Thus by (14) and (13)

$$\pi(x) > \frac{2}{5}\sum_{h=1}^{[(1-\varepsilon)c_1\log_2 x]}\frac{1}{h!}\log_2^h(x^{c_1(1-\varepsilon)}) > \tfrac{1}{10}e^{(1-\varepsilon)c_1\log_2 x} = \tfrac{1}{10}x^{(1-\varepsilon)c_1(\log 2)^{-1}}$$

for $x > x_0''$. By choosing $\varepsilon = 1/100$ the proof is completed.

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INSTITUT FÜR MATHEMATIK
UNIVERSITÄT HANNOVER
Welfengarten 1, 3000 Hannover 1, BRD

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The average order of $d_k(n)$ over integers free of large prime factors

by

TI ZUO XUAN (Beijing)

1. Introduction and statement of results. Let us define $p(n)$ as the largest prime factor of $n \geqslant 2$, $p(1) = 1$, and let

$$\Psi_f(x, y) = \sum_{n \leqslant x, p(n) \leqslant y} f(n),$$

where f is an additive or a multiplicative function. Recently, in [1], [2], Alladi established asymptotic formulas for $\Psi_f(x, y)$ when $f = \mu$ and $f = \omega$, respectively, where $\mu(n)$ is the Möbius function and $\omega(n)$ denotes the number of distinct prime factors of n. In [9], [10], Ivić established asymptotic formulas for $\Psi_f(x, y)$ when $f = \mu^2$ and $f = \Omega - \omega$, respectively, where $\Omega(n)$ denotes the total number of prime factors of n.

In [11], we estimate $\Psi_f(x, y)$ for $f = d$, where $d(n)$ denotes the divisor function.

The purpose of this note is to estimate $\Psi_f(x, y)$ for $f = d_k$, where $d_k(n)$ denotes the number of ways n can be written as a product of k factors, in particular $d_2(n) = d(n)$.

Let $\Psi(x, y)$ denote the number of positive integers not exceeding x, all of whose prime factors do not exceed y. Let the "Dickman function" ϱ be defined for $u \geqslant 0$ as the continuous solution of the equations

$$\begin{cases} \varrho(u) = 1, & 0 \leqslant u \leqslant 1, \\ u\varrho'(u) = -\varrho(u-1), & u > 1. \end{cases} \tag{1.1}$$

In 1951, Hua [8] established the asymptotic relation

$$\varrho(u) = \exp\left\{-u\left(\log u + \log_2 u - 1 + \frac{\log_2 u}{\log u} + O\left(\frac{1}{\log u}\right)\right)\right\}, \tag{1.2}$$

where $\log_2 u = \log\log u$. (1.2) is also due to de Bruijn [3], who actually proves a stronger result.

In the sequel, we write systematically $u = \log x/\log y$. The following is proved in [11]:

THEOREM A. *For any fixed* $\varepsilon > 0$ *and*

$$x \geqslant 3, \quad \exp\{(\log_2 x)^{5/3+\varepsilon}\} \leqslant y \leqslant x, \tag{1.3}$$

we have uniformly

$$\sum_{n \leqslant x, p(n) \leqslant y} d(n) = \frac{\pi^{1/2}\varrho^2(u/2)}{(\xi'(u/2))^{1/2}\varrho(u)} \Psi(x, y)\log y\left(1+O\left(\frac{1}{u}\right)+O\left(\frac{\log(u+1)}{\log y}\right)\right),$$

where $\xi(u)$ *is given by* (2.1) *below.*

To give a clearer impression of Theorem A, in [11] we also point out the following corollary.

COROLLARY A. *For* x, y *satisfying* (1.3) *we have uniformly*

$$\sum_{n \leqslant x, p(n) \leqslant y} d(n) = 2^{u+O(u/\log u)}\Psi(x, y)\log y.$$

In the present note, the following result is proved.

THEOREM 1. *For* x, y *satisfying* (1.3) *and for* $k \geqslant 2$ *fixed we have uniformly*

$$\sum_{n \leqslant x, p(n) \leqslant y} d_k(n) = \frac{(2\pi)^{(k-1)/2}\varrho^k(u/k)}{k^{1/2}(\xi'(u/k))^{(k-1)/2}\varrho(u)} \Psi(x, y)(\log y)^{k-1}$$
$$\times\left(1+O\left(\frac{1}{u}\right)+O\left(\frac{\log(u+1)}{\log y}\right)\right),$$

where the constant implied by "O" depends on k *and* ε.

