here $e_j(\varrho)$, $k_j(\varrho)$ and h are suitable constants which may be explicitly evaluated. Thus for most sequences $g(n)$, μ_n that are encountered in applications the series in (2.3) converges absolutely for ϱ sufficiently large, which is a fundamental fact, used in essentially every application of formulas of this type. Hafner [11] actually shows that (2.3) holds if $\varrho > 2\alpha\sigma_a - \alpha r - 3/2$, providing that an additional technical condition is satisfied, which occurs for many examples interesting in applications, including $d(n)$, $r(n)$ and $a(n)$.

Finally we may define (for $\varrho \geqslant 0$)

$$P_\varrho(x) = F_\varrho(x) - Q_\varrho(x) \tag{2.9}$$

as the error term in the asymptotic formula for $F_\varrho(x)$ (we avoid the case $\varrho < 0$ when $P_\varrho(x)$ is undefined for $x = \lambda_n$). Since $F_\varrho(x)$ (as a function of x) is continuous for $\varrho > 0$ and all $x > 0$, and the same is also true for $Q_\varrho(x)$ by the integral representation (2.4), it follows that $P_\varrho(x)$ (as a function of x) is continuous for $\varrho > 0$ and $x > 0$. Note that when $\varrho = 0$ it may happen that $P_0(x)$ is not continuous, because it has jumps for $x = \lambda_n$. Such is the case of $\Delta(x)$ and $P(x)$ (both are of the form $P_0(x)$), where the functions in question have jumps at integral x which may be as large as $\exp(C\log x/\log\log x)$.

From (2.9) it follows that $P_{\varrho+1}(x)$ is a primitive function of $P_\varrho(x)$ for $\varrho \geqslant 0$, which we shall write as

$$\int P_\varrho(x)\,dx = P_{\varrho+1}(x) \quad (\varrho \geqslant 0). \tag{2.10}$$

This important property follows from the same property for $F_\varrho(x)$ and $Q_\varrho(x)$. In view of $s\Gamma(s) = \Gamma(s+1)$ we have by (2.4)

$$\int Q_\varrho(x)\,dx = \frac{1}{2\pi i}\int_C \frac{\Gamma(s)\varphi(s)x^{\varrho+s+1}}{\Gamma(s+\varrho+1)(\varrho+s+1)}\,ds = \frac{1}{2\pi i}\int_C \frac{\Gamma(s)\varphi(s)x^{\varrho+s+1}}{\Gamma(s+\varrho+2)}\,ds = Q_{\varrho+1}(x),$$

and also

$$\int_0^x F_\varrho(t)\,dt = \frac{1}{\Gamma(\varrho+1)}\int_0^x \sum_{\lambda_n\leqslant t}{}' f(n)(t-\lambda_n)^\varrho\,dt$$
$$= \frac{1}{\Gamma(\varrho+1)}\sum_{\lambda_n\leqslant x}{}' \int_{\lambda_n}^x f(n)(t-\lambda_n)^\varrho\,dt = \frac{1}{\Gamma(\varrho+1)(\varrho+1)}\sum_{\lambda_n\leqslant x} f(n)(x-\lambda_n)^{\varrho+1}$$
$$= F_{\varrho+1}(x),$$

which establishes (2.10).

The relation (2.3) may be cast in the form

$$P_\varrho(x) = \sum_{n=1}^\infty g(n)\mu_n^{-r-\varrho} f_\varrho(x\mu_n) \quad (\varrho > 2\alpha\sigma_a - \alpha r - 1/2), \tag{2.11}$$

with the remark that in case of the function $a(n)$ and similar functions where there is no main term one has $P_\varrho(x) = F_\varrho(x)$.

The situation with the function $E(T)$ is, as was pointed out earlier, markedly different. There is no functional equation resembling (2.1), and there is no asymptotic series expansion analogous to (2.11). However Hafner–Ivić [15], [16] proved a sharp asymptotic formula for the integral of $E(t)$. This is

$$\int_2^T E(t)\,dt = \pi T + G(T), \tag{2.12}$$

$$G(T) = 2^{-3/2}\sum_{n\leqslant N}(-1)^n d(n) n^{-1/2}\left(\operatorname{arsinh}\sqrt{\frac{\pi n}{2T}}\right)^{-2}\times\left(\frac{T}{2\pi n}+\frac{1}{4}\right)^{-1/4}\sin(f(T,n)) \tag{2.13}$$
$$-2\sum_{n\leqslant N'} d(n) n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-2}\sin\left(T\log\left(\frac{T}{2\pi n}\right)-T+\frac{\pi}{4}\right)+O(T^{1/4}),$$

where for any fixed $0 < A < A'$ we have

$$AT < N < A'T, \quad N' = \frac{T}{2\pi}+\frac{N}{2}-\left(\frac{N^2}{4}+\frac{NT}{2\pi}\right)^{1/2},$$

$$f(T,n) = 2T\operatorname{arsinh}\sqrt{\frac{\pi n}{2T}}+(2\pi nT+\pi^2n^2)^{1/2}-\frac{\pi}{4}, \quad \operatorname{arsinh}x = \log(x+\sqrt{x^2+1}).$$

By Taylor's formula (2.13) simplifies to

$$G(T) = \frac{1}{2}\left(\frac{2T}{\pi}\right)^{3/4}\sum_{n=1}^\infty(-1)^n d(n) n^{-5/4}\sin(\sqrt{8\pi nT}-\pi/4)+O(T^{2/3}\log T). \tag{2.14}$$

Note that, apart from the unimportant oscillating factor $(-1)^n$, the absolutely convergent series in (2.14) is completely analogous to the one that will appear in the asymptotic expansion of $P_1(x) = \int_0^x \Delta(t)\,dt$ for $d(n)$, given by (2.11), if the appropriate Bessel functions are approximated by trigonometric functions. The fact that the series in (2.14) is absolutely convergent is crucial in establishing sharp omega results for $E(T)$ (see Hafner–Ivić [15], [16]). Likewise, in what follows, the asymptotic formula (2.14) will enable us to treat $\Delta(x)$ and $E(T)$ in a very similar way.

3. Omega results. Statistical results on the occurrence of large values of $\Delta(x)$, $P(x)$ and $E(T)$ were obtained by the author [20], [21], [22]. The main goal of this work is to establish two-sided omega results in short intervals for $P_\varrho(x)$ (given by (2.9)), related to a suitable subclass of functions satisfying the functional equation (2.1), and for $E(T)$. We recall the definition of the omega symbols: if $g(x) > 0$ for $x \geqslant x_0$, then $f(x) = \Omega_+(g(x))$ means that $f(x) > Cg(x)$ for some constant $C > 0$ and some arbitrarily large values of x, and $f(x) = \Omega_-(g(x))$ if $f(x) < -Cg(x)$ for some arbitrarily large values of x. Finally $f(x) = \Omega_\pm(g(x))$ means that both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ hold, while $f(x) = \Omega(g(x))$ means that $|f(x)| = \Omega_+(g(x))$. In general, the error term

$P_\varrho(x)$ will be a complex-valued function of x, and the methods used in [7], [8], [13] provide Ω_+ and Ω_- results for Re $P_\varrho(x)$ and Im $P_\varrho(x)$. For the sake of simplicity we shall henceforth assume that the sequences $f(n)$ and $g(n)$ in Section 2 are real, and moreover we shall impose some (moderately restrictive) conditions on $g(n)$ and μ_n which will be satisfied in most important applications such as $d(n)$, $r(n)$ and (real) $a(n)$. Other conditions of similar nature could be also assumed, but the following ones are fairly simple and easy to check in practice. We shall suppose that

(i) $g(n) \ll_\varepsilon n^{K+\varepsilon}$ for some $K \geqslant 0$ and $g(n) \gg n^K$ for infinitely many n,

(ii) $n^H \ll \mu_n \ll n^H$ for some $H > 0$, and $\mu_m^c - \mu_n^c \gg m^{Hc} - n^{Hc}$ for $0 < c < 1/H$ and $m > n$.

