concerne le théorème 2, il faut voir que le cas où l'on n'avait pas de fréquence dominante était le cas difficile du théorème du quotient de Hadamard. La méthode utilisée dans ce cas ne semble pas devoir s'appliquer ici, et par suite il est probable qu'un tel résultat sera assez compliqué à démontrer.

Pour ce qui concerne le lemme 1, il semble qu'une généralisation nécessite des renseignements précis sur la croissance d'une suite récurrente linéaire. Des résultats sur ce sujet ont été annoncés par H. P. Schlickewei et A. J. van der Poorten ([9]), mais les démonstrations n'ont pas été publiées semble-t-il à la connaissance de l'auteur.

Comme nous l'a fait remarquer le professeur A. Schinzel le théorème 2 de l'article de J. H. Evertse *On sums of S-units and linear recurrences*, Compositio Math. 53 (1984), 225–244, fournit dans le cas général une minoration de la forme (6), ce qui fait que le lemme 1 est en fait vrai sans l'hypothèse de la fréquence dominante.

4) Enfin remarquer que le résultat du théorème 2 peut se traduire comme une propriété de certains opérateurs fonctionnels; dans un autre cadre, nous avons fait une étude semblable dans [3].

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On the $(3N+1)$-conjecture*

by

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1. Introduction. We define for all odd positive integers b

$$C(b) = (3b+1)/2^{e(b)}$$

where $2^{e(b)}$ is the greatest power of 2 dividing $3b+1$. Obviously, $C(b)$ is odd again, thus we are able to define recursively

$$C^0(b) = b, \quad C^{k+1}(b) = C(C^k(b)) \quad \text{for } k \geqslant 0.$$

The following conjecture has been made.

$(3N+1)$-CONJECTURE. *For all $b \in N$, $2 \nmid b$, there is a $k \in N_0$ such that $C^k(b) = 1$.*

Several results concerning the conjecture are known, for references see [4], [5] and [7]. The problem itself is still open.

An odd positive integer b will be called *descending* if $C^k(b) = 1$ for some $k \in N_0$. In the first part of this paper we show that the conjecture is true, if one can prove for a certain "arbitrarily thin" congruence class that it contains only descending integers. In the second part we improve a lower bound of the number of descending integers below a given x by R. E. Crandall [2].

2. A sufficient condition. Let b be an odd positive integer. If b is descending, we define

$$S(b) = \min\{k \in N_0 \colon C^k(b) = 1\},$$

otherwise $S(b) = \infty$.

LEMMA 1. *For $b \in N$, $2 \nmid b$, we have $S(b) = S(4b+1)$.*

Proof.

$$C(4b+1) = \frac{3(4b+1)+1}{2^{e(4b+1)}} = \frac{3b+1}{2^{e(b)}} = C(b).$$

* Theorem 2 was first presented by the author at the Journées Arithmétiques 1987, Ulm, West Germany.

THEOREM 1. *Let $S(b) < \infty$ for some $m \in N$ and all $b \equiv (2^{2m}-1)/3 \mod 2^{2m-1}$. Then $S(b) < \infty$ for all $b \in N$, $2 \nmid b$.*

Proof. For odd positive b let $D(b) = 4b+1$ and

$$D^0(b) = b, \quad D^{n+1}(b) = D(D^n(b)) \quad \text{for } n \geqslant 0.$$

We claim

$$D^n(b) \equiv \sum_{j=0}^{n} 2^{2j} \mod 2^{2n+1}. \tag{1}$$

Obviously, the claim holds for $n = 0$ since $2 \nmid b$. We have

$$D^{n+1}(b) = 4D^n(b)+1 = 4\Big(\sum_{j=0}^{n} 2^{2j}+2^{2n+1}t\Big)+1$$

for some integer t by induction hypothesis. Thus

$$D^{n+1}(b) = \sum_{j=0}^{n+1} 2^{2j}+2^{2(n+1)+1}t \equiv \sum_{j=0}^{n+1} 2^{2j} \mod 2^{2(n+1)+1},$$

which proves the claim.

Since $\sum_{j=0}^{n} 2^{2j} = (2^{2n+2}-1)/3$, we get from (1)

$$D^n(b) \equiv (2^{2n+2}-1)/3 \mod 2^{2n+1}.$$

Let the condition of the theorem hold for some given $m \in N$. We choose $n = m-1$, so we have $S(D^{m-1}(b)) < \infty$. By applying Lemma 1 repeatedly we get $S(b) = S(D^{m-1}(b))$, and the theorem is proved.