COROLLARY 1. *For* x, y *satisfying* (1.3) *and for* $k \geqslant 2$ *fixed we have uniformly*

$$\sum_{n \leqslant x, p(n) \leqslant y} d_k(n) = k^{u+O(u/\log u)}\Psi(x, y)(\log y)^{k-1}.$$

Moreover, it was proved in [4] that

$$\sum_{n \leqslant x} \frac{1}{p(n)} = x\delta(x)\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right), \tag{1.4}$$

where

$$\delta(x) = \int_2^x \varrho\left(\frac{\log x}{\log t}\right)\frac{dt}{t^2}. \tag{1.5}$$

It was proved in [10] that

$$\sum_{n \leqslant x} \frac{\omega(n)}{p(n)} = \left(\frac{2\log x}{\log_2 x}\right)^{1/2}\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right)\sum_{n \leqslant x}\frac{1}{p(n)}, \tag{1.6}$$

$$\sum_{n \leqslant x} \frac{\Omega(n)-\omega(n)}{p(n)} = \left(\sum_p \frac{1}{p^2-p}+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\sum_{n \leqslant x}\frac{1}{p(n)}, \tag{1.7}$$

$$\sum_{n \leqslant x} \frac{\mu^2(n)}{p(n)} = \left(\frac{6}{\pi^2}+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\sum_{n \leqslant x}\frac{1}{p(n)}. \tag{1.8}$$

In [12], we proved that

$$\sum_{n \leqslant x} \frac{\sigma(n)}{p(n)} = \frac{\pi^2}{12}x\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\sum_{n \leqslant x}\frac{1}{p(n)}, \tag{1.9}$$

$$\sum_{n \leqslant x} \frac{\varphi(n)}{p(n)} = \frac{3}{\pi^2}x\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\sum_{n \leqslant x}\frac{1}{p(n)}, \tag{1.10}$$

where $\sigma(n)$ denotes the sum of all divisors of n, and $\varphi(n)$ is Euler's totient function. Using Theorem A we proved in [11] that

THEOREM B.

$$\sum_{n \leqslant x} \frac{d(n)}{p(n)} = 2^{-3/4}\pi^{1/2}x(\log x)^{3/4}(\log_2 x)^{1/4}\Delta(x)\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right),$$

where

$$\Delta(x) = \int_2^x \varrho^2\left(\frac{\log x}{2\log t}\right)\frac{dt}{t^2}.$$

COROLLARY B.

$$\sum_{n \leqslant x} \frac{d(n)}{p(n)} = 2^{(2\log x/\log_2 x)^{1/2}(1+O(\log_3 x/\log_2 x))}\sum_{n \leqslant x}\frac{1}{p(n)}.$$

In the present note, using Theorem 1, we prove the following

THEOREM 2.

$$\sum_{n \leqslant x} \frac{d_k(n)}{p(n)} = 2^{(k-1)/4}\pi^{(k-1)/2}k^{-k/2}x(\log x)^{3(k-1)/4}(\log_2 x)^{(k-1)/4}$$
$$\times \Delta_k(x)\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right),$$

where

$$\Delta_k(x) = \int_2^x \varrho^k\left(\frac{\log x}{k\log t}\right)\frac{dt}{t^2}.$$

COROLLARY 2.

$$\sum_{n \leqslant x} \frac{d_k(n)}{p(n)} = k^{(2\log x/\log_2 x)^{1/2}(1+O(\log_3 x/\log_2 x))}\sum_{n \leqslant x}\frac{1}{p(n)}.$$

2. Several lemmas

LEMMA 1 [5]. *For any fixed* $\varepsilon > 0$ *and* $x \geqslant 3$, $\exp\{(\log_2 x)^{5/3+\varepsilon}\} \leqslant y \leqslant x$, *we have uniformly*

$$\Psi(x, y) = x\varrho(u)\left(1+O\left(\frac{\log(u+1)}{\log y}\right)\right).$$

LEMMA 2. *Uniformly for $u \geqslant 1$ and $1 \leqslant t \leqslant u^{2/3}$ we have*

(i) $\varrho(u-t) = \varrho(u)\exp\left\{t\xi(u)-\frac{t^2}{2}\xi'(u)+\frac{t^3}{6}\xi''(u)\right\}\left(1+O\left(\frac{t}{u}\right)\right)$,

(ii) $\varrho(u+t) = \varrho(u)\exp\left\{-t\xi(u)-\frac{t^2}{2}\xi'(u)-\frac{t^3}{6}\xi''(u)\right\}\left(1+O\left(\frac{t}{u}\right)\right)$,

where $\xi = \xi(u)$ denotes the positive solution of the equation

(2.1) $$e^{\xi} = u\xi+1 \quad (u>1)$$

and $\xi(u) = 0$ $(u \leqslant 1)$, and satisfies

(2.2) $$\xi(u) = \log u+\log_2 u+O(\log_2 u/\log u), \quad u\to\infty.$$

Proof. From Corollary 2 of [6], we know that as $u\to\infty$,

(2.3) $$\varrho(u) = \left(1+O\left(\frac{1}{u}\right)\right)\left(\frac{\xi'(u)}{2\pi}\right)^{1/2}\exp\left\{\gamma-u\xi(u)+\int_0^{\xi(u)}\frac{e^s-1}{s}\,ds\right\}.$$