For the classical error terms mentioned in Section 1 the best known omega results at present are:

$$\Delta(x) = \Omega_+\{(x\log x)^{1/4}(\log\log x)^{(3+\log 4)/4}\exp(-B\sqrt{\log\log\log x})\} \quad (B>0), \tag{3.1}$$

$$\Delta(x) = \Omega_-\left\{x^{1/4}\exp\left(\frac{C(\log\log x)^{1/4}}{(\log\log\log x)^{3/4}}\right)\right\} \quad (C>0), \tag{3.2}$$

$$P(x) = \Omega_+\left\{x^{1/4}\exp\left(\frac{D(\log\log x)^{1/4}}{(\log\log\log x)^{3/4}}\right)\right\} \quad (D>0), \tag{3.3}$$

$$P(x) = \Omega_-\{(x\log x)^{1/4}(\log\log x)^{(\log 2)/4}\exp(-E\sqrt{\log\log\log x})\} \quad (E>0), \tag{3.4}$$

$$\sum_{n\leqslant x}{}' a(n) = \Omega_\pm(x^{\frac{1}{2}\varkappa-\frac{1}{4}}\log\log\log x). \tag{3.5}$$

Of these, (3.1) and (3.4) are due to J. L. Hafner [12], [13], (3.2) and (3.3) were proved by Corrádi–Kátai [9], and (3.5) was proved by H. Joris [25] (the $a(n)$'s are assumed to be real). These papers also contain references to earlier results on the same problems. Omega results for the error term $P_\varrho(x)$ involving the functional equation (2.1) were obtained by numerous authors, including B. C. Berndt [5], Chandrasekharan–Narasimhan [7], D. Redmond [30] and J. L. Hafner [13]. The results of Hafner appear to be hitherto the sharpest and most general ones. Instead of our assumptions (i) and (ii), Hafner makes different assumptions on $g(n)$ and μ_n, which are (like in our case) fulfilled for most examples interesting in applications. Under his assumptions he proves that

$$\operatorname{Re} P_\varrho(x) = \Omega_\pm(x^{\theta_\varrho}g(x)) \tag{3.6}$$

with

$$\theta_\varrho = \frac{r}{2} - \frac{1}{4\alpha} + \varrho\left(1-\frac{1}{2\alpha}\right), \quad \alpha = \sum_{j=1}^N \alpha_j, \tag{3.7}$$

where r is the constant appearing in (2.1), and $g(x)$ is a certain function tending to ∞ as $x\to\infty$ (slower than any power of x). The explicit construction of $g(x)$ is rather involved, and the reader is referred to [13] for details (this function has no connection with the $g(n)$ in the definition of (2.1)). The results of [13] are relevant for

$$\varkappa = \sigma_a - \frac{1}{2}r - \frac{1}{4\alpha} - \frac{\varrho}{2\alpha} \geqslant 0,$$

as the case $\varkappa < 0$ was previously treated by Chandrasekharan–Narasimhan [7]. One may conjecture that $P_\varrho(x) \ll x^{\theta_\varrho+\varepsilon}$, which reduces to the classical (hitherto unproved) conjectures $\Delta(x) \ll x^{1/4+\varepsilon}$ and $P(x) \ll x^{1/4+\varepsilon}$ in case of the divisor and circle problem, respectively. That is, one expects (3.6) to be close to the true order of magnitude of $P_\varrho(x)$.

In [15] Hafner–Ivić showed that the analogues of (3.1) and (3.2) hold also for $E(T)$, and one expects $E(T) \ll T^{1/4+\varepsilon}$ to hold by analogy with the divisor problem (more on this topic in Section 6). The proof depended on the asymptotic formula (2.14) and was technically more complicated than the corresponding proof in the divisor problem, although the basic ideas of the proof were naturally the same. Similar ideas were used by J. L. Hafner [14] in deriving sharp omega results in a two-dimensional divisor problem. As in the case of $E(T)$, there was no functional equation of the type (2.1), but instead there was a good asymptotic formula for the relevant integral.

The preceding results have the shortcoming that there is no localization of the values for which e.g. (3.6) is attained. The form of localization given by Theorem C of [13] is not effective. The approximation theorems of Dirichlet and Kronecker that are used in the proofs would lead to very poor localization results (intervals for which the omega result in question is attained at least once). What we seek is a slightly poorer result than (3.6), but with good localization. Our main result in this direction is

THEOREM 1. *Let $P_\varrho(x)$ be defined by (2.9) and let* (i) *and* (ii) *on $g(n)$ and μ_n hold. Then for fixed $\varrho \geqslant 0$ there exist two constants B, $C > 0$ such that for $x \geqslant x_0$ every interval $[x, x+Cx^{1-1/(2\alpha)}]$ contains two values x_1, x_2 for which*

$$P_\varrho(x_1) > Bx_1^{\theta_\varrho}, \quad P_\varrho(x_2) < -Bx_2^{\theta_\varrho}, \tag{3.8}$$

where θ_ϱ is given by (3.7).

Recall that, for $\varrho > 0$, $P_\varrho(x)$ is a continuous function of x, so that in this case (3.8) implies the existence of a zero of odd order of $P_\varrho(x)$ in $[x, x+Cx^{1-1/(2\alpha)}]$. For $P_0(x)$ such a zero does not necessarily exist, but only a change of sign in the aforementioned interval. Hence for $\varrho \geqslant 0$ there are at least $C_1 T^{1/(2\alpha)}$ changes of sign of $P_\varrho(x)$ for $0 < x \leqslant T$, $T \geqslant T_0$, where $C_1 > 0$ is a suitable constant. This corollary was also obtained by J. Steinig (Th. 4.1 of [32]) with an explicit value of C_1. His method, which requires ϱ to be an integer, was based on a refinement of a classical method of G. Pólya. It is different from ours, and does not seem to be able to yield (3.8).

