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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ISTANBUL UNIVERSITY
34459 Vezneciler, Istanbul
Turkey

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Lüroth-type alternating series representations for real numbers

by

SOFIA KALPAZIDOU (Thessaloniki),
ARNOLD KNOPFMACHER (Johannesburg)
and JOHN KNOPFMACHER (Johannesburg)

Introduction. In this paper, we introduce an algorithm that leads to a general alternating series expansion for real numbers in terms of rationals. In particular, this algorithm is used to show the existence and uniqueness of two alternating series expansions which are analogous to the positive series of *Lüroth*, and to a modified *Engel* expansion, respectively. In addition, the representation of rational numbers by means of these algorithms is investigated. Thereafter, stochastic properties of the sequence of digits in the Lüroth-type alternating representation are studied. In particular, we solve the *Gauss-type measure problem* for this expansion.

1. A general alternating series algorithm. We first define a general alternating series algorithm, analogous to a positive one of Oppenheim [7], as follows:

Given any real number A, let $a_0 = [A]$, $A_1 = A - a_0$. Then recursively define

$$a_n \doteq [1/A_n] \geqslant 1 \quad \text{for } n \geqslant 1,\ A_n > 0,$$

where $A_{n+1} = (1/a_n - A_n)(c_n/b_n)$ for $a_n > 0$. Here

$$b_i = b_i(a_1, \ldots, a_i), \quad c_i = c_i(a_1, \ldots, a_i)$$

are positive numbers (usually integers), chosen so that $A_n \leqslant 1$ for $n \geqslant 1$. Note that $A_{n+1} \geqslant 0$, since $a_n \leqslant 1/A_n$ for $A_n > 0$.

Using this algorithm we now prove:

THEOREM 1. *Every real number has unique representations in the forms*

$$\text{(i)}\quad A = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)a_1}\cdot\frac{1}{a_2} + \frac{1}{(a_1+1)a_1(a_2+1)a_2}\cdot\frac{1}{a_3} - \ldots$$

$$= ((a_0, a_1, \ldots, a_n, \ldots)), \quad \textit{say, where } a_n \geqslant 1\ (n \geqslant 1), \textit{ and}$$

$$\text{(ii)}\ A = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)}\cdot\frac{1}{a_2} + \frac{1}{(a_1+1)(a_2+1)}\cdot\frac{1}{a_3} - \dots$$
$$= (a_0, a_1, \dots, a_n, \dots),\ \text{say, where } a_{n+1} \geqslant a_n,\ a_1 \geqslant 1.$$

Proof. Repeated application of our alternating series algorithm yields

$$A = a_0 + A_1 = a_0 + \frac{1}{a_1} - \frac{b_1}{c_1} A_2 = \dots$$
$$= a_0 + \frac{1}{a_1} - \frac{b_1}{c_1}\cdot\frac{1}{a_2} + \frac{b_1 b_2}{c_1 c_2}\cdot\frac{1}{a_3} - \dots + (-1)^{k-1}\frac{b_1 b_2 \dots b_{k-1}}{c_1 c_2 \dots c_{k-1}} A_k.$$

Now $a_n = [1/A_n]$ implies $1/(a_n+1) < A_n \leqslant 1/a_n$ for $0 < A_n \leqslant 1$. Thus

$$A_{n+1} = \left(\frac{1}{a_n} - A_n\right)(c_n/b_n) < \left(\frac{1}{a_n} - \frac{1}{a_n+1}\right)(c_n/b_n) = \frac{1}{a_n(a_n+1)}(c_n/b_n), \quad \text{if } 0 < A_n \leqslant 1.$$

In particular, by setting $b_n = 1$, $c_n = (a_n+1)a_n$ for all n we obtain $a_{n+1} = [1/A_{n+1}] \geqslant 1$, provided $A_i > 0$ for $i \leqslant n$. Furthermore,

$$\frac{A_{n+1}}{a_1(a_1+1)\dots a_n(a_n+1)} \leqslant \frac{1}{2^n} \to 0 \quad \text{as } n \to \infty,$$

since $a_n \geqslant 1$ for all $n \geqslant 1$. It follows that A has a Lüroth-type alternating expansion

$$(1)\qquad A = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)a_1}\cdot\frac{1}{a_2} + \frac{1}{(a_1+1)a_1(a_2+1)a_2}\cdot\frac{1}{a_3} - \dots,$$

$a_n \geqslant 1$ $(n \geqslant 1)$, which perhaps may terminate.