Hence

(2.4) $$\frac{\varrho(u-t)}{\varrho(u)} = \left(1+O\left(\frac{1}{u-t}\right)\right)\left(\frac{\xi'(u-t)}{\xi'(u)}\right)^{1/2}\exp\{F(u,t)\},$$

where

(2.5) $$F(u,t) = u\xi(u)-(u-t)\xi(u-t)+\int_{\xi(u)}^{\xi(u-t)}\frac{e^s-1}{s}\,ds.$$

By (2.1) we have

(2.6) $$\frac{\partial}{\partial t}F(u,t) = \xi(u-t)+(u-t)\xi'(u-t)-(u-t)\xi'(u-t) = \xi(u-t).$$

So

$$\frac{\partial^2}{\partial t^2}F(u,t) = -\xi'(u-t), \quad \frac{\partial^3}{\partial t^3}F(u,t) = \xi''(u-t), \quad \frac{\partial^4}{\partial t^4}F(u,t) = -\xi'''(u-t).$$

By (2.1) we have also

(2.7) $$\xi'(u) = \frac{\xi}{u\xi-u+1} = \frac{1}{u}\left(1+O\left(\frac{1}{\xi(u)}\right)\right),$$

(2.8) $$\xi''(u) = -\frac{e^{\xi}(\xi^2-2\xi+2)-2}{(e^{\xi}-u)^3} = -\frac{1}{u^2}\left(1+O\left(\frac{1}{\xi(u)}\right)\right),$$

$$\xi'''(u) = O(1/u^3).$$

Therefore, for $1 \leqslant t \leqslant u^{2/3}$, we obtain

(2.9) $$F(u,t) = t\xi(u)-\frac{t^2}{2}\xi'(u)+\frac{t^3}{6}\xi''(u)+O\left(\frac{t^4}{u^3}\right).$$

Obviously

(2.10) $$(\xi'(u-t)/\xi'(u))^{1/2} = 1+O(t/u).$$

From (2.4), (2.9) and (2.10), part (i) of Lemma 2 is derived at once. The proof of (ii) is similar. Moreover, it is easy to deduce (2.2) using (2.1). This completes the proof of the lemma.

LEMMA 3. *Uniformly for $u \geqslant 1$ and $0 \leqslant t \ll 1$ we have*

(i) $\varrho(u-t) = \varrho(u)e^{t\xi(u)}(1+O(t/u))$,

(ii) $\varrho(u+t) = \varrho(u)e^{-t\xi(u)}(1+O(t/u))$.

Proof. (i) is a slightly stronger form of [2, Lemma 3]. By (3.11) of [2] we have

$$\frac{\varrho(u-t)}{\varrho(u)} = \left(1+O\left(\frac{t}{u\xi(u)}\right)\right)\left(\frac{\xi'(u-t)}{\xi'(u)}\right)^{1/2}\exp\{F(u,t)\},$$

where $F(u,t)$ is given by (2.5). We now obtain (i) as in the proof of Lemma 2. The proof of (ii) is similar. This completes the proof of Lemma 3.

LEMMA 4. *Let $k \geqslant 2$ be a fixed integer. Uniformly for $u \geqslant 1$ and $1 \leqslant t \leqslant u^{2/3}$ we have*

(i) $$\varrho\left(\frac{u}{k}-t\right)\varrho^{k-1}\left(\frac{u}{k}+\frac{t}{k-1}\right) = \varrho^k\left(\frac{u}{k}\right)\exp\left\{-\frac{k}{2(k-1)}t^2\xi'\left(\frac{u}{k}\right)\right.$$
$$\left.+\frac{t^3}{6}\left(1-\frac{1}{(k-1)^2}\right)\xi''\left(\frac{u}{k}\right)\right\}\left(1+O\left(\frac{1}{u}\right)+O\left(\frac{t^4}{u^3}\right)\right),$$

(ii) $$\varrho\left(\frac{u}{k}+t\right)\varrho^{k-1}\left(\frac{u}{k}-\frac{t}{k-1}\right) = \varrho^k\left(\frac{u}{k}\right)\exp\left\{-\frac{k}{2(k-1)}t^2\xi'\left(\frac{u}{k}\right)\right.$$
$$\left.-\frac{t^3}{6}\left(1-\frac{1}{(k-1)^2}\right)\xi''\left(\frac{u}{k}\right)\right\}\left(1+O\left(\frac{1}{u}\right)+O\left(\frac{t^4}{u^3}\right)\right).$$

Proof. For simplicity put $\bar{u} = u/k$, $\bar{t} = t/(k-1)$. From (2.4) and (2.9) we have