4. The necessary lemmas. The proof of Theorem 1 depends on a simple mean-value estimate. This is

LEMMA 1. *Let $a > 1$, A, B be given real constants and*

$$f(t) = \sum_{n=1}^{\infty} g(n)\mu_n^{-u}\cos\left(A(t\mu_n)^{1/a}+B\right), \tag{4.1}$$

where $g(n)$, μ_n are real and satisfy (i) *and* (ii) *of Section 3 with $H < a$, $u > 0$ and $Hu > 1+K$. Then there exists a constant $C > 0$ such that uniformly for $1 \ll G \leqslant T$*

$$\int_T^{T+G} f^2(t)\,dt = CG+O(T^{1-1/a}). \tag{4.2}$$

Proof. The condition $Hu > 1+K$ ensures that the series in (4.1) is absolutely convergent. Therefore it may be squared and integrated termwise, and the left-hand side of (4.2) equals

$$\sum_{n=1}^{\infty} g^2(n)\mu_n^{-2u}\int_T^{T+G}\cos^2\left(A(t\mu_n)^{1/a}+B\right)dt \tag{4.3}$$

$$+O\Big\{\sum_{m,n=1,m\neq n}^{\infty} |g(m)g(n)|(\mu_m\mu_n)^{-u}\Big|\int_T^{T+G}\exp\left(iAt^{1/a}(\mu_m^{1/a}\pm\mu_n^{1/a})\right)dt\Big|\Big\}.$$

The first integral in (4.3) is

$$\tfrac{1}{2}\int_T^{T+G}\{1+\cos(2A(t\mu_n)^{1/a}+2B)\}\,dt = \tfrac{1}{2}G+O(T^{1-1/a}\mu_n^{-1/a}).$$

Here we used the elementary estimate (see Lemma 2.1 of [22])

$$\Big|\int_A^B e^{iF(x)}\,dx\Big| \leqslant 4m^{-1}, \tag{4.4}$$

if $F(x)$ is a real, differentiable function such that $F'(x)$ is monotonic on $[A, B]$ and $|F'(x)| \geqslant m > 0$. Therefore

$$\sum_{n=1}^{\infty} g^2(n)\mu_n^{-2u}\int_T^{T+G}\cos^2\left(A(t\mu_n)^{1/a}+B\right)dt = \tfrac{1}{2}\sum_{n=1}^{\infty} g^2(n)\mu_n^{-2u}G+O(T^{1-1/a})$$

uniformly in G, and the last series is absolutely convergent. By (ii) (with $c = 1/a$) and the mean value theorem we have, for $m > n$,

$$\mu_m^{1/a}-\mu_n^{1/a} \geqslant m^{H/a}-n^{H/a} = (m-n)M^{H/a-1},$$

where $n \leqslant M \leqslant m$ and $H/a < 1$. Hence, using again (4.4) and (i) it follows that the sum in the O-term in (4.3) is majorized by

$$T^{1-1/a}\sum_{1\leqslant n<m} m^{K+\varepsilon}n^{K+\varepsilon}(mn)^{-Hu}(\mu_m^{1/a}-\mu_n^{1/a})^{-1}$$

$$\ll T^{1-1/a}\sum_{1\leqslant n<m} n^{K+\varepsilon-Hu}\frac{m^{K+\varepsilon-Hu+1-H/a}}{m-n}$$

$$= T^{1-1/a}\sum_{n=1}^{\infty} n^{K+\varepsilon-Hu}\sum_{n<m\leqslant 2n}\frac{m^{K+1+\varepsilon-Hu-H/a}}{m-n}$$

$$+T^{1-1/a}\sum_{n=1}^{\infty} n^{K+\varepsilon-Hu}\sum_{m>2n}\frac{m^{K+1+\varepsilon-Hu-H/a}}{m-n}$$

$$\ll T^{1-1/a}\Big(\sum_{n=1}^{\infty} n^{2K+1+2\varepsilon-2Hu-H/a}\log(n+1)$$

$$+\sum_{n=1}^{\infty} n^{K+\varepsilon-Hu}\sum_{m>2n} m^{K+\varepsilon-Hu-H/a}\Big),$$

since $1/(m-n) \leqslant 2/m$ for $m > 2n$. By hypothesis $Hu > 1+K$, hence

$$2Hu+H/a-2K-1-2\varepsilon > 2+H/a-1-2\varepsilon > 1$$

for $0 < \varepsilon < H/(2a)$, implying

$$\sum_{n=1}^{\infty} n^{2K+1+2\varepsilon-2Hu-H/a}\log(n+1) \ll 1 \qquad (0 < \varepsilon < H/(2a)),$$

and similarly

$$\sum_{n=1}^{\infty} n^{K+\varepsilon-Hu}\sum_{m>2n} m^{K+\varepsilon-Hu-H/a} \ll \sum_{n=1}^{\infty} n^{K+\varepsilon-Hu}n^{K+\varepsilon-Hu-H/a+1} \ll 1.$$

Thus we have shown that the error term in (4.3) is $O(T^{1-1/a})$ independently of G, so that (4.2) follows with

$$C = \tfrac{1}{2}\sum_{n=1}^{\infty} g^2(n)\mu_n^{-2u} > 0,$$

since $g(n)$ is not identically equal to zero by hypothesis.

From Lemma 1 it follows that $|f(t)| > B$ for some $t\in[T, T+DT^{1-1/a}]$ and suitable $B, D > 0$, if we take $G = DT^{1-1/a}$, $T \geqslant T_0$. But we can take advantage of the fact that the series in (4.1) is absolutely convergent and deduce even more. Namely we have

LEMMA 2. *Under the hypotheses of Lemma 1 there exist two constants B, $D > 0$ such that for $T \geqslant T_0$ every interval $[T, T+DT^{1-1/a}]$ contains two points t_1, t_2 for which*

$$f(t_1) > B, \quad f(t_2) < -B. \tag{4.5}$$

Proof. Both inequalities in (4.5) are proved analogously, so the details will be given only for the second one. Suppose that $f(t) > -\varepsilon$ for any $\varepsilon > 0$, and $t\in[T, T+DT^{1-1/a}]$ for $T \geqslant T_0(\varepsilon)$ with arbitrary $D>0$. If $C_1, C_2, \ldots$ denote absolute, positive constants, then for D sufficiently large and $G = DT^{1-1/a}$ we have from (4.2)

$$C_1 G \leqslant \int_T^{T+G} f^2(t)\,dt = \sum_k \int_{I_k} f^2(t)\,dt + \sum_l \int_{J_l} f^2(t)\,dt,$$

where the I_k's denote subintervals of $[T, T+G]$ in which $f(t) > 0$, and the J_l's denote the subintervals where $f(t) < 0$. In each J_l we have $f^2(t) < \varepsilon^2$, and since

$$|f(t)| \leqslant \sum_{n=1}^{\infty} |g(n)|\,\mu_n^{-u} = C_2,$$

we have

$$\begin{aligned} C_1 G &\leqslant C_2 \sum_k \int_{I_k} f(t)\,dt + G\varepsilon^2 \\ &= C_2 \int_T^{T+G} f(t)\,dt + C_2 \sum_l \int_{J_l} (-f(t))\,dt + G\varepsilon^2 \\ &\leqslant C_2 \int_T^{T+G} f(t)\,dt + C_2 G\varepsilon + G\varepsilon^2. \end{aligned}$$

But using (4.4) it follows that

$$\begin{aligned} \int_T^{T+G} f(t)\,dt &= \sum_{n=1}^{\infty} g(n)\,\mu_n^{-u} \int_T^{T+G} \cos\big(A(t\mu_n)^{1/a} + B\big)\,dt \\ &\ll T^{1-1/a} \sum_{n=1}^{\infty} |g(n)|\,\mu_n^{-u-1/a} \leqslant C_3 T^{1-1/a}, \end{aligned}$$

hence

$$C_1 G \leqslant C_4 T^{1-1/a} + C_2 G\varepsilon + G\varepsilon^2. \tag{4.6}$$

If we take $G = DT^{1-1/a}$, $D > C_4/C_1$ and ε sufficiently small, then (4.6) yields a contradiction which proves the second inequality in (4.5), and the first one is proved analogously.