3. Definitions and preliminaries. We define for positive integers a and rational r

$$B_a(r) = (2^a r-1)/3$$

and for positive integers $a_k, a_{k-1}, \ldots, a_1$

$$B_{a_k,\ldots,a_1}(r) = B_{a_k}(B_{a_{k-1},\ldots,a_1}(r)).$$

Moreover, let

$$B(a_k, \ldots, a_1) = B_{a_k,\ldots,a_1}(1).$$

$B(a_k, \ldots, a_1)$ is called a number with *height* k.

Let $G = \bigcup_{j=1}^{\infty} \{(a_j, \ldots, a_1) \colon a_i \in N,\ a_1 > 2,\ (2) \text{ holds}\}$, where

$$\begin{cases} 2^{a_1} \equiv 4,7 \mod 9, \\ 2^{a_i}B(a_{i-1}, \ldots, a_1) \equiv 4,7 \mod 9 \quad (2 \leqslant i \leqslant j-1), \\ 2^{a_j}B(a_{j-1}, \ldots, a_1) \equiv 1 \mod 3. \end{cases} \tag{2}$$

LEMMA 2. *$B(a_j, \ldots, a_1)$ is an integer (with height j) iff $(a_j, \ldots, a_1) \in G$.*

Proof. [2], Lemma (4.3).

LEMMA 3. *Let $M = \{b \in N\colon 2 \nmid b,\ S(b) < \infty\}$. Then for each $b \in M$, $b > 1$, there is a unique $(a_h, \ldots, a_1) \in G$ with $h = S(b)$ such that $b = B(a_h, \ldots, a_1)$, and conversely, for each $(a_j, \ldots, a_1) \in G$ we have $B(a_j, \ldots, a_1) \in M$.*

Proof. [2], Theorem (4.1).

An easy induction argument proves

LEMMA 4. $B(a_j, \ldots, a_1) < 2^{a_1+\ldots+a_j}/3^j$.

LEMMA 5. $\lim_{t\to\infty} e^{-t} \sum_{k=1}^{[t]} t^k/k! = \frac{1}{2}$.

Proof. We define

$$y_n(x) = e^{-x} \sum_{k=0}^{n-1} x^k/k!.$$

By [3] we have for any real α

$$y_n(n-\alpha) = \tfrac{1}{2}+O_\alpha(n^{-1/2}).$$

We set $n = [t]+1$, $\alpha = [t]+1-t$; obviously, $0 < \alpha \leqslant 1$. Hence

$$\lim_{t\to\infty} e^{-t} \sum_{k=1}^{[t]} \frac{t^k}{k!} = \lim_{t\to\infty} e^{-t} \sum_{k=0}^{[t]} \frac{t^k}{k!} = \lim_{n\to\infty} y_n(n-\alpha) = \tfrac{1}{2}.$$

It should be remarked that the proof given earlier by Ramanujan [6] is not satisfactory.

4. The main lemma. For positive integers h and z let

$$\varkappa_h(z) = \operatorname{card}\{(a_h, \ldots, a_1) \in G\colon a_1+\ldots+a_h \leqslant z\}.$$

Furthermore, let $c = \frac{1}{4}(13+\sqrt{153})$.

LEMMA 6. *Let h and z be positive integers satisfying $z \equiv h \mod 3$ and $z \geqslant ch$. Then*

$$\varkappa_h(z) \geqslant \frac{2}{5}\binom{(z-h)/3}{h}.$$

Proof. We will consider the following multiplication table mod 9:

(3)

a	2^a	1	2	4	5	7	8
0	1	1	2	$\underline{4}$	5	$\underline{7}$	8
1	2	2	$\underline{4}$	8	1	5	$\underline{7}$
2	4	$\underline{4}$	8	$\underline{7}$	2	1	5
3	8	8	$\underline{7}$	5	$\underline{4}$	2	1
4	7	$\underline{7}$	5	1	8	$\underline{4}$	2
5	5	5	1	2	$\underline{7}$	8	$\underline{4}$

For $k \in N$ let

$$I_k = \{6k-5,\, 6k-4,\, \ldots,\, 6k\};$$

also for $h \in N$ let

$$\tilde{G}_h = \{(a_h, \ldots, a_1):\ a_i \in N;\ 2^{a_1} \equiv 4, 7 \bmod 9;\ \text{for } j = 2, \ldots, h$$
$$2^{a_j} B(a_{j-1}, \ldots, a_1) \equiv 4, 7 \bmod 9\}.$$