(2.11) $$\frac{\varrho(\bar{u}-t)}{\varrho(\bar{u})} = \left(1+O\left(\frac{1}{u}\right)+O\left(\frac{t^4}{u^3}\right)\right)\left(\frac{\xi'(\bar{u}-t)}{\xi'(\bar{u})}\right)^{1/2}$$
$$\times\exp\left\{t\xi(\bar{u})-\frac{t^2}{2}\xi'(\bar{u})+\frac{t^3}{6}\xi''(\bar{u})\right\}.$$

Similarly

$$\frac{\varrho^{k-1}(\bar u+\bar t)}{\varrho^{k-1}(\bar u)}=\left(1+O\left(\frac{1}{u}\right)+O\left(\frac{t^4}{u^3}\right)\right)\left(\frac{\xi'(\bar u+\bar t)}{\xi'(\bar u)}\right)^{(k-1)/2}\times\exp\left\{-t\xi(\bar u)-\frac{t^2}{2(k-1)}\xi'(\bar u)-\frac{t^3}{6(k-1)^2}\xi''(\bar u)\right\}. \tag{2.12}$$

It is easy to prove that

$$\left(\frac{\xi'(\bar u-t)}{\xi'(\bar u)}\right)^{1/2}=1-\frac{t}{2}\cdot\frac{\xi''(\bar u)}{\xi'(\bar u)}+O\left(\frac{t^2}{u^2}\right). \tag{2.13}$$

Similarly

$$\left(\frac{\xi'(\bar u+\bar t)}{\xi'(\bar u)}\right)^{(k-1)/2}=1+\frac{t}{2}\cdot\frac{\xi''(\bar u)}{\xi'(\bar u)}+O\left(\frac{t^2}{u^2}\right). \tag{2.14}$$

Part (i) of the lemma then follows from (2.11)–(2.14), when we note that $t^2/u^2 \ll 1/u+t^4/u^3$ for $1\leqslant t\leqslant u^{2/3}$. Part (ii) is proved analogously.

LEMMA 5. (i) *Uniformly for $u\geqslant 2k$ and $1\leqslant t\leqslant u/k-1$ we have*

$$\varrho\left(\frac{u}{k}-t\right)\varrho^{k-1}\left(\frac{u}{k}+\frac{t}{k-1}\right)\ll u^{1/2}\varrho^k\left(\frac{u}{k}\right)\exp\left\{-\frac{t^2}{2}\xi'\left(\frac{u}{k}\right)\right\}.$$

(ii) *Uniformly for $u\geqslant 2k$ and $1\leqslant t\leqslant (1-1/k)u-k$ we have*

$$\varrho\left(\frac{u}{k}+t\right)\varrho^{k-1}\left(\frac{u}{k}-\frac{t}{k-1}\right)\ll u^{1/2}\varrho^k\left(\frac{u}{k}\right)\exp\left\{-\frac{t^2}{2}\xi'\left(\frac{u}{k}\right)\right\}.$$

Proof. We use $\bar u$, $\bar t$ as in the preceding proof. By (2.3) we have

$$\frac{\varrho(u+t)}{\varrho(u)}=\left(1+O\left(\frac{1}{u}\right)\right)\left(\frac{\xi'(u+t)}{\xi'(u)}\right)^{1/2}\exp\{F(u,-t)\}, \tag{2.15}$$

where $F(u,t)$ is given by (2.5). By (2.7) we obtain, for $1\leqslant t\leqslant \bar u-1$,

$$\left(\frac{\xi'(\bar u-t)}{\xi'(\bar u)}\right)^{1/2}\ll u^{1/2},\quad \left(\frac{\xi'(\bar u+\bar t)}{\xi'(\bar u)}\right)^{(k-1)/2}\ll 1.$$

From this and (2.4), (2.15) we have

$$\frac{\varrho(\bar u-t)}{\varrho(\bar u)}\left(\frac{\varrho(\bar u+\bar t)}{\varrho(\bar u)}\right)^{k-1}\ll u^{1/2}\exp\{F(\bar u,t)+(k-1)F(\bar u,-\bar t)\}. \tag{2.16}$$

Let

$$G(u,t)=F(\bar u,t)+(k-1)F(\bar u,-\bar t)+\tfrac12 t^2\xi'(\bar u).$$

By (2.5) and (2.6) we have

$$\frac{\partial}{\partial t}G(u,t)=\xi(\bar u-t)-\xi(\bar u+\bar t)+t\xi'(\bar u),$$

whence

$$\frac{\partial^2}{\partial t^2}G(u,t)=-\xi'(\bar u-t)-\frac{1}{k-1}\xi'(\bar u+\bar t)+\xi'(\bar u).$$

From (2.7) and (2.8) we know that $\xi'(u)>0$, $\xi''(u)<0$, for $u>1$. Thus $\xi'(u)$ is decreasing, and $\xi'(\bar u)-\xi'(\bar u-t)<0$. Hence $(\partial^2/\partial t^2)G(u,t)<0$. From this we obtain, for $t>0$, $(\partial/\partial t)G(u,t)<(\partial/\partial t)G(u,0)=0$ and $G(u,t)<G(u,0)=0$. Part (i) of the lemma now follows from (2.16). Part (ii) is proved analogously.