5. Proof of Theorem 1 and some applications. To prove Theorem 1 we start from (2.11) and use (2.7) with $m = 0$, supposing that ϱ is fixed and satisfies

$$\varrho > \max(2\alpha\sigma_a - \alpha r - 1/2,\ 0). \tag{5.1}$$

Then we have, for suitable constants A, B, E,

$$P_\varrho(x) = Ex^{\theta_\varrho} \sum_{n=1}^{\infty} g(n)\,\mu_n^{-r-\varrho+\theta_\varrho} \cos\big(A(x\mu_n)^{1/(2\alpha)} + B\big) + O(x^{\theta_\varrho - 1/(2\alpha)}). \tag{5.2}$$

By definition σ_a is the abscissa of absolute convergence of $\sum_{n=1}^{\infty} g(n)\mu_n^{-s}$, and moreover the condition on ϱ in (5.1) ensures that $\sigma_a - r - \varrho + \theta_\varrho < 0$. Hence the series in (5.2) is absolutely convergent, and for ϱ satisfying (5.1) the assertion of Theorem 1 follows immediately from Lemma 2 and (5.2). For other values of $\varrho \geqslant 0$ we proceed recursively, deducing the truth of the assertion for ϱ from the truth of the assertion for $\varrho+1$. If M is such an integer that (5.1) is true with ϱ replaced by $\bar\varrho = \varrho + M$, then it will follow that the assertion of Theorem 1 is true for $\bar\varrho - 1, \bar\varrho - 2, \ldots, \bar\varrho - M = \varrho$. Now to see that the truth of Theorem 1 for $\varrho+1$ implies the same for ϱ, note that for $1 \ll G \leqslant T$ (2.10) gives

$$P_{\varrho+1}(T+G) - P_{\varrho+1}(T) = \int_T^{T+G} P_\varrho(t)\,dt \qquad (\varrho \geqslant 0). \tag{5.3}$$

Since $P_{\varrho+1}(x)$ is continuous and Theorem 1 holds in this case, it follows that $P_{\varrho+1}(x)$ always has a zero in $[x, x+Dx^{1-1/(2\alpha)}]$ for $D>0$ sufficiently large. Suppose $P_\varrho(t) < \varepsilon t^{\theta_\varrho}$ for all $t\in[T, T+G]$, $G = DT^{1-1/(2\alpha)}$ and any given $\varepsilon > 0$. Then choose T ($\geqslant T_0(\varepsilon)$) and D such that $P_{\varrho+1}(T) = 0$ and $P_{\varrho+1}(T+G) > B(T+G)^{\theta_{\varrho+1}}$. By (5.3) we infer that

$$B(T+G)^{\theta_{\varrho+1}} \leqslant \varepsilon G (2T)^{\theta_\varrho}. \tag{5.4}$$

Since by definition

$$\theta_\varrho = \frac{r}{2} - \frac{1}{4\alpha} + \varrho\left(1 - \frac{1}{2\alpha}\right), \qquad G = DT^{1-1/(2\alpha)},$$

it follows that

$$\theta_{\varrho+1} = \theta_\varrho + 1 - 1/(2\alpha),$$

and consequently (5.4) is impossible if ε is sufficiently small and $T \geqslant T_0(\varepsilon)$. By the preceding discussion it follows that $P_\varrho(t_1) > Bt_1^{\theta_\varrho}$ for some suitable $B > 0$, $t_1 \in [T, T+CT^{1-1/(2\alpha)}]$, $C = C(\varrho) > 0$ and $\varrho \geqslant 0$. The inequality $P_\varrho(t_2) < -Bt_2^{\theta_\varrho}$ is proved analogously.

As an illustrative application of Theorem 1 consider

$$\Delta_k(x) = \sum_{n\leqslant x}{}' d_k(n) - \operatorname*{Res}_{s=1} \zeta^k(s) x^s s^{-1} - 2^{-k},$$

where $k \geqslant 2$ is a fixed integer and $d_k(n)$ is the number of ways n can be written as a product of k factors. Thus $d_2(n) = d(n)$ and $\Delta_2(x) = \Delta(x)$. The estimation of $\Delta_k(x)$ is known as the (general) Dirichlet divisor problem (see Ch. 13 of [22] and Ivić–Ouellet [23] for recent results). For $k \geqslant 3$ there is no explicit series representation for $\Delta_k(x)$ analogous to (1.1), but nevertheless (2.11) holds for $\varrho = \varrho(k)$ sufficiently large (for $k = 3$ one may take $\varrho > 0$). We have

THEOREM 2. *For $k \geqslant 2$ fixed there exist constants B, $C > 0$ such that for $T \geqslant T_0$ the interval $[T, T+CT^{(k-1)/k}]$ always contains two points t_1, t_2 which satisfy*

$$\Delta_k(t_1) > Bt_1^{(k-1)/(2k)}, \qquad \Delta_k(t_2) < -Bt_2^{(k-1)/(2k)}. \tag{5.5}$$

Proof. For $\sigma = \operatorname{Re} s > 1$

$$\sum_{n=1}^{\infty} d_k(n) n^{-s} = \zeta^k(s),$$

so that to $d_k(n)$ there corresponds the functional equation (2.1) with

$$\Delta(s) = \Gamma^k(s/2), \quad \alpha = \sum_{\nu=1}^{k} \alpha_\nu = k/2, \quad \beta_\nu = 0, \quad r = 1, \quad \varphi(s) = \psi(s) = \zeta^k(s)\pi^{-ks/2},$$

$$\sigma_a = \sigma_a^* = 1, \qquad K = 0, H = 1 \qquad \text{((i) and (ii) of § 3).}$$

In the notation of (2.9)

$$\Delta_k(x) = P_0(x) = F_0(x) - Q_0(x)$$

with

$$F_0(x) = \sum_{n \leqslant x}{}' d_k(n), \qquad Q_0(x) = \operatorname*{Res}_{s=1} \zeta^k(s) x^s s^{-1} - 2^{-k} = x P_{k-1}(\log x) - 2^{-k},$$

where $P_{k-1}(t)$ is a polynomial in t of degree $k-1$, whose coefficients may be explicitly evaluated. Moreover

$$\theta_0 = \frac{r}{2} - \frac{1}{4\alpha} = \frac{k-1}{2k}, \qquad 1 - \frac{1}{2\alpha} = \frac{k-1}{k},$$

so that (5.5) follows from Theorem 1. As mentioned in the discussion of the general case, (5.5) is slightly weaker than the best omega result of the type (3.6) (in this case $g(x)$ will be bounded by a log-power, as given explicitly by J. L. Hafner [13]), but the strength of (5.5) lies in producing large positive and large negative values of $\Delta_k(x)$ in short intervals.