Secondly, by setting $c_n = a_n + 1$, $b_n = 1$ for all n, we obtain $a_{n+1} = [1/A_{n+1}] \geqslant a_n$, provided $A_i > 0$ for $i \leqslant n$, and so

$$\frac{A_n}{(a_1+1)(a_2+1)\dots(a_n+1)} < \frac{1/a_n}{2^n} \to 0 \quad \text{as } n \to \infty,$$

since $a_1 \geqslant 1$. Thus A has the "modified Engel-type" alternating expansion

$$A = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)}\cdot\frac{1}{a_2} + \frac{1}{(a_1+1)(a_2+1)}\cdot\frac{1}{a_3} - \dots,$$

$a_{i+1} \geqslant a_i \geqslant 1$ $(i \geqslant 1)$, which also may terminate. ∎

In the same manner as above, by setting $b_n = c_n = 1$ for all n in the alternating-Oppenheim algorithm, we obtain the known "alternating-Sylvester" expansion

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3} - \dots,$$

where $a_{n+1} \geqslant a_n(a_n+1)$; cf. [5], [9], [11].

Similarly, by setting $c_n = a_n$, $b_n = 1$ for all n we obtain the "alternating-Engel" (or "Pierce") expansion

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \dots,$$

where $a_{n+1} \geqslant a_n + 1$ $(n \geqslant 1)$; cf. [5], [9], [10], [11].

We remark here also that another suitable choice of the numbers b_n, c_n above leads to an interesting alternating *product* expansion for real numbers, which will be treated in detail elsewhere.

2. Expansions for rational numbers. As in the case of the positive Lüroth series, rational numbers have special types of alternating-Lüroth expansions.

THEOREM 2. *The alternating-Lüroth expansion* $((a_0, a_1, a_2, \dots))$ *is periodic or terminates if and only if* A *is rational.*

Proof. Let $A = p/q$ be rational (with $p, q \in N$). Then each A_n is also rational, with

$$A_n = a_{n-1} + 1 - a_{n-1}(a_{n-1}+1)A_{n-1}$$
$$= a_{n-1} + 1 - a_{n-1}(a_{n-1}+1)\{a_{n-2} + 1 - a_{n-2}(a_{n-2}+1)A_{n-2}\}$$
$$= \dots = aA_1 + b = p_n/q,$$

where $a, b \in Z$. Now since $0 \leqslant A_n < 1$ for all n, either $A_n = 0$ for some n, in which case the expansion terminates, or else for every n,

$$A_n \in \left\{\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}\right\}$$

and so $\exists k, n \in N$ such that $A_n = A_{n+k}$. Then the algorithm applied to A_{n+k} gives the same successive digits as when applied to A_n, i.e. the digits become periodic.

Conversely, suppose eventually $a_n = a_{n+k}$ for some $k \in N$. Using the notation

$$X_n = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)a_1}\cdot\frac{1}{a_2} + \dots$$
$$+ (-1)^{n-2}\frac{1}{(a_1+1)a_1\dots(a_{n-2}+1)a_{n-2}}\cdot\frac{1}{a_{n-1}},$$

and letting $\alpha_n = (a_1+1)a_1\dots(a_n+1)a_n$ and $\alpha_* = \alpha_{n+k-1}/\alpha_{n-1}$, we have

$$A = X_n + \frac{(-1)^{n-1}}{\alpha_{n-1}}\left\{\left(\frac{1}{a_n} - \frac{1}{(a_n+1)a_n}\cdot\frac{1}{a_{n+1}} + \dots\right.\right.$$
$$\left. + \frac{(-1)^{k-1}}{(a_n+1)a_n\dots(a_{n+k-2}+1)a_{n+k-2}}\cdot\frac{1}{a_{n+k-1}}\right)$$

$$+(-1)^k\left(\frac{1}{\alpha_* a_n}-\frac{1}{\alpha_*(a_n+1)a_n}\cdot\frac{1}{a_{n+1}}+\dots+\frac{(-1)^{k-1}}{\alpha_*(a_n+1)a_n\dots(a_{n+k-2}+1)a_{n+k-2}}\cdot\frac{1}{a_{n+k-1}}\right)+(-1)^{2k}\left(\frac{1}{\alpha_*^2 a_n}-\dots\right)+\dots\Big\}.$$