LEMMA 6. *Let*

$$L_1=\exp\left\{(\tfrac12\log x\log_2 x)^{1/2}\left(1-2\frac{\log_3 x}{\log_2 x}\right)\right\},$$

$$L_2=\exp\left\{(\tfrac12\log x\log_2 x)^{1/2}\left(1+2\frac{\log_3 x}{\log_2 x}\right)\right\}.$$

Then for any fixed $A>0$,

$$\sum_{n\leqslant x}\frac{1}{p(n)}=1+\sum_{p\leqslant x}\frac{1}{p}\Psi\left(\frac{x}{p},p\right)=(1+O(\log^{-A}x))\sum_{L_1<p\leqslant L_2}\frac{1}{p}\Psi\left(\frac{x}{p},p\right).$$

Proof. See (4.3) of [10].

3. Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. In the proof, we do not use Theorem A, which actually will be proven again. We proceed by induction. Let $d_1(n)\equiv 1$. In the case $k=1$, Theorem 1 is trivial. Now assume that Theorem 1 is true for $k-1$ (here $k\geqslant 2$); we shall show it is also true for k.

When $x^{1/(k+1)}<y\leqslant x$, the conclusion of Theorem 1 becomes

$$\sum_{n\leqslant x,\,p(n)\leqslant y} d_k(n)\ll \Psi(x,y)(\log x)^{k-1}.$$

Obviously, this is true. Now suppose $y\leqslant x^{1/(k+1)}$. We have

$$\begin{aligned} D_k(x):&=\sum_{n\leqslant x,\,p(n)\leqslant y} d_k(n)=\sum_{n\leqslant x,\,p(n)\leqslant y}\sum_{\delta\mid n} d_{k-1}(\delta)\\ &=\sum_{y<m\leqslant x/y^k,\,p(m)\leqslant y}\ \sum_{\delta\leqslant x/m,\,p(\delta)\leqslant y} d_{k-1}(\delta)\\ &\quad+\sum_{m\leqslant y}\ \sum_{\delta\leqslant x/m,\,p(\delta)\leqslant y} d_{k-1}(\delta)\\ &\quad+\sum_{x/y^k<m\leqslant x,\,p(m)\leqslant y}\ \sum_{\delta\leqslant x/m,\,p(\delta)\leqslant y} d_{k-1}(\delta)\\ &=D_1+D_2+D_3,\quad\text{say.}\end{aligned} \tag{3.1}$$

Put

$$w_m := \frac{1}{k-1}\left(u - \frac{\log m}{\log y}\right).$$

By the inductive hypothesis we have

$$D_1 = \frac{(2\pi)^{(k-2)/2}}{(k-1)^{1/2}} x(\log y)^{k-2} \sum_{y<m\leqslant x/y^k, p(m)\leqslant y} \frac{\varrho^{k-1}(w_m)m^{-1}}{(\xi'(w_m))^{(k-2)/2}} \times\left(1+O\left(\frac{1}{u-\log m/\log y}\right)+O\left(\frac{\log(u+1)}{\log y}\right)\right). \tag{3.2}$$

We first estimate the sum on the right-hand side of (3.2). We shall use the following elementary partial summation identity:

$$\sum_{M<n\leqslant N} a_n(b_n-b_{n-1}) = \sum_{M<n\leqslant N-1} b_n(a_n-a_{n+1}) + a_N b_N - a_{M+1}b_M, \tag{3.3}$$

where M, N are positive integers. By (3.3) and Lemma 1 we have

$$\begin{aligned}\textstyle\sum := &\sum_{y<m\leqslant x/y^k, p(m)\leqslant y} \frac{\varrho^{k-1}(w_m)m^{-1}}{(\xi'(w_m))^{(k-2)/2}} \\ = &\sum_{y<m\leqslant x/y^k} \Psi(x,y)\left\{\frac{\varrho^{k-1}(w_m)m^{-1}}{(\xi'(w_m))^{(k-2)/2}} - \frac{\varrho^{k-1}(w_{m+1})(m+1)^{-1}}{(\xi'(w_{m+1}))^{(k-2)/2}}\right\} \\ &+O(\varrho(u-k))+O\left(\varrho^{k-1}\left(\frac{u-1}{k-1}\right)u^{(k-2)/2}\right).\end{aligned} \tag{3.4}$$