A result analogous to (5.5) for $k = 2$ holds also for $P(x)$ (circle problem) and $A(x) = \sum_{n \leqslant x}' a(n)$ if the $a(n)$'s are real (in this case $K = (\varkappa - 1)/2$, $H = 1$ in (i) and (ii) of Section 3). Namely, the interval $[T, T + C\sqrt{T}]$ contains two points t_1, t_2 satisfying

$$P(t_1) > B t_1^{1/4}, \qquad P(t_2) < -B t_2^{1/4}$$

and two points t_3, t_4 satisfying

$$A(t_3) > B t_3^{\varkappa/2 - 1/4}, \qquad A(t_4) < -B t_4^{\varkappa/2 - 1/4}. \tag{5.6}$$

Several other applications of Theorem 1, such as the general divisor problem in algebraic number fields (functional equation for the Dedekind zeta-function), the number of lattice-points in many-dimensional ellipsoids (functional equation for the Epstein zeta-function) etc., can be obtained by following the detailed discussion of these problems given by e.g. J. L. Hafner [13] and D. Redmond [30].

6. The mean square of the Riemann zeta-function. In this section we shall discuss in detail the occurrence of large values of $E(T)$ in short intervals. For $T > 0$ the function $E(T)$ is continuous (it has in fact also derivatives of any order). Thus from Ω_+ and Ω_- analogues of (3.1) and (3.2) for $E(T)$ (proved by Hafner–Ivić [15], [16]), it follows that $E(T)$ has an infinity of zeros. From (2.12) and (2.14) it transpires that $E(T)$ has the mean value π, so that it seems more natural to investigate the zeros of $E(T) - \pi$ than directly the zeros of $E(T)$ itself. This viewpoint was adopted by Ivić–te Riele [24], where a detailed study (both theoretical and numerical) of t_n, the nth zero of $E(T) - \pi$, is made. Likewise, let u_n be the nth distinct zero of $G(T)$ (defined by (2.12)). The existence of an infinity of u_n's follows by continuity and the fact that $G(T) = \Omega_\pm(T^{3/4})$, proved by Hafner–Ivić [15]. We can rewrite (2.14) as

$$G(T) = \frac{1}{2}\left(\frac{2T}{\pi}\right)^{3/4} f(T) + O(T^{2/3}\log T), \tag{6.1}$$
$$f(T) = \sum_{n=1}^{\infty} (-1)^n d(n) n^{-5/4} \cos\left(\sqrt{8\pi n T} - (3\pi)/4\right).$$

Clearly $f(T)$ is a continuous function of the form considered in Lemmas 1 and 2 of Section 4 with $g(n) = (-1)^n d(n)$, $\mu_n = n$, $K = 0$, $H = 1$, $u = 5/4$, $a = 1/2$. Lemma 2 shows that every interval $[T, T + C\sqrt{T}]$ contains for $T \geqslant T_0$ points τ_1, τ_2 such that $f(\tau_1) > B$, $f(\tau_2) < -B$. From (6.1) it follows then that $G(\tau_1) > B\tau_1^{3/4}$ and $G(\tau_2) < -B\tau_2^{3/4}$ in every interval $[T, T + C\sqrt{T}]$ for some values τ_1, τ_2 from that interval. By continuity $G(t)$ must vanish at least once in the same interval, implying $u_{n+1} - u_n \ll u_n^{1/2}$. If we define

$$\beta = \limsup_{n \to \infty} \frac{\log(u_{n+1} - u_n)}{\log u_n}, \tag{6.2}$$

then we have shown that $\beta \leqslant 1/2$. If we further define

$$\varkappa = \limsup_{n \to \infty} \frac{\log(t_{n+1} - t_n)}{\log t_n}, \tag{6.3}$$

then analogously $\varkappa \leqslant 1/2$, or more precisely $t_{n+1} - t_n \ll t_n^{1/2}$ (this has been obtained in a different way by Ivić–te Riele [24]). Namely we have

$$G(T+H) - G(T) = \int_T^{T+H} (E(t) - \pi)\, dt, \tag{6.4}$$

and since we can choose T and H ($\leqslant C\sqrt{T}$) in such a way that $G(T+H) = G(T) = 0$, it follows that $E(t) - \pi$ must vanish at least once in $[T, T+H]$. On the basis of numerical evidence and some heuristic arguments we conjecture that $\varkappa = 1/4$. If true, this must be deep, since it implies the hitherto unproved bound $\zeta(1/2 + it) \ll t^{1/8 + \varepsilon}$. Now we prove $\beta = 1/2$ by showing that $\beta < 1/2$ cannot hold. We start from (6.4), choosing T and H ($\ll T^{\beta+\varepsilon}$) in such a way that $G(T) \gg T^{3/4}$, $G(T+H) = 0$. Then we have

$$T^{3/4} \ll \int_T^{T+H} |E(t)|\,dt \ll H^{1/2}\Big(\int_T^{T+H} E^2(t)\,dt\Big)^{1/2}. \tag{6.5}$$

By a recent result of T. Meurman [28] (proved also independently by Y. Motohashi)

$$\int_2^T E^2(t)\,dt = CT^{3/2} + F(T), \quad F(T) = O(T\log^5 T), \tag{6.6}$$

where $C = 2\zeta^4(3/2)/(3\sqrt{2\pi}\,\zeta(3))$. Inserting (6.6) in (6.5) we obtain

$$T^{3/2} \ll H(HT^{1/2} + F(T+H) - F(T)) \ll T^{2\beta+2\varepsilon+1/2} + T^{1+\beta+2\varepsilon}.$$

But if $\beta < 1/2$, then for $\varepsilon > 0$ sufficiently small the above estimate is impossible, so that $\beta = 1/2$ has to hold. In fact, the above proof shows that

$$u_{n+1} - u_n = \Omega(u_n^{1/2}(\log u_n)^{-5}),$$

and if the hypothetical estimate $F(T) = o(T)$ holds as $T \to \infty$, then even

$$u_{n+1} - u_n = \Omega(u_n^{1/2}).$$

In that case the maximal order of $u_{n+1} - u_n$ is precisely determined, up to the values of the constants involved. The preceding discussion may be summarized in

THEOREM 3. *There exist constants B, $C > 0$ such that every interval $[T, T+C\sqrt{T}]$ for $T \geq T_0$ contains numbers $\tau_1, \tau_2, \tau_3, \tau_4$ for which*

$$E(\tau_1) > B\tau_1^{1/4}, \quad E(\tau_2) < -B\tau_2^{1/4}, \quad G(\tau_3) > B\tau_3^{3/4}, \quad G(\tau_4) < -B\tau_4^{3/4},$$

so that every interval $[T, T+C\sqrt{T}]$ contains a zero of $E(T) - \pi$ and a zero of $G(T)$. Moreover, if u_n is the n-th zero of $G(T)$, then

$$\limsup_{n\to\infty} \frac{\log(u_{n+1} - u_n)}{\log u_n} = \frac{1}{2}.$$

7. Fourier coefficients of cusp forms. We recall that (5.6) provides the localization of large values of $A(x) = \sum_{n\leq x}' a(n)$, and that these results are only by a factor of $\log\log\log x$ weaker than the best omega result (3.5) for $A(x)$. Henceforth we suppose the $a(n)$'s to be real and we proceed to derive some further results on mean values of $A(x)$, which seem to be new.