If k is even,

$$A = X_n+\frac{(-1)^{n-1}}{\alpha_{n-1}}\left\{\frac{1}{a_n}-\frac{1}{(a_n+1)a_n}\cdot\frac{1}{a_{n+1}}+\dots+\frac{(-1)^{k-1}}{(a_n+1)a_n\dots(a_{n+k-2}+1)a_{n+k-2}}\cdot\frac{1}{a_{n+k-1}}\right\}\left(1+\frac{1}{\alpha_*}+\frac{1}{\alpha_*^2}+\dots\right)$$

$$= \text{a rational number.}$$

If k is odd, the expression is the same except for the last factor in round brackets which is to be replaced by $1-1/\alpha_*+1/\alpha_*^2-\dots$

Obviously if the expansion terminates then A is rational. ∎

We note that for rational numbers with a finite expansion there is a possible ambiguity in the final term, analogous to that for continued fractions. We eliminate this as follows:

CONVENTION 3. We replace the finite sequence $((a_0, a_1, \dots, a_{n-1}, 1))$ by the sequence $((a_0, a_1, \dots, a_{n-2}, a_{n-1}+1))$ in the case $a_n = 1$.

For the above "modified Engel-type" expansion $A = (a_0, a_1, a_2, \dots)$, the question of whether or not all rationals have a finite or recurring expansion has not been settled. However, we note:

THEOREM 4. *If the expansion $A = (a_0, a_1, a_2, \dots)$ terminates or if the digits a_i satisfy $a_i = a \geqslant 1$ for $i \geqslant n+1$, then A is rational.*

Proof. The terminating case is obvious. Suppose now $a_i = a \geqslant 1$ for $i \geqslant n+1$. Then, with the notation

$$X_n = a_0+\frac{1}{a_1}-\frac{1}{(a_1+1)}\cdot\frac{1}{a_2}+\dots+(-1)^{n-1}\frac{1}{(a_1+1)\dots(a_{n-1}+1)}\cdot\frac{1}{a_n},$$

we have

$$A = X_n+\frac{(-1)^n}{(a_1+1)\dots(a_n+1)}\left\{\frac{1}{a}-\frac{1}{(a+1)a}+\frac{1}{(a+1)^2 a}-\dots\right\}$$

$$= X_n+\frac{(-1)^n}{(a_1+1)\dots(a_n+1)a}\cdot\left\{\frac{a+1}{a+2}\right\} = \text{a rational.}$$

Note that in the special case $A = (a+1)/(a(a+2))$ for some $a \in N$, then A has expansion $(0, a, a, a, \dots)$. ∎

Once again, to eliminate the possibility of ambiguity in a finite representation we have

CONVENTION 5. We replace the finite sequence $(a_0, a_1, \dots, a_n)$ by the sequence $(a_0, a_1, \dots, a_{n-2}, a_{n-1}+1)$ in the case $a_n = a_{n-1}$.

3. Uniqueness and order properties. In order to be able to compare finite expansions of different lengths in size we introduce the symbol ω with the property $n < \omega$, for any $n \in N$. We can now represent finite sequences by infinite sequences as follows: For every $A = ((a_0, a_1, \dots, a_n))$ let $a_j = \omega$ for $j > n$ and hence $A = ((a_0, a_1, \dots, a_n, \omega, \omega, \dots))$. We do the same in the case $A = (a_0, a_1, \dots, a_n)$.

THEOREM 6 (Uniqueness and order). *Let $A = ((a_0, a_1, \dots)) \neq B = ((b_0, b_1, \dots))$, or $A = (a_0, a_1, \dots) \neq B = (b_0, b_1, \dots)$. In both these cases, the condition $A < B$ is equivalent to:*

(i) $a_{2n} < b_{2n}$ *or* (ii) $a_{2n+1} > b_{2n+1}$,

where $i = 2n$ or $i = 2n+1$ is the first index $i \geqslant 0$ such that $a_i \neq b_i$.