By Lemma 3(i) we have

$$\varrho(w_{m+1}) = \varrho(w_m)\left(1+O\left(\frac{1}{m}\frac{\log(u+1)}{\log y}\right)\right), \tag{3.5}$$

and by (2.7) and (2.8) we have

$$\xi'(w_{m+1}) = \xi'(w_m)\left(1+O\left(\frac{1}{m}\frac{\log(u+1)}{\log y}\right)\right). \tag{3.6}$$

Using Lemma 1 and (3.4)–(3.6) we have

$$\textstyle\sum = \displaystyle\sum_{y<m\leqslant x/y^k} \frac{\varrho(\log m/\log y)\varrho^{k-1}(w_m)}{m(\xi'(w_m))^{(k-2)/2}}\left(1+O\left(\frac{\log(u+1)}{\log y}\right)\right) +O(\varrho(u-k))+O\left(\varrho^k\left(\frac{u-1}{k-1}\right)u^{(k-2)/2}\right). \tag{3.7}$$

It is easy to prove that the summatory function on the right-hand side of (3.7) is decreasing. Using Theorem 8.2 of [7, Ch. 5] we see that the sum on the right-hand side of (3.7) is

$$\log y \int_1^{u-k} \frac{\varrho(w)\varrho^{k-1}((u-w)/(k-1))}{(\xi'((u-w)/(k-1)))^{(k-2)/2}}dw + O\left(\varrho^{k-1}\left(\frac{u-1}{k-1}\right)u^{(k-2)/2}\right). \tag{3.8}$$

Now we estimate the integral in (3.8). We have

$$I(u) := \int_1^{u-k} = \int_1^{u/k} + \int_{u/k}^{u-k} = I_1 + I_2, \quad \text{say}. \tag{3.9}$$

Let $w = u/k - t$. Then we have

$$I_1 = \int_0^{u/k-1} \frac{\varrho\left(\frac{u}{k}-t\right)\varrho^{k-1}\left(\frac{u}{k}+\frac{t}{k-1}\right)}{\left(\xi'\left(\frac{u}{k}+\frac{t}{k-1}\right)\right)^{(k-2)/2}}dt = \int_0^{t_0} + \int_{t_0}^{u/k-1},$$

where $t_0 = u^{1/2}\log u$. Set as before $\bar{u} = u/k$, $\bar{t} = t/(k-1)$. Using Lemma 5(i) and (2.7) we have

$$\int_{t_0}^{\bar{u}-1} \ll \int_{t_0}^{\infty} u^{(k-2)/2}u^{1/2}\varrho^k(\bar{u})e^{-(1/2)t^2\xi'(\bar{u})}dt \ll \varrho^k(\bar{u})u^{-2}.$$

Similarly

$$I_2 = \int_0^{t_0} \frac{\varrho(\bar{u}+t)\varrho^{k-1}(\bar{u}-\bar{t})}{(\xi'(\bar{u}-\bar{t}))^{(k-2)/2}}dt + O\left(\frac{1}{u^2}\varrho^k(\bar{u})\right).$$

By (3.9) we have

$$I(u) = \int_0^{t_0}\left\{\frac{\varrho(\bar{u}-t)\varrho^{k-1}(\bar{u}+\bar{t})}{(\xi'(\bar{u}+\bar{t}))^{(k-2)/2}} + \frac{\varrho(\bar{u}+t)\varrho^{k-1}(\bar{u}-\bar{t})}{(\xi'(\bar{u}-\bar{t}))^{(k-2)/2}}\right\}dt + O\left(\frac{1}{u^2}\varrho^k(\bar{u})\right). \tag{3.10}$$

It is easy to prove that

$$(\xi'(\bar{u}\pm\bar{t}))^{-(k-2)/2} = (\xi'(\bar{u}))^{-(k-2)/2}\left(1 \mp \frac{t(k-2)\xi''(\bar{u})}{2(k-1)\xi'(\bar{u})} + O\left(\frac{t^2}{u^2}\right)\right)$$

and we note that $t^2/u^2 \ll 1/u + t^4/u^3$ for $1 \leqslant t \leqslant t_0$, so that Lemma 4 gives

$$\begin{aligned}&\frac{\varrho(\bar{u}\mp t)\varrho^{k-1}(\bar{u}\pm\bar{t})}{(\xi'(\bar{u}\pm\bar{t}))^{(k-2)/2}} \\ &= \frac{\varrho^k(\bar{u})}{(\xi'(\bar{u}))^{(k-2)/2}}e^{-(kt^2/2(k-1))\xi'(\bar{u})}\left(1 \pm \frac{t^3}{6}\left(1-\frac{1}{(k-1)^2}\right)\xi''(\bar{u}) \mp \frac{t(k-2)\xi''(\bar{u})}{2(k-1)\xi'(\bar{u})} \right. \\ &\qquad \left. + O\left(\frac{1}{u}\right) + O\left(\frac{t^4}{u^3}\right) + O\left(\frac{t^6}{u^4}\right)\right).\end{aligned}$$