From (1.4) and (1.7) with $\varrho = 0$ we have

$$A(x) = \frac{1}{\pi 2^{1/2}} x^{\varkappa/2-1/4} \sum_{n=1}^{\infty} a(n) n^{-\varkappa/2-1/4} \cos(4\pi\sqrt{xn} - \pi/4) + O(x^{\varkappa/2-3/4}). \tag{7.1}$$

This formula is analogous to the classical formula of Voronoi for $\Delta(x)$, namely

$$\Delta(x) = \frac{1}{\pi 2^{1/2}} x^{1/4} \sum_{n=1}^{\infty} d(n) n^{-3/4} \cos(4\pi\sqrt{xn} - \pi/4) + O(x^{-1/4}), \tag{7.2}$$

which follows from (1.1) and the asymptotic formulas for Y_1 and K_1. Also by a deep result of P. Deligne [10]

$$a(n) \ll n^{(\varkappa-1)/2} d(n). \tag{7.3}$$

For practical purposes one often uses a truncated form of (7.2) (see Ch. 3 of [22]), namely

$$\Delta(x) = \frac{1}{\pi 2^{1/2}} x^{1/4} \sum_{n\leq N} d(n) n^{-3/4} \cos(4\pi\sqrt{xn} - \pi/4) + O(x^{1/2+\varepsilon} N^{-1/2}) \quad (1 \ll N \ll x), \tag{7.4}$$

where the error term is uniform in N. An analogue of (7.4) for $A(x)$ also exists, and a proof is sketched by M. Jutila [27]. This is

$$A(x) = \frac{1}{\pi 2^{1/2}} x^{\varkappa/2-1/4} \sum_{n\leq N} a(n) n^{-\varkappa/2-1/4} \cos(4\pi\sqrt{xn} - \pi/4) + O(x^{\varkappa/2+\varepsilon} N^{-1/2}) \quad (1 \ll N \ll x). \tag{7.5}$$

The results on mean values of $A(x)$ are contained in

THEOREM 4. *We have*

$$\int_1^X A^2(x)\,dx = CX^{\varkappa+1/2} + B(X) \quad \Big(C = \frac{1}{(4\varkappa+2)\pi^2} \sum_{n=1}^{\infty} a^2(n) n^{-\varkappa-1/2}\Big)$$

with

$$B(X) = O(X^{\varkappa}\log^5 X), \quad B(X) = \Omega\Big(X^{\varkappa-1/4} \frac{(\log\log\log X)^3}{\log X}\Big), \tag{7.6}$$

and

$$\int_1^X A^8(x)\,dx \ll T^{4\varkappa-1+\varepsilon}. \tag{7.7}$$

Proof. (7.6) is the analogue of

$$\int_1^X \Delta^2(x)\,dx = CX^{3/2} + R(X) \quad \Big(C = \frac{1}{6\pi^2} \sum_{n=1}^{\infty} d^2(n) n^{-3/2}\Big),$$

with

$$R(X) = O(X\log^5 X), \quad R(X) = \Omega\{X^{3/4}(\log X)^{-1/4}(\log\log X)^{(9+3\log 4)/4}\exp(-D\sqrt{\log\log\log X})\}. \tag{7.8}$$

The O-result in (7.8) is due to K.-C. Tong [33], and the Ω-result to Ivić–Ouellet [23] (a somewhat weaker Ω-result for $R(x)$ is proved by Ivić in Ch. 13 of [22]). Recently T. Meurman [28] obtained a simple new proof of the O-result in (7.8) by working with (7.2). Applying Meurman's method to (7.1) one will eventually obtain the O-result in (7.6), as was pointed out by M. Jutila [27]. To prove the Ω-result in (7.6) note first that, uniformly for $x^\varepsilon \ll H \leqslant x$,

$$A(x) = H^{-1}\int_x^{x+H} A(t)\,dt + O(x^{(\varkappa-1)/2} H\log x). \tag{7.9}$$

Namely by (7.3)

$$A(x)-H^{-1}\int_x^{x+H} A(t)\,dt = H^{-1}\int_x^{x+H} (A(x)-A(t))\,dt$$
$$\ll H^{-1}x^{(\varkappa-1)/2}\int_x^{x+H}\sum_{x<n\leqslant t} d(n)\,dt$$
$$\leqslant H^{-1}x^{(\varkappa-1)/2}\int_x^{x+H}\sum_{x<n\leqslant x+H} d(n)\,dt \ll x^{(\varkappa-1)/2} H\log x.$$

Here we used the fact that, uniformly for $x^\varepsilon \ll y \leqslant x$,

$$\sum_{x<n\leqslant x+y} d(n) \ll y\log x,$$

which follows from a general result of P. Shiu [31] on multiplicative functions in short intervals.

Suppose now that the Ω-estimate in (7.6) does not hold, that is,

$$B(X) = o\left(X^{\varkappa-1/4}\frac{(\log\log\log X)^3}{\log X}\right) \quad (X\to\infty). \tag{7.10}$$

Choose for X the sequence of points where the Ω_+-result in (3.5) is attained. Then from (7.9) and the Cauchy–Schwarz inequality we have, for any $\varepsilon > 0$ and $X \geqslant X_0(\varepsilon)$,

$$C_1 X^{\varkappa/2-1/4}\log\log\log X \leqslant H^{-1/2}\Big(\int_X^{X+H} A^2(t)\,dt\Big)^{1/2} + C_2 HX^{\varkappa/2-1/2}\log X$$
$$\leqslant C_3 X^{\varkappa/2-1/4} + \varepsilon H^{-1/2} X^{(\varkappa-1/4)/2}\frac{(\log\log\log X)^{3/2}}{(\log X)^{1/2}} + C_2\delta HX^{\varkappa/2-1/2}\log X.$$

Take now

$$H = \delta X^{1/4}\frac{\log\log\log X}{\log X} \quad (\delta > 0).$$

Then we obtain, for $X \geqslant X_0$,

$$\tfrac{1}{2}C_1 X^{\varkappa/2-1/4}\log\log\log X \leqslant \varepsilon\delta^{-1/2}X^{\varkappa/2-1/4}\log\log\log X + C_2 X^{\varkappa/2-1/4}\log\log\log X,$$

or

$$C_1/2 \leqslant \varepsilon\delta^{-1/2} + C_2\delta$$

for some absolute constants C_1, $C_2 > 0$, which is impossible e.g. for $\delta = C_1/(3C_2)$, $\varepsilon = \frac{1}{8}\delta^{1/2}C_1$. This proves that (7.10) cannot be true, hence the Ω-estimate in (7.6) must hold.