Proof. We shall use the notation

$$A'_n = \frac{1}{a_n}-\frac{1}{(a_n+1)}\cdot\frac{1}{a_{n+1}}+\frac{1}{(a_n+1)(a_{n+1}+1)}\cdot\frac{1}{a_{n+2}}-\dots$$

for $A = ((a_0, a_1, a_2, \dots))$, and

$$A'_n = \frac{1}{a_n}-\frac{1}{(a_n+1)a_n}\cdot\frac{1}{a_{n+1}}+\frac{1}{(a_n+1)a_n(a_{n+1}+1)a_{n+1}}\cdot\frac{1}{a_{n+2}}-\dots$$

for $A = (a_0, a_1, a_2, \dots)$. (Note we do *not* assume at this stage that $A'_n = A_n$ as defined by the algorithm.)

Now suppose (i) holds. If firstly $a_0 < b_0$ then

$$A = a_0+A_1 < a_0+1 \leqslant b_0 \leqslant b_0+B_1 = B,$$

in either case.

Next suppose $a_{2n} < b_{2n}$, $n > 0$, in the Lüroth-type case. Since $a_n \geqslant 1$, $n \geqslant 1$, we have

$$A'_{2n} = \frac{1}{a_{2n}}-\frac{1}{(a_{2n}+1)a_{2n}}\cdot\frac{1}{a_{2n+1}}+\frac{1}{(a_{2n}+1)a_{2n}(a_{2n+1}+1)a_{2n+1}}\cdot\frac{1}{a_{2n+2}}-\dots$$

$$\geqslant \frac{1}{a_{2n}}\left(1-\frac{1}{a_{2n}+1}\right)+\frac{1}{(a_{2n}+1)a_{2n}(a_{2n+1}+1)a_{2n+1}a_{2n+2}}\left(1-\frac{1}{a_{2n+2}+1}\right)+\dots$$

$$> \frac{1}{a_{2n}}\left(1-\frac{1}{a_{2n}+1}\right) = \frac{1}{a_{2n}+1}.$$

Strict inequality holds above, since we use Convention 3 to eliminate the case $A = ((a_0, a_1, \ldots, a_{2n}, 1))$. Also,

$$A'_{2n} \leqslant \frac{1}{a_{2n}} - \frac{1}{(a_{2n}+1)a_{2n}a_{2n+1}}\left(1-\frac{1}{a_{2n+1}+1}\right) - \ldots \leqslant \frac{1}{a_{2n}}.$$

Thus again $A'_{2n} > 1/(a_{2n}+1) \geqslant 1/b_{2n} \geqslant B'_{2n}$, and the result $A < B$ now follows from

$$A = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)a_1}\cdot\frac{1}{a_2} + \ldots - \frac{1}{(a_1+1)a_1 \ldots (a_{2n-1}+1)a_{2n-1}} A'_{2n},$$

$$B = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)a_1}\cdot\frac{1}{a_2} + \ldots - \frac{1}{(a_1+1)a_1 \ldots (a_{2n-1}+1)a_{2n-1}} B'_{2n}.$$

Note that if $b_{2n} = \omega$ then $B'_{2n} = 0$, and the result remains valid in this case. The result is proved in a similar fashion if (ii) holds.

For the other expansion, if $a_{2n} < b_{2n}$, $n > 0$, the fact that $a_{n+1} \geqslant a_n$, $n \geqslant 1$, implies that

$$A'_{2n} = \frac{1}{a_{2n}} - \frac{1}{(a_{2n}+1)}\cdot\frac{1}{a_{2n+1}} + \frac{1}{(a_{2n}+1)(a_{2n+1}+1)}\cdot\frac{1}{a_{2n+2}} - \ldots$$

$$\geqslant \frac{1}{a_{2n}}\left(1-\frac{1}{a_{2n}+1}\right) + \frac{1}{(a_{2n}+1)(a_{2n+1}+1)a_{2n+2}}\left(1-\frac{1}{a_{2n+2}+1}\right) + \ldots$$

$$> \frac{1}{a_{2n}}\left(1-\frac{1}{a_{2n}+1}\right) = \frac{1}{a_{2n}+1},$$

since $a_n \geqslant 1$ for $n \geqslant 1$, and by observing Convention 5: we use this in order to eliminate the possibility that

$$A = (a_0, a_1, \ldots, a_{2n}, a_{2n+1}), \quad a_{2n+1} = a_{2n},$$

as this implies that

$$A'_{2n} = \frac{1}{a_{2n}} - \frac{1}{a_{2n}+1}\cdot\frac{1}{a_{2n}} = \frac{1}{a_{2n}+1}.$$