We define the function $\varphi_h\colon \tilde{G}_h \to N_3^h$, where $N_3 = \{3k-1:\ k \in N\}$, in the following way: Let $(a_h, \ldots, a_1) \in \tilde{G}_h$; we write $\varphi_h(a_h, \ldots, a_1) = (b_h, \ldots, b_1)$. For j, $0 < j \leqslant h$, let $b_1, \ldots, b_{j-1}$ be defined already. For a_j there is a unique k with $a_j \in I_k$ (namely $k = \lceil a_j/6 \rceil$). In addition, for $(a_h, \ldots, a_1)$ there exist a unique $a_j' \in I_k$, $a_j' \neq a_j$, and some $a_{j+1}', \ldots, a_h'$ with $(a_h', \ldots, a_j', a_{j-1}, \ldots, a_1) \in \tilde{G}_h$, since we may observe in (3) that given a fixed $B \not\equiv 0 \bmod 3$, out of six consecutive integers $a^{(1)}, \ldots, a^{(6)}$ exactly two satisfy the congruence $2^{a^{(i)}} B \equiv 4, 7 \bmod 9$, and it is easily seen from the definition of $\tilde{G}_h$ that $B(a_i, \ldots, a_1) \not\equiv 0 \bmod 3$ for the occurring B. Now let

$$b_j = \begin{cases} 6k-4 & \text{for } a_j < a_j', \\ 6k-1 & \text{for } a_j > a_j'. \end{cases}$$

Obviously, φ_h is bijective. Using table (3) we find $|a_j - a_j'| \geqslant 2$, thus $|b_j - a_j| \leqslant 2$. This yields

$$\begin{aligned} \varkappa_h(z) &\geqslant \operatorname{card}\{(a_h, \ldots, a_1) \in G_h:\ a_1 > 2,\ a_1 + \ldots + a_h \leqslant z\} \\ &\geqslant \operatorname{card}\{(b_h, \ldots, b_1) \in N_3^h:\ b_1 > 2,\ (b_1+2) + \ldots + (b_h+2) \leqslant z\} \\ &= \operatorname{card}\{(b_h, \ldots, b_1) \in N_3^h:\ b_1 > 2,\ b_1 + \ldots + b_h \leqslant z - 2h\} \\ &= \operatorname{card}\{(k_h, \ldots, k_1) \in N^h:\ k_1 > 1,\ (3k_1 - 1) + \ldots + (3k_h - 1) \leqslant z - 2h\} \\ &= \operatorname{card}\{(k_h, \ldots, k_1) \in N^h:\ k_1 > 1,\ k_1 + \ldots + k_h \leqslant (z-h)/3\}. \end{aligned}$$

It is an elementary result in combinatorics (see [1]) that

$$\operatorname{card}\{(k_h, \ldots, k_1) \in N^h:\ k_1 + \ldots + k_h \leqslant t\} = \binom{t}{h}.$$

Hence

$$\begin{aligned} \operatorname{card}\{(k_h, \ldots, k_1) \in N^h:\ k_1 > 1,\ k_1 + \ldots + k_h \leqslant t\} &= \operatorname{card}\{(k_h, \ldots, k_1) \in N^h:\ k_1 + \ldots + k_h \leqslant t\} \\ &\quad - \operatorname{card}\{(k_h, \ldots, k_2) \in N^{h-1}:\ k_2 + \ldots + k_h \leqslant t-1\} \\ &= \binom{t}{h} - \binom{t-1}{h-1} \geqslant \frac{2}{5}\binom{t}{h} \quad \text{for } t \geqslant \frac{5}{3}h. \end{aligned}$$

This yields for $z \geqslant ch$ the desired result.