As a consequence of the O-estimate in (7.6) and (7.9) one easily obtains

$$A(x) = H^{-1}\int_x^{x+H} A(t)\,dt + O(Hx^{(\varkappa-1)/2}\log x)$$
$$\ll H^{-1/2}\Big(\int_x^{x+H} A^2(t)\,dt\Big)^{1/2} + Hx^{(\varkappa-1)/2}\log x$$
$$\ll H^{-1/2}(Hx^{\varkappa-1/2} + x^\varkappa\log^5 x)^{1/2} + Hx^{(\varkappa-1)/2}\log x$$
$$\ll x^{\varkappa/2-1/4} + H^{-1/2}x^{\varkappa/2}\log^{5/2} x + Hx^{(\varkappa-1)/2}\log x$$
$$\ll x^{(\varkappa-1)/2+1/3}\log^2 x$$

for $H = x^{1/3}\log x$. Thus the order estimate (analogous to the classical bound $\Delta(x) \ll x^{1/3}\log^2 x$)

$$A(x) \ll x^{(\varkappa-1)/2+1/3}\log^2 x \tag{7.11}$$

follows easily from the mean-square formula for $A(x)$. In analogy with the conjecture $\Delta(x) \ll x^{1/4+\varepsilon}$ one expects

$$A(x) \ll x^{(\varkappa-1)/2+1/4+\varepsilon} \tag{7.12}$$

to hold. Note that (7.12) is true in a mean power sense by (7.7), and from (7.7) and Hölder's inequality it follows that

$$\int_1^X |A(x)|^B\,dx \ll X^{1+\varepsilon+B((\varkappa-1)/2+1/4)} \quad (0 \leqslant B \leqslant 8). \tag{7.13}$$

This means that the crucial estimate is (7.13) with $B = 8$, which we proceed to prove now. Write the truncated formula (7.5) as

$$A(x) \ll X^{(\varkappa-1)/2}\Big(x^{1/4}\Big|\sum_{n\leqslant X^{1+\varepsilon}H^{-2}} a(n)n^{(1-\varkappa)/2}n^{-3/4}\cos(4\pi\sqrt{xn}-\pi/4)\Big| + H\Big) \tag{7.14}$$

for $X^\varepsilon \ll H \leqslant X^{1/2}$, $X \leqslant x \leqslant 2X$. Now we are going to employ the large values technique, similarly to what was done by the author in [20], [21], [22] for the estimation of large values of $\Delta(x)$. Namely, therein it is shown that if $t_1, \ldots, t_R$ are points from $[T, 2T]$ for which $|\Delta(t_i)| \geqslant V > 0$ and $|t_i - t_j| \geqslant V$ $(i \neq j)$, then

$$R \ll T^{\varepsilon}(TV^{-3} + T^{15/4}V^{-12}), \tag{7.15}$$

which easily leads to power moment estimates for $\Delta(x)$. But note that (7.14) is analogous to (7.4) with $N = X^{1+\varepsilon}H^{-2}$ since (7.3) holds, and therefore the application of the Halász–Montgomery inequality used in the proof of (7.15) will remove $a(n)n^{(1-\varkappa)/2}$ in the same way that it removed $d(n)$ in the case of $\Delta(x)$. Thus eventually the same exponential sums as in the case of $\Delta(x)$ will remain, and these will be estimated by the technique of exponent pairs. The only substantial difference between (7.14) and (7.4) is that (7.14) contains $X^{(\varkappa-1)/2}$ on the right-hand side, so that $A(x)$ is in a certain sense by a factor of $X^{(\varkappa-1)/2}$ "larger" than $\Delta(x)$. If we impose the spacing condition $|t_i - t_j| \geqslant 1$ (instead of $|t_i - t_j| \geqslant V$ $(i \neq j)$), then (7.15) becomes

$$R \ll T^{\varepsilon}(TV^{-2} + T^{15/4}V^{-11}). \tag{7.16}$$

In fact, (7.15) and (7.16) with their respective spacing conditions imply one another in view of the relation

$$\Delta(x+H) - \Delta(x) = O(|H| \log x) \quad (x^{\varepsilon} \leqslant |H| \leqslant x),$$

which is proved analogously to (7.9). Hence if we suppose that $x_1, x_2, \ldots, x_R$ are points from $[X, 2X]$ such that $|A(x_i)| \geqslant V$ and $|x_i - x_j| \geqslant 1$ for $i \neq j$, V satisfies

$$X^{(\varkappa-1)/2+1/4+\varepsilon} \ll V \ll X^{(\varkappa-1)/2+1/3+\varepsilon}, \tag{7.17}$$

then by the above mentioned analogy between $\Delta(x)$ and $A(x)$ we shall obtain (7.16) with V replaced by $VX^{(1-\varkappa)/2}$ and T replaced by X. Thus

$$R \ll X^{\varepsilon}(X^{\varkappa}V^{-2} + X^{15/4+11(\varkappa-1)/2}V^{-11}), \tag{7.18}$$

with the remark that the upper bound for V in (7.17) must hold in view of (7.11). Suppose now $2 \leqslant B \leqslant 11$, and write

$$\int_X^{2X} |A(t)|^B \, dt = I_1 + I_2,$$

where in I_1 we have $|A(t)| \leqslant X^{(\varkappa-1)/2+1/4+\varepsilon}$, and to prove (7.13) it will be sufficient to prove the corresponding estimate for I_1 and I_2. Trivially

$$I_1 \leqslant X^{1+\varepsilon+B((\varkappa-1)/2+1/4)}, \tag{7.19}$$

which is the desired upper bound. It remains to estimate I_2, where we have $|A(t)| > X^{(\varkappa-1)/2+1/4+\varepsilon}$. Divide I_2 into $O(\log X)$ subintegrals $I_2(V)$, where in each $I_2(V)$

$$V \leqslant |A(t)| \leqslant 2V,$$

and V satisfies (7.17). By (7.18) each $I_2(V)$ is estimated as

$$\begin{aligned} I_2(V) &\ll RV^B \ll X^{\varepsilon}(X^{\varkappa}V^{B-2} + X^{15/4+11(\varkappa-1)/2}V^{B-11}) \\ &\ll X^{\varepsilon}(X^{\varkappa+(B-2)((\varkappa-1)/2+1/3)} + X^{15/4+11(\varkappa-1)/2+(B-11)((\varkappa-1)/2+1/4)}) \\ &= X^{\varepsilon}(X^{1+(B-2)/3+B(\varkappa-1)/2} + X^{1+B/4+B(\varkappa-1)/2}). \end{aligned}$$

For $B \leqslant 8$ we have $(B-2)/3 \leqslant B/4$, hence

$$I_2(V) \ll X^{1+B/4+B(\varkappa-1)/2+\varepsilon} \quad (2 \leqslant B \leqslant 8). \tag{7.20}$$

Consequently (7.19) and (7.20) imply (7.13) for $2 \leqslant B \leqslant 8$, while for $0 \leqslant B < 2$ this follows easily from Hölder's inequality and the mean-square formula for $A(x)$.

Although (7.3) holds for all n, on the average $|a(n)|n^{(1-\varkappa)/2}$ is smaller than $d(n)$, whose average order is $\log n$. Namely, if one sets

$$\sum_{n \leqslant x} a^2(n) = Ax^{\varkappa} + D(x) \quad (A > 0), \tag{7.21}$$

then by a classical result of R. A. Rankin [29]

$$D(x) = O(x^{\varkappa-2/5}), \tag{7.22}$$

and the value of A may be written explicitly. I conjecture that

$$D(x) = O(x^{\varkappa-5/8+\varepsilon}), \quad D(x) = \Omega(x^{\varkappa-5/8}), \tag{7.23}$$

although proving even

$$D(x) = O(x^{\varkappa-1/2+\varepsilon}) \tag{7.24}$$

would be very interesting. Rankin's proof of (7.22) is based on a convolution argument due to E. Landau and the functional equation (see pp. 180–182 of G. H. Hardy [19] for the special case of the function $\tau(n)$, corresponding to $\varkappa = 12$)

$$Z(s) = \Delta(s)Z(1-s), \quad \Delta(s) = (2\pi)^{4s-2}\frac{\Gamma(1-s)\Gamma(\varkappa-s)}{\Gamma(s)\Gamma(s+\varkappa-1)}. \tag{7.25}$$