Also

$$A'_{2n} \leqslant \frac{1}{a_{2n}} - \frac{1}{(a_{2n}+1)a_{2n+1}}\left(1-\frac{1}{a_{2n+1}+1}\right) - \ldots \leqslant \frac{1}{a_{2n}}.$$

Thus $A'_{2n} > 1/(a_{2n}+1) \geqslant 1/b_{2n} \geqslant B'_{2n}$. It now follows from

$$A = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)}\cdot\frac{1}{a_2} + \ldots - \frac{A'_{2n}}{(a_1+1)\ldots(a_{2n-1}+1)},$$

$$B = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)}\cdot\frac{1}{a_2} + \ldots - \frac{B'_{2n}}{(a_1+1)\ldots(a_{2n-1}+1)},$$

that $A < B$. The case (ii) is similar again. ∎

4. Stochastic properties of the sequence of digits. We now focus on the Lüroth-type expansion (1) for an irrational number $x \in [0, 1] = I$, and define

(2)
$$b_n(x) = \frac{1}{a_n(x)} - \frac{1}{(a_n(x)+1)a_n(x)}\cdot\frac{1}{a_{n-1}(x)} + \frac{1}{(a_n(x)+1)a_n(x)(a_{n-1}(x)+1)a_{n-1}(x)}\cdot\frac{1}{a_{n-2}(x)} + \ldots + (-1)^{n-1}\frac{1}{(a_n(x)+1)a_n(x)\ldots(a_2(x)+1)a_2(x)}\cdot\frac{1}{a_1(x)}.$$

Then the sequence $(b_n(\cdot))_{n\in N}$ is a sequence of real random variables defined on I and satisfying the recursive relation

(3)
$$b_{n+1} = \frac{1}{a_{n+1}} - \frac{1}{(a_{n+1}+1)a_{n+1}} b_n, \quad n \in N.$$

We first prove

THEOREM 7. *The digits $a_n(\cdot)$, $n \in N$, are stochastically independent and identically distributed random variables with respect to Lebesgue measure λ, with*

$$\lambda(a_n = k) = \frac{1}{k(k+1)} \quad (k \in N).$$

Proof. We first consider the sets

$$I_{k_1 \ldots k_r} = \{x \in I\colon a_1(x) = k_1, \ldots, a_r(x) = k_r\},$$

where $k_1, \ldots, k_r \in N$. Then we have

$$I_{k_2 \ldots k_r k_{r+1}} = \{x\colon a_2(x) = k_2, \ldots, a_r(x) = k_r, a_{r+1}(x) = k_{r+1}\} = \bigcup_{k_1 \in N} I_{k_1 k_2 \ldots k_r k_{r+1}}.$$

Also

$$I_{k_1 k_2 \ldots k_r k_{r+1}} = \Big\{x\colon x = \frac{1}{k_1} - \frac{1}{(k_1+1)k_1}\cdot\frac{1}{k_2} + \ldots + \frac{(-1)^{r-1}}{k_1(k_1+1)\ldots k_{r+1}(k_{r+1}+1)}\cdot\frac{1}{k_{r+2}} + \ldots, \text{ where } k_{r+2}, k_{r+3}, \ldots \text{ range over } N\Big\}$$

$$= \Big\{x\colon x = \frac{1}{k_1} - \frac{1}{(k_1+1)k_1} y,\ y \in I_{k_2 \ldots k_{r+1}}\Big\}.$$

Thus $I_{k_1 \ldots k_r} = (m_r, M_r]$ or $[m_r, M_r)$, where

$$m_r(M_r) = \min(\max)\left\{\frac{1}{k_1} - \frac{1}{(k_1+1)k_1}\cdot\frac{1}{k_2} + \ldots + \frac{(-1)^{r-1}}{k_1(k_1+1)\ldots k_{r-1}(k_{r-1}+1)}\cdot\frac{1}{k_r},\right.$$

$$\left.\frac{1}{k_1} - \frac{1}{(k_1+1)k_1}\cdot\frac{1}{k_2} + \ldots + (-1)^r \frac{1}{k_1(k_1+1)\ldots k_r(k_r+1)}\right\}.$$

Therefore we get

$$\lambda(\{x \in I:\ a_2(x) = k_1, \ldots, a_r(x) = k_{r-1}, a_{r+1}(x) = k_r\})$$

$$= \sum_{k \in N} \lambda(I_{kk_1 \ldots k_r}) = \sum_{k \in N} \frac{1}{k(k+1)k_1(k_1+1)\ldots k_r(k_r+1)}$$

$$= \prod_{i=1}^{r} \frac{1}{k_i(k_i+1)}.$$

Repeating the same argument we see in general that the value

$$\lambda(\{x \in I:\ a_n(x) = k_1, \ldots, a_{n+r-1}(x) = k_r\}) = \prod_{i=1}^{r} \frac{1}{k_i(k_i+1)}$$

does not depend on $n \in N$ for any $r \in N$, $i_1, \ldots, i_r \in N$, and the proof is complete. ∎

Further, following the same reasonings as in [3] and [4], relations (2) and (3) and Theorem 7 allow us to consider the *random system with complete connections* (for short RSCC) below:

(4) $$\{(I, B_I), (N, P_0(N)), u, P\}$$

where

$$u(x, n) = \frac{1}{n} - \frac{1}{n(n+1)}x, \quad P(x, n) = \frac{1}{n(n+1)} \quad (x \in [0, 1],\ n \in N),$$

and $P_0(N)$ denotes the power set of N.

Let us now associate with each real-valued function f defined on $I = [0, 1]$ the following two positive numbers:

$$|f|_0 = \sup_{x \in I} |f(x)|, \quad |f|_1 = \sup_{x_1 \neq x_2 \in I} \left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right|.$$

Then the set $L(I)$ of all real-valued functions defined on I for which $|f|_i < \infty$, $i = 0, 1$, is a Banach space with respect to the norm $\|f\| = |f|_0 + |f|_1$. We now prove:

THEOREM 8. *The system (4) is a RSCC with contraction in the sense of Norman, and its Markov operator is regular with respect to $L(I)$.* (Cf. the definitions in [2], [6].)

Proof. Since

$$\frac{d}{dx}P(x, n) \equiv 0, \quad \frac{d}{dx}u(x, n) = -\frac{1}{n(n+1)} < 0, \quad n \in N,$$

the system (4) is a RSCC with contraction according to Norman's definition given in [6] (see also [2]).

We further show that the associated Markov operator U is regular with respect to $L(I)$ (see Theorem 3.2.23 of Grigorescu and Iosifescu [2]). This means that we have to prove the existence of a point $x_0 \in (0, 1)$ such that

$$\lim_{n \to \infty} |\sigma_n(x) - x_0| = 0,$$

for any $x \in I$, where $\sigma_n(x)$ denotes the support of the measure $Q^n(x, \cdot)$ and Q is the transition probability measure of the associated Markov chain (see [2], p. 31).

If x is arbitrarily fixed in I, then by defining recursively

(5) $$x_1 = x, \quad x_{n+1} = \tfrac{1}{3} - \tfrac{1}{6}x_n, \quad n \in N,$$

we have $x_n \in (0, 1)$. Then letting $n \to \infty$ in (5) we find $x_0 = \frac{2}{7}$, which was to be proved. ∎

An immediate consequence of Theorem 8 is the fact that, on account of Theorem 3.4.5 of Grigorescu and Iosifescu [2], the RSCC (4) is uniformly ergodic and that there exists a limit probability on B_I. It is this limit probability that we shall identify in the following statement.

THEOREM 9. *The limit probability of the RSCC (4) is identical to Lebesgue measure λ.*

Proof. We shall show that

(6) $$\int_0^1 Q(x, B)\,\lambda(dx) = \lambda(B),$$

for all $B \in B_I$, where $Q(\cdot,\cdot)$ is defined by

$$Q(x, B) = \sum_{\substack{n \in N \\ u(x,n) \in B}} P(x, n).$$

In fact it suffices to prove (6) only for $B_0 = (0, u]$, where u ranges over the rationals of $(0, 1]$.