Therefore

$$I(u)=\varrho^k(\bar u)(\xi'(\bar u))^{-(k-2)/2}\{2\int_0^\infty e^{-(k/2(k-1))t^2\xi'(\bar u)}\,dt$$
$$+O(\int_{t_0}^\infty e^{-(k/2(k-1))t^2\xi'(\bar u)}\,dt)$$
$$+O\left(\int_0^{t_0}\left(\frac{1}{u}+\frac{t^4}{u^3}+\frac{t^6}{u^4}\right)e^{-(k/2(k-1))t^2\xi'(\bar u)}\,dt\right)+O\left(\frac{1}{u^2}\varrho^k(\bar u)\right).$$

A simple calculation shows that

$$I(u)=\left(\frac{2\pi(k-1)}{k}\right)^{1/2}\frac{\varrho^k(\bar u)}{(\xi'(\bar u))^{(k-1)/2}}\left(1+O\left(\frac{1}{u}\right)\right). \tag{3.11}$$

Also, it follows from (1.2) and Lemma 3(i) that

$$\varrho(u-k)\ll\varrho^k(\bar u)u^{-1},\quad \varrho^{k-1}\left(\frac{u-1}{k-1}\right)u^{(k-2)/2}\ll\varrho^k(\bar u)u^{-1}. \tag{3.12}$$

From this and (3.2), (3.7), (3.8) and (3.11) we have

$$D_1=\frac{(2\pi)^{(k-1)/2}}{k^{1/2}}\frac{\varrho^k(\bar u)}{(\xi'(\bar u))^{(k-1)/2}\varrho(u)}\Psi(x,y)(\log y)^{k-1} \tag{3.13}$$
$$\times\left(1+O\left(\frac{1}{u}\right)+O\left(\frac{\log(u+1)}{\log y}\right)\right).$$

To finish the proof of the theorem it remains to show $D_2, D_3\ll D_1u^{-1}$. By the inductive hypothesis we have

$$D_2\ll\sum_{m\le y}\frac{\varrho^{k-1}(w_m)}{(\xi'(w_m))^{(k-2)/2}}\cdot\frac{x}{m}(\log y)^{k-2}.$$

Because $\varrho(u)$ is decreasing, it follows from (2.7) that

$$D_2\ll x(\log y)^{k-1}\varrho^{k-1}\left(\frac{u-1}{k-1}\right)u^{(k-2)/2}.$$

By (1.2) we have

$$\varrho^k(u/k)=k^{u+O(u/\log u)}\varrho(u). \tag{3.14}$$

Thus we have $D_2\ll D_1u^{-1}$. Now we turn to the estimation of D_3. By the inductive hypothesis we have

$$D_3\ll x(\log y)^{k-2}\sum_{x/y^k<m\le x}m^{-1}(\Psi(m,y)-\Psi(m-1,y)).$$

Using (3.3) we have

$$D_3\ll x(\log y)^{k-2}\sum_{x/y^k<m\le x}\Psi(m,y)m^{-2}\ll x(\log y)^{k-1}\varrho(u-k).$$

Using (3.12) we have $D_3\ll D_1u^{-1}$. This completes the proof of Theorem 1.

Corollary 1 follows from Theorem 1 and (3.14).

4. Proofs of Theorem 2 and Corollary 2. The proofs of Theorem 2 and Corollary 2 are similar to those of Theorem B and Corollary B (cf. [11]). Therefore we shall only sketch the proof of Theorem 2.

Let $Z=\exp\{(\log x\log_2 x)^{1/2}\}$, $Z_1=Z^{1/10}$, and $Z_2=Z^{10}$. Using Corollary 1, Lemma 1 and (1.2) we have

$$\sum_{n\le x}\frac{d_k(n)}{p(n)}=\sum_{n\le x,Z_1<p(n)\le Z_2}\frac{d_k(n)}{p(n)}+O(xZ^{-4})=G_1+O(xZ^{-4}),\quad\text{say.} \tag{4.1}$$

Further we have

$$G_1=\sum_{p(n)\|n}+\sum_{p^2(n)|n}=\sum_{Z_1<p\le Z_2}\frac{k}{p}\sum_{m\le x/p,p(m)\le p}d_k(m) \tag{4.2}$$
$$+O\Big(\sum_{Z_1<p\le Z_2}\frac{1}{p}\sum_{m\le x/p^2,p(m)\le p}d_k(m)\Big)=G_2+O(G_3),\quad\text{say.}$$