Here for $\mathrm{Re}\, s > 1$

$$Z(s) = \zeta(2s)f(s+\varkappa-1) = \sum_{n=1}^{\infty} c_n n^{-s}, \tag{7.26}$$

say, where

$$f(s) = \sum_{n=1}^{\infty} a^2(n) n^{-s} \quad (\mathrm{Re}\, s > \varkappa).$$

Note that (7.25) is a functional equation of the type (2.1) with $r = 1$, $\alpha = 2$, and moreover by Stirling's formula for the gamma-function

$$|\Delta(s)| = \left(\frac{t}{2\pi}\right)^{2-4\sigma}\left(1 + O\left(\frac{1}{t}\right)\right) \quad (s = \sigma + it,\ t \geqslant t_0).$$

If we set, with a suitable $\beta > 0$,

$$\sum_{n \leqslant x} c_n = \beta x + C(x),$$

then by applying a general theorem of E. Landau Rankin deduced that

$$C(x) = O(x^{3/5}), \tag{7.27}$$

which easily leads to (7.22). The estimation of $C(x)$ bears resemblance to the estimation of $\Delta_4(x)$, since $d_4(n)$ is generated by $\zeta^4(s)$, and one has the functional equation

$$\zeta^4(s) = \chi^4(s)\zeta^4(1-s), \quad |\chi^4(s)| = \left(\frac{t}{2\pi}\right)^{2-4\sigma}\left(1+O\left(\frac{1}{t}\right)\right).$$

Thus $|\Delta(s)|$ and $|\chi^4(s)|$ are asymptotically equal, and (7.27) essentially corresponds to $\Delta_4(x) \ll x^{3/5+\varepsilon}$. The last bound may be considered as the trivial estimate for $\Delta_4(x)$, since it may be obtained by trivial estimation of the truncated formula for $\Delta_4(x)$, analogous to the truncated Voronoi formula for $\Delta(x)$ (see (3.23) of [22]). But using

$$\int_0^T |\zeta(1/2+it)|^4\, dt \ll T^{1+\varepsilon}$$

one obtains easily (Ch. 13 of [22]) $\Delta_4(x) \ll x^{1/2+\varepsilon}$, which is (still!) the sharpest known estimate for $\Delta_4(x)$, and which corresponds to (7.24). The reason for which one cannot at present obtain (7.24) lies crudely in the fact that $Z(s)$ is not the square of a "nice" Dirichlet series, while $\zeta^4(s) = (\zeta^2(s))^2$ is. Hence at present there is no analogy between the above fourth power moment for the zeta-function and

$$\int_0^T |Z(1/2+it)|^2\, dt \ll T^{1+\varepsilon},$$

since the latter is not known to hold yet. However from (7.25) and Theorem 1 we can obtain (in this case $\alpha = 2$, $r = 1$) by specialization some results on $C(x)$ and thus indirectly on $D(x)$. This is

THEOREM 5. *There exist two constants A, $B > 0$ such that for $T \geqslant T_0$ every interval $[T, T+AT^{3/4}]$ contains two points t_1, t_2 such that*

$$C(t_1) > Bt_1^{3/8}, \quad C(t_2) < -Bt_2^{3/8}. \tag{7.28}$$

Moreover,

$$\int_1^X C^2(x)\, dx \ll X^{2+\varepsilon}, \tag{7.29}$$

and for $D(x)$ defined by (7.21) we have

$$\int_1^X D^2(x)\, dx \ll X^{2\varkappa+\varepsilon}. \tag{7.30}$$

Proof. The estimates in (7.28) are a direct corollary of Theorem 1 with $\alpha = 2$, $r = 1$, $\varrho = 0$. The mean-square estimate (7.29) follows by using the metod of Lemma 13.1 of [22] and the fact that

$$\int_T^{2T} |Z(\sigma+it)|^2\, dt \ll T^{2+\varepsilon} \quad (1/2 \leqslant \sigma \leqslant 1). \tag{7.31}$$

One obtains (7.31) by using the mean value theorem for Dirichlet polynomials (Theorem 5.2 of [22]) and the fact that by the functional equation (7.25) one can write $Z(\sigma+it)$ for $\sigma \geqslant 1/2$ as a sum of Dirichlet polynomials of length $\ll T^2$ (the technique of the "reflection principle", expounded in Ch. 4 of [22]). One expects (7.28) to be fairly sharp, so that we may conjecture $C(x) = \Omega(x^{3/8})$ and $C(x) = O(x^{3/8+\varepsilon})$ to be true. This conjecture heuristically explains the reasons for believing that (7.23) might also be true. Moreover it shows that (7.29) can be replaced by

$$\int_1^X C^2(x)\, dx \ll X^{7/4+\varepsilon},$$

and probably an asymptotic formula for this integral should also exist.

It remains to prove (7.30). From $Z(s) = \zeta(2s) f(s+\varkappa-1)$ one obtains that (7.29) holds if $C(x)$ is replaced by $C_1(x)$, which we define as the error term in the asymptotic formula

$$U(x) := \sum_{n \leqslant x} a^2(n) n^{1-\varkappa} = \alpha x + C_1(x),$$

where $\alpha > 0$ is a suitable constant. This assertion follows similarly to the proof of (7.29) by the technique of Lemma 13.1 of [22], since for $1/2 < \sigma \leqslant 1$ we have, by (7.31),

$$\int_T^{2T} |f(\sigma+\varkappa-1+it)|^2\, dt = \int_T^{2T} |\zeta(2\sigma+2it)|^{-2} |Z(\sigma+it)|^2\, dt$$
$$\ll (2\sigma-1)^{-2} \int_T^{2T} |Z(\sigma+it)|^2\, dt \ll_{\varepsilon,\sigma} T^{2+\varepsilon}.$$

Partial summation gives

$$\sum_{n \leqslant x} a^2(n) = U(x)x^{\varkappa-1} - (\varkappa-1)\int_1^x U(t)t^{\varkappa-2}\, dt$$
$$= (\alpha/\varkappa)x^{\varkappa} + C_0 + x^{\varkappa-1} C_1(x) - (\varkappa-1)\int_1^x C_1(t)t^{\varkappa-2}\, dt$$

for some constant C_0. Comparing with (7.21) ($A = \alpha/\varkappa$) we obtain

$$D(x) = C_0 + x^{\varkappa-1} C_1(x) - (\varkappa-1)\int_1^x C_1(t)t^{\varkappa-2}\, dt.$$

This gives

$$\int_T^{2T} D^2(t)\,dt \ll T + \int_T^{2T} T^{2\varkappa-2} C_1^2(t)\,dt + \int_T^{2T} \Big(\int_1^t u^{\varkappa-2} C_1(u)\,du\Big)^2 dt$$

$$\ll T^{2\varkappa+\varepsilon} + \int_T^{2T} t^{2\varkappa-3} \int_1^t C_1^2(u)\,du\,dt \ll T^{2\varkappa+\varepsilon}.$$

Replacing T by $X2^{-j}$ and summing over $j = 1, 2, \ldots$ we obtain (7.30).

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