Corollary. *Let $h \in N$, $z \in R$ with $z \geqslant ch + 3$. Then there is $z' \in N$ satisfying $z - 3 < z' \leqslant z$, $z' \equiv h \bmod 3$, and*

$$\varkappa_h(z) \geqslant \frac{2}{5}\binom{(z'-h)/3}{h}.$$

Proof. For $z \in R$, $z \geqslant ch + 3$, we can find $z' \in N$ with $z - 3 < z' \leqslant z$ and $z' \equiv h \bmod 3$. Applying Lemma 6 we have

$$\varkappa_h(z) \geqslant \varkappa_h(z') \geqslant \frac{2}{5}\binom{(z'-h)/3}{h}.$$

5. The functions $\pi_h(x)$ and $\pi(x)$; the main theorem. We define for real x

$$\pi(x) = \operatorname{card}\{b \leqslant x:\ 2 \nmid b,\ b \text{ is descending}\};$$

$$\pi_h(x) = \operatorname{card}\{b \leqslant x:\ 2 \nmid b,\ b \text{ is descending with height } h\}.$$

Lemma 7. *For $\log_2 x \geqslant (c - \log_2 3)h + 3$, we have*

$$\pi_h(x) > \frac{2}{5h!}\log_2^h\left(x^{c_1(1 - 3(\log_2 x)^{-1})}\right),$$

where

$$c = \tfrac{1}{4}(13 + \sqrt{153}), \qquad c_1 = (c - \log_2 3)^{-1}.$$

Proof. Combining Lemmas 2–4, we get

$$\pi_h(x) \geqslant \operatorname{card}\{(a_h, \ldots, a_1) \in G:\ a_1 + \ldots + a_h \leqslant \log_2(3^h x)\} = \varkappa_h(\log_2(3^h x)).$$

By assumption $\log_2 x \geqslant (c - \log_2 3)h + 3$, therefore $\log_2(3^h x) \geqslant ch + 3$. The Corollary guarantees the existence of an $x' \in N$, $x' \equiv h \bmod 3$, with

(4) $$\log_2(3^h x) - 3 < x' \leqslant \log_2(3^h x),$$

(5) $$\pi_h(x) \geqslant \frac{2}{5}\binom{(x'-h)/3}{h}.$$

Let z be an integer with $z > h > 0$. We have

$$\log\left(\frac{z!}{(z-h)!}\right) = \sum_{j=0}^{h-1} \log(z-j) > \int_{z-h}^{z} \log t\,dt \tag{6}$$

$$= z\log z - (z-h)\log(z-h) - h.$$

Since

$$\log(z-h) = \log z + \log\left(1-\frac{h}{z}\right) < \log z - \frac{h}{z} - \frac{1}{2}\left(\frac{h}{z}\right)^2,$$

we get from (6)

$$\log\left(\frac{z!}{(z-h)!}\right) > h\left(\log z - \frac{1}{2}\left(1+\frac{h}{z}\right)\frac{h}{z}\right). \tag{7}$$

Setting

$$z = (x'-h)/3, \tag{8}$$

and by (4), we have $z > \frac{c-1}{3}h$ (*a fortiori* $z > h$). Hence, applying (7),

$$\log\left(\frac{z!}{(z-h)!}\right) > h\left(\log z - \frac{1}{2}\left(1+\frac{3}{c-1}\right)\frac{h}{z}\right) = h\left(\log z - c_2\frac{h}{z}\right), \tag{9}$$

where $c_2 = (c+2)/(2c-2)$. The Taylor expansion of $\log(1-h/z)$ gives

$$\frac{h}{z} < -\log\left(1-\frac{h}{z}\right),$$

which implies by (9)

$$\log\left(\frac{z!}{(z-h)!}\right) > h\left(\log z - c_2\log\left(1-\frac{h}{z}\right)\right) = h\log\left(z\left(1-\frac{h}{z}\right)^{c_2}\right). \tag{10}$$

By Bernoulli's inequality, we have

$$\left(1-\frac{h}{z}\right)^{c_2} > 1 - c_2\frac{h}{z}.$$

Applying this to (10), we get

$$\log\left(\frac{z!}{(z-h)!}\right) > h\log(z - c_2 h).$$

This yields by (8)

$$h!\binom{(x'-h)/3}{h} > \left(\frac{x'-h}{3} - c_2 h\right)^h. \tag{11}$$

By (4), the assumption of the lemma and the definition of c,

$$3\left(\frac{x'-h}{3} - c_2 h\right) \geqslant \log_2(3^h x) - 3 - (3c_2+1)h \tag{12}$$

$$= \log_2 x - h(3c_2 + 1 - \log_2 3) - 3$$

$$\geqslant (\log_2 x - 3)(1 - c_1(3c_2 + 1 - \log_2 3))$$

$$= 3c_1\log_2 x - 9c_1.$$

Joining (5), (11) and (12), we finally get

$$\pi_h(x) > \frac{2}{5h!}(c_1\log_2 x - 3c_1)^h$$

$$= \frac{2}{5h!}(c_1\log_2 x - 3c_1\log_2(x^{(\log_2 x)^{-1}}))^h$$

$$= \frac{2}{5h!}(\log_2(x^{c_1 - 3c_1(\log_2 x)^{-1}}))^h,$$

which proves the lemma.