Let us fix $u_0 \in (0, 1]$. Then the solutions $n \in N$ of the inequality

$$\frac{n+1-x}{n(n+1)} \leqslant u_0$$

are

$$n \geqslant \left[\frac{1-u_0+\sqrt{(1-u_0)^2+4u_0(1-x)}}{2u_0}\right]+1, \qquad x \in (0, 1].$$

Let

$$n_0 = n_0(x) = \min\left\{n \in N: \frac{n+1-x}{n(n+1)} \leqslant u_0\right\}, \qquad x \in (0, 1].$$

Thus

(7) $$\int_0^1 Q(x, (0, u_0])\,dx = \int_0^1 \left(\sum_{n \geqslant n_0(x)} \frac{1}{n(n+1)}\right) dx = \int_0^1 \frac{1}{n_0(x)}\,dx.$$

Moreover, since

(8) $$\frac{1}{u_0}-1 < \frac{1-u_0+\sqrt{(1-u_0)^2+4u_0(1-x)}}{2u_0} \leqslant \frac{1}{u_0},$$

we may find x_0 such that

$$\frac{1-u_0+\sqrt{(1-u_0)^2+4u_0(1-x_0)}}{2u_0} = \left[\frac{1}{u_0}\right],$$

and therefore obtain $u_0 = (n_0+1-x_0)/(n_0(n_0+1))$. Then, by further equalities like (7), we get

$$\int_0^{x_0} \frac{1}{n_0+1}\,dx + \int_{x_0}^1 \frac{1}{n_0}\,dx = \frac{1}{n_0+1}x_0 + \frac{1}{n_0}(1-x_0)$$
$$= \frac{n_0+1-x_0}{n_0(n_0+1)} = u_0 = \lambda((0, u_0]).$$

This is what was to be proved. ∎

5. The Gauss-type problem. Let us consider the *r-th rank remainder* of the Lüroth-type alternating series representation (1):

$$r_n(x) = \frac{1}{a_n(x)} - \frac{1}{(a_n(x)+1)a_n(x)} \cdot \frac{1}{a_{n+1}(x)} + \ldots$$

Also, let μ be an arbitrary nonatomic probability measure on B_I. Set

(9) $$F_n(x) = F_n(x, \mu) = \mu(r_{n+1} < x)$$

for any $n \in N \cup \{0\} = N_0$, $x \in I$. Then we plainly have $F_0(x) = \mu([0, x])$. Since $0 < r_{n+2} < x$ if and only if

$$\frac{1}{a_{n+1}} - \frac{1}{(a_{n+1}+1)a_{n+1}}x < r_{n+1} < \frac{1}{a_{n+1}},$$

we may consider the *Gauss equation*

(10) $$F_{n+1}(x) = \sum_{i \in N}\left(F_n\left(\frac{1}{i}\right) - F_n\left(\frac{1}{i} - \frac{1}{i(i+1)}x\right)\right).$$

Assuming that F_0' exists and is bounded (i.e. μ has a bounded density), it follows by induction that F_n', $n \in N$, exist and are bounded too.

Then the derivative operation transforms (10) into

(11) $$F_{n+1}'(x) = \sum_{i \in N} \frac{1}{i(i+1)} F_n'\left(\frac{1}{i} - \frac{1}{i(i+1)}x\right),$$

for any $n \in N_0$, $x \in [0, 1]$. By writing $F_n'(x) = f_n(x)$, $n \in N_0$, $x \in [0, 1]$, relation (11) becomes

(12) $$f_{n+1}(x) = \sum_{i \in N} \frac{1}{i(i+1)} f_n\left(\frac{1}{i} - \frac{1}{i(i+1)}x\right).$$

Therefore $f_{n+1} = Uf$, where U denotes the Markov operator associated to the RSCC (4). Then

(13) $$F_n(x) = \int_0^x U^n f_0(u)\,du, \qquad n \in N_0,$$

where f_0 is given by the equality $f_0(x) = F_0'(x)$, $x \in [0, 1]$.

Now we are ready to solve the *Gauss-type measure problem* for the Lüroth-type alternating series representation:

COROLLARY 10. *If $F_0' \in L([0, 1])$, then there exist two positive constants c and $q < 1$ such that for each $x \in [0, 1]$, $n \in N$, we have*

(14) $$\mu(r_n < x) = (1 + \theta_0 q^n)x,$$

where $\theta_0 = \theta_0(n, x)$ with $|\theta_0| \leqslant c$.

Proof. We have

$$U^\infty f_0 = \int_0^1 f_0(x)\lambda(dx) = \int_0^1 F_0'(x)\,dx = F_0(1) = 1.$$

Thus, on account of Theorem 8, Theorem 9 and Lemma 3.1.22 of Grigorescu and Iosifescu [2], and on writing $V = U - U^\infty$, we get

$$U^n f_0(x) = U^\infty f_0(x) + V^n f_0(x) = 1 + \theta_0(n, x) q^n,$$

where $0 < q < 1$ and $|\theta_0| \leqslant c$ for $c > 0$. Then (14) follows from (9) and (13) and the proof is complete. ∎

Finally, we remark that the independence of the digits a_n, $n \in N$, with infinite mean value allows us to apply the various *extended classical limit* theorems to this case.

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FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS
ARISTOTLE UNIVERSITY OF THESSALONIKI
54006 Thessaloniki, Greece

DEPARTMENT OF APPLIED MATHEMATICS (A. Knopfmacher)
DEPARTMENT OF MATHEMATICS (J. Knopfmacher)
UNIVERSITY OF WITWATERSRAND
Johannesburg, Wits 2050, South Africa

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ACTA ARITHMETICA
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Über die zahlentheoretische Funktion $\omega(n)$

von

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1. Einleitung und Ergebnisse. Das mittlere Verhalten der Funktion

$$\omega(n) = \sum_{p|n} 1 \qquad (p \text{ prim})$$

ist sehr gut bekannt. Für $q \in N$ gilt

$$(1.1)\qquad \sum_{n \leqslant x} \omega^q(n) = x \sum_{j=0}^{N} R_{jq}(\ln\ln x)(\ln x)^{-j} + O\left(x(\ln x)^{-N-1}(\ln\ln x)^{q-1}\right).$$

Dabei ist N beliebig aus N, R_{0q} ist ein Polynom vom Grad q, die R_{jq} ($j \geqslant 1$) sind Polynome vom Grad $\leqslant q-1$. (Für $N = 1$ s. Hardy und Ramanujan [3]. Für beliebige N s. Delange [2], Théorème 2. Hier wird die von Selberg [7] angegebene Methode benutzt. Für eine ausführliche Darstellung s. Ivić [4], Ch. 15.) Mit Hilfe der Dirichletschen Hyperbelmethode erzielte Saffari [6] eine Verschärfung von (1.1). Für $q = 1$ lautet sein Ergebnis

$$(1.2)\qquad \sum_{n \leqslant x} \omega(n) = x\ln\ln x + Bx - x\int_1^{x^{1/2}} \frac{\{t\}}{t^2(\ln x - \ln t)}\,dt + O\left(\exp\left(-c(\ln x)^{3/5}(\ln\ln x)^{-1/5}\right)\right).$$

Unter Annahme der Riemannschen Vermutung kann nach Saffari der Fehler sogar durch $O(x^{2/3}(\ln x)^{1/3})$ abgeschätzt werden. Es scheint, als ob die zu erwartende Schranke $O(x^{1/2+\varepsilon})$ mit der Hyperbelmethode nicht erzielt werden könnte. Kolesnik und Straus [5] untersuchten mittels Integration über die erzeugende Dirichlet-Reihe die Summen $\sum_{n \leqslant x, \omega(n) = k} 1$. Sie deuteten an, daß unter Annahme der Riemannschen Vermutung Fehlerglieder der Ordnung $O(x^{1/2+\varepsilon})$ erreicht werden können. Dies Verfahren soll hier auf das erstgenannte Problem angewandt werden.

SATZ 1. *Sei $q \in N$. Notwendig und hinreichend für die Gültigkeit der Riemannschen Vermutung ist die asymptotische Formel*

$$\sum_{n \leqslant x} \omega^q(n) = x \sum_{0 \leqslant j \leqslant (1/2)\ln x} R_{jq}(\ln\ln x)(\ln x)^{-j} + O(x^{1/2+\varepsilon})$$

(*für jedes* $\varepsilon > 0$).