Using Theorem 1 and (2.7) we obtain

$$G_2=\frac{(2\pi)^{(k-1)/2}}{k^{k/2-1}}\sum_{Z_1<p\le Z_2}\frac{u_1^{(k-1)/2}\varrho^k((u_1-1)/k)(\log p)^{k-1}}{p\varrho(u_1-1)} \tag{4.3}$$
$$\times\Psi\left(\frac{x}{p},p\right)\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right),$$

where $u_1=\log x/\log p$. To further estimate G_2 we write simply

$$G_2=\sum_{Z_1<p\le L_1}+\sum_{L_1<p\le L_2}+\sum_{L_2<p\le Z_2}=G_{21}+G_{22}+G_{23},\quad\text{say} \tag{4.4}$$

where L_1 and L_2 are defined in Lemma 6. By (3.14) we have

$$\frac{u_1^{(k-1)/2}\varrho^k((u_1-1)/k)}{\varrho(u_1-1)}=k^{u_1+O(u_1/\log u_1)}.$$

Similarly for G_{21}, G_{23} we obtain analogously to (4.3) of [10]

$$G_{21}, G_{23}\ll(\log^{-A}x)G_{22}, \tag{4.5}$$

where A is any fixed positive number. Using Lemma 3(i), (2.1), and (2.2) we obtain

$$G_{22}=2^{(k-1)/4}\pi^{(k-1)/2}k^{-k/2}x(\log x)^{3(k-1)/4}(\log_2 x)^{(k-1)/4} \tag{4.6}$$
$$\times\sum_{L_1<p\le L_2}\frac{\log p}{p^2}\varrho^k\left(\frac{u_1}{k}\right)\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right).$$

Further we have

$$(4.7)\qquad \sum_{L_1<p\le L_2}\frac{\log p}{p^2}\varrho^k\left(\frac{u_1}{k}\right)=\sum_{Z_1<p\le Z_2}\frac{\log p}{p^2}\varrho^k\left(\frac{u_1}{k}\right)\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right)$$
$$=\int_{Z_1}^{Z_2}\frac{\log z}{z^2}\varrho^k\left(\frac{\log x}{k\log z}\right)d\pi(z)\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right).$$

As for G_2 we have similarly $G_3 \ll (\log_3 x/\log_2 x)G_2$. Combining (4.1)–(4.7) completes the proof of Theorem 2.

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DEPARTMENT OF MATHEMATICS
BEIJING NORMAL UNIVERSITY
Beijing, China

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B_2-sequences whose terms are squares

by

J. Cilleruelo (Madrid)

Introduction. A sequence of integers $1 \le a_1 < a_2 < \ldots$ is called a *B_2-sequence* if the sums $a_i + a_j$ are all different. Sidon asked for a B_2- sequence for which a_k increases as slowly as possible. There is a trivial argument which allows us to construct such a B_2-sequence with $a_k \ll k^3$ for all k. For a long time, this bound was the best known one until Ajtai, Komlós and Szemerédi [1] showed, with an ingenious method, the existence of a B_2-sequence such that $a_k/k^3 \to 0$. However, this result is far from Erdős' conjecture on the existence, for each $\varepsilon > 0$, of a B_2-sequence with $a_k \ll k^{2+\varepsilon}$ [3].

In this paper we deal with B_2-sequences of squares, in other words, sequences of integers $1 \le a_1 < a_2 < \ldots$ where the sums $a_i^2 + a_j^2$ are all distinct.

Again, there is an easy argument giving us, for each $\varepsilon > 0$, a sequence such that $a_k \ll k^{2+\varepsilon}$ and where the sums $a_i^2 + a_j^2$ are all different. Apparently, there is not a simple argument to improve this result.

The purpose of this paper is to remove ε, using a new method developed by Javier Cilleruelo and Antonio Córdoba in [2].

Theorem. *There exists a sequence $A = \{a_k\}$, $a_k \ll k^2$, such that the sums $a_i^2 + a_j^2$ are all different.*

Proof. Consider the sets $I_j = \{a;\ 6^j \le a < 6^j + 6^{j/2},\ a \equiv 2 \pmod 6\}$ and $I = \bigcup_{j=1}^{\infty} I_j$. The sequence A will be given by the set I except for a few numbers that we have to eliminate: $A = \bigcup_{j=1}^{\infty} A_j$, $A_j \subset I_j$.

Construction of A_k. Once we have chosen the A_j, $j < k$, we shall pick the members of A_k from among the elements of I_k, with a few exceptions, to avoid $a^2 + b^2 = c^2 + d^2$, with $a, b, c, d \in \bigcup_{j=1}^{k} A_j$.

Lemma 1. *Let a, b, c, d belong respectively to I_k, I_l, I_j, I_m, where $k \ge j \ge m \ge l$, and suppose $a^2 + b^2 = c^2 + d^2$, $a > c > d > b$. Then we have:*

(i) $k = j$.

(ii) *If $l < m$, $k/2 \le m \le 3k/4$.*