In [2] R. E. Crandall proves that there is a constant x_0 such that $\pi(x) > x^\gamma$ for some $\gamma > 0$ and $x > x_0$. Explicit calculations in Crandall's proof show that you can choose $\gamma \approx 0.057$.

THEOREM 2. *There is an absolute constant x_0 such that for $x > x_0$*

$$\pi(x) > x^{3/10}.$$

Proof. Let $0 < \varepsilon < 1$. By Lemma 5, we have for all $x > x_0'$

$$\sum_{h=1}^{[(1-\varepsilon)c_1\log_2 x]} \frac{1}{h!}\log_2^h(x^{(1-\varepsilon)c_1}) > \frac{1}{4}e^{\log_2(x^{(1-\varepsilon)c_1})}, \tag{13}$$

where x_0' is some absolute constant. Lemma 7 yields

$$\pi(x) = \sum_{h=1}^{\infty} \pi_h(x) > \frac{2}{5}\sum_{h=1}^{[c_1(\log_2 x - 3)]} \frac{1}{h!}\log_2^h(x^{c_1(1-3(\log_2 x)^{-1})}). \tag{14}$$

Choose $x_0'' = \max(x_0', 2^{3/\varepsilon})$. Then for $x > x_0''$

$$c_1(\log_2 x - 3) \geqslant (1-\varepsilon)c_1\log_2 x$$

and

$$c_1(1 - 3(\log_2 x)^{-1}) \geqslant c_1(1-\varepsilon).$$

Thus by (14) and (13)

$$\pi(x) > \frac{2}{5}\sum_{h=1}^{[(1-\varepsilon)c_1\log_2 x]}\frac{1}{h!}\log_2^h(x^{c_1(1-\varepsilon)}) > \tfrac{1}{10}e^{(1-\varepsilon)c_1\log_2 x} = \tfrac{1}{10}x^{(1-\varepsilon)c_1(\log 2)^{-1}}$$

for $x > x_0''$. By choosing $\varepsilon = 1/100$ the proof is completed.

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The average order of $d_k(n)$ over integers free of large prime factors

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1. Introduction and statement of results. Let us define $p(n)$ as the largest prime factor of $n \geqslant 2$, $p(1) = 1$, and let

$$\Psi_f(x, y) = \sum_{n \leqslant x, p(n) \leqslant y} f(n),$$

where f is an additive or a multiplicative function. Recently, in [1], [2], Alladi established asymptotic formulas for $\Psi_f(x, y)$ when $f = \mu$ and $f = \omega$, respectively, where $\mu(n)$ is the Möbius function and $\omega(n)$ denotes the number of distinct prime factors of n. In [9], [10], Ivić established asymptotic formulas for $\Psi_f(x, y)$ when $f = \mu^2$ and $f = \Omega - \omega$, respectively, where $\Omega(n)$ denotes the total number of prime factors of n.

In [11], we estimate $\Psi_f(x, y)$ for $f = d$, where $d(n)$ denotes the divisor function.

The purpose of this note is to estimate $\Psi_f(x, y)$ for $f = d_k$, where $d_k(n)$ denotes the number of ways n can be written as a product of k factors, in particular $d_2(n) = d(n)$.

Let $\Psi(x, y)$ denote the number of positive integers not exceeding x, all of whose prime factors do not exceed y. Let the "Dickman function" ϱ be defined for $u \geqslant 0$ as the continuous solution of the equations

$$\begin{cases} \varrho(u) = 1, & 0 \leqslant u \leqslant 1, \\ u\varrho'(u) = -\varrho(u-1), & u > 1. \end{cases} \tag{1.1}$$

In 1951, Hua [8] established the asymptotic relation

$$\varrho(u) = \exp\left\{-u\left(\log u + \log_2 u - 1 + \frac{\log_2 u}{\log u} + O\left(\frac{1}{\log u}\right)\right)\right\}, \tag{1.2}$$

where $\log_2 u = \log\log u$. (1.2) is also due to de Bruijn [3], who actually proves a stronger result.

In the sequel, we write systematically $u = \log x/\log y$. The following is proved in [11]: