[16] R. Sita Rama Chandra Rao and G. Sri Rama Chandra Murty, *On Mirsky's generalisation of a problem of Evelyn-Linfoot and Page in additive theory of numbers*, J. Reine Angew. Math. 309 (1979), 92–99.
[17] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1966.

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On the greatest prime factor of $\prod_{k=1}^{x} f(k)$

by

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In memory of Trygve Nagell

Let $P(n)$ denote the greatest prime factor of n. T. Nagell was the first to give a non-trivial lower bound for $P(\prod_{k=1}^{x} f(k))$, where f is an arbitrary irreducible polynomial of degree greater than 1. In [5] he proved

$$P(\prod_{k=1}^{x} f(k)) > c(f, \varepsilon)\, x(\log x)^{1-\varepsilon} \quad \text{for all } \varepsilon > 0.$$

In 1951 the first named author improved considerably the above inequality by proving that for $x > x_0(f)$

$$P(\prod_{k=1}^{x} f(k)) > x(\log x)^{c(f)\log\log\log x} \quad \text{with } c(f) > 0. \tag{1}$$

In the same paper [1] he has also claimed that

$$P(\prod_{k=1}^{x} f(k)) > x\exp((\log x)^{\delta(f)}) \quad \text{with } \delta(f) > 0. \tag{2}$$

Our efforts to reconstruct the proof of the latter estimate have been unsuccessful. Instead we have proved the following

Theorem 1. *Let $f \in \mathbf{Z}[x]$ be an irreducible polynomial of degree $l > 1$. There exists an absolute constant $c_1 > 0$ such that for $x > x_1(f)$*

$$P(\prod_{k=1}^{x} f(k)) > x\exp\exp(c_1(\log\log x)^{1/3}).$$

In the sequel we shall denote the nth iterate of $\log x$ by $\log_n x$, the number of solutions of the congruence $f(k) \equiv 0 \pmod m$ in the interval $1 \leqslant k \leqslant x$ by

$\varrho_x(m)$, the number of divisors of an integer n in a set S by $d(n, S)$ and we shall put:

$$\varrho_m(m) = \varrho(m).$$

$c_1, c_2, \ldots, x_1, x_2, \ldots$ will denote positive constants, in general depending on f, p will denote primes.

Theorem 1 is an immediate consequence of the following two theorems.

THEOREM 2. *Under the assumptions of Theorem 1 the number $N(x)$ of positive integers $k \leqslant x$ such that*

$$d\left(f(k), \left[\frac{x}{2}, x\right]\right) \geqslant 1$$

satisfies for $x > x_2$

$$N(x) > \frac{x}{\log x}\exp\left(c_2(\log_2 x)^{1/3}\right),$$

where c_2 is an absolute constant.

THEOREM 3. *Under the assumptions of Theorems 1 and 2*

$$P\left(\prod_{k=1}^{x} f(k)\right) > x\exp\left(\frac{\log x}{lx}N(x)\right)$$

for $x > x_3$.

The proof of Theorem 3 follows closely the proof of (1) given in [1]. It is clear from this theorem that in order to prove (2) it would be enough to show that

$$N(x) > \frac{x}{(\log x)^{1-\delta(f)}} \quad \text{for } x > x_4. \tag{2a}$$

In this connection it is interesting to note that G. Tenenbaum [7] has obtained the asymptotic equality

$$H_f(x, y, 2y) = x/(\log y)^{1-\delta+o(1)}$$

where the left-hand side is the number of positive integers $k \leqslant x$ such that $d(f(k), [y, 2y]) \geqslant 1$; x, y tend to infinity in the domain $y \leqslant x^{c_0}$ $(c_0 < 1)$ and

$$\delta = \frac{1+\log\log 2}{\log 2}.$$

Note added on April 27, 1989. Recently G. Tenenbaum has established by a different method the inequality (2a), which implies (2) via Theorem 3.

For the proof of Theorem 2 we require four lemmata.

LEMMA 1. *If $x \geqslant m$ we have*

$$2\frac{x}{m}\varrho(m) \geqslant \varrho_x(m) \geqslant \frac{1}{2}\frac{x}{m}\varrho(m).$$

Proof. We have for $x \geqslant m$

$$\frac{x}{2m}\varrho(m) \leqslant \left[\frac{x}{m}\right]\varrho(m) \leqslant \varrho_x(m) \leqslant \left[\frac{x}{m}\right]\varrho(m)+\varrho(m) \leqslant \frac{2x}{m}\varrho(m).$$

LEMMA 2. *If $z \geqslant 2y$, $y > y_1$ we have*

$$2\log\frac{\log z}{\log y} \geqslant \sum_{y \leqslant p \leqslant z}\frac{\varrho(p)}{p} > \frac{1}{2}\log\frac{\log z}{\log y}.$$

Proof. We shall use the prime ideal theorem in the form

$$\sum_{p \leqslant y}\varrho(p)\log p = y + O(ye^{-c_3\sqrt{\log y}})$$

(see [4], Satz 190).

By partial summation we obtain

$$\sum_{p \leqslant y}\frac{\varrho(p)}{p} = \log\log y + c_4 + O(e^{-c_3\sqrt{\log y}}),$$

hence

$$\sum_{y \leqslant p \leqslant z}\frac{\varrho(p)}{p} = \log\frac{\log z}{\log y} + O(e^{-c_3\sqrt{\log y}})$$

and since for $z \geqslant 2y$ the main term dominates the error we get the desired bounds.

LEMMA 3. *Assume that f is primitive. If P runs through all integers composed of n distinct prime factors we have for $y \geqslant y_2$*

$$\sum_{y/4 < P \leqslant y}\frac{\varrho(P)}{P} < \frac{c_5\, l^n(\log_2 y + c_6)^{n-1}}{(n-1)!\log y}.$$

Proof. Since $\varrho(m)$ is multiplicative, we have $\varrho(P) \leqslant l^n$. On the other hand, for the number $\pi_n(x)$ of positive integers $\leqslant x$ composed of n distinct prime factors we have the inequality (see [3])

$$\pi_n(x) \leqslant \frac{c_7\, x(\log_2 x + c_6)^{n-1}}{(n-1)!\,\log x}.$$

Hence

$$\sum_{y/4 < P \leqslant y}\frac{1}{P} < \frac{4}{y}\pi_n(y) \leqslant \frac{c_5(\log_2 y + c_6)^{n-1}}{(n-1)!\,\log y}.$$

Remark. The formulation of Lemma 3 and its proof have been corrected following a suggestion from G. Tenenbaum.

LEMMA 4. *Let* $c > 0$, $r = [c(\log_2 x)^{1/3}]$, $A(c)$ *be the set of all integers in the interval* $[x/2, x]$ *of the form*

$$pq_1 \cdots q_r,$$

where $p, q_1, \ldots, q_r$ *are primes and for* $i = 1, 2, \ldots, r$

$$\exp\left(\frac{1}{2(2r+7)}(\log x)^{1-i/2r}\right) < q_i < \exp\left(\frac{1}{2(2r+7)}(\log x)^{1-(i-1)/2r}\right). \tag{3}$$

The number $N_0(x)$ *of positive integers* $k \leqslant x$ *such that*

$$d(f(k), A(c)) > 2lr!(2r+7)^{r+1} \tag{4}$$

is $o(x/(\log x)^{r+2})$.

Proof. We shall assume throughout that x is sufficiently large and without loss of generality that f is primitive. Then if $pq_1 \cdots q_r \in [x/2, x]$ and q_i satisfy the inequalities (3) we have $p > x^{1/2}$. On the other hand for $k \leqslant x$

$$|f(k)| < c_8 x^l, \tag{5}$$

hence $f(k)$ can have at most $2l$ prime factors greater than $x^{1/2}$. Therefore, (4) implies that $f(k)$ has more than

$$R = r!(2r+7)^{r+1}$$

divisors in $A(c)$, of the form $pq_1^{(\sigma)} \cdots q_r^{(\sigma)}$, where p is fixed.

Consider the family of sets $\{q_1^{(\sigma)}, \ldots, q_r^{(\sigma)}\}$ $(1 \leqslant \sigma \leqslant R)$. By the theorem of Erdős and Rádo [2] the family contains a Δ-system of cardinality $2r+7$. Let the common intersection of any two distinct sets of this Δ-system be $\{p_1, \ldots, p_\delta\}$, where $0 \leqslant \delta < r$. Let s be the integer defined by

$$2^{s-1} < \frac{x}{pp_1 \cdots p_\delta} \leqslant 2^s.$$

By the condition $pq_1^{(\sigma)} \cdots q_r^{(\sigma)} \in A(c)$ $f(k)$ has at least $2r+7$ pairwise coprime divisors in the interval $(2^{s-2}, 2^s]$, each divisor consisting of $r-\delta$ distinct prime factors all in the interval

$$\left(\exp\left(\frac{1}{2(2r+7)}(\log x)^{1/2}\right), \exp\left(\frac{1}{2(2r+7)}(\log x)\right)\right)$$

and all but one less than

$$\exp\left(\frac{1}{2(2r+7)}(\log x)^{1-1/2r}\right).$$

Hence

$$s \geqslant \frac{(\log x)^{1/2}}{2(2r+7)\log 2} = s_0$$

and

$$N_0(x) \leqslant \sum_{s=0}^{r-1} \sum_{s \geqslant s_0} \sum{}^* \varrho_x(P_1 P_2 \cdots P_{2r+7}), \tag{6}$$

where the sum $\sum^*$ is taken over all sets of $2r+7$ pairwise coprime integers $P_1, P_2, \ldots, P_{2r+7}$ in the interval $(2^{s-2}, 2^s]$, each integer consisting of $r-\delta$ distinct prime factors of the size described above. For every such set we have

$$P_1 \cdots P_{2r+7} < x^{1/2} \exp\left(\frac{r-\delta-1}{2}(\log x)^{1-1/2r}\right) < x,$$

thus by Lemma 1

$$\varrho_x(P_1 \cdots P_{2r+7}) < 2\frac{\varrho(P_1 \cdots P_{2r+7})}{P_1 \cdots P_{2r+7}} x = 2x \prod_{\nu=1}^{2r+7} \frac{\varrho(P_\nu)}{P_\nu}$$

and by (6)

$$N_0(x) \leqslant \sum_{\delta=0}^{r-1} \sum_{s \geqslant s_0} \sum{}^* 2x \prod_{\nu=1}^{2r+7} \frac{\varrho(P_\nu)}{P_\nu} \leqslant 2x \sum_{\delta=0}^{r-1} \sum_{s \geqslant s_0} \left(\sum \frac{\varrho(P)}{P}\right)^{2r+7},$$

where P runs through all integers P in the interval $(2^{s-2}, 2^s]$ consisting of $r-\delta$ distinct prime factors. By Lemma 3 we obtain

$$\sum \frac{\varrho(P)}{P} \leqslant \frac{c_5 l^{r-\delta}(\log\log 2^s + c_6)^{r-\delta-1}}{(r-\delta-1)!\log 2^s} \leqslant \frac{c_9 l^{r-\delta}(\log s)^{r-\delta-1}}{(r-\delta-1)!\, s},$$

hence

$$N_0(x) \leqslant 2x \sum_{\delta=0}^{r-1} \left(\frac{c_9 l^{r-\delta}}{(r-\delta-1)!}\right)^{2r+7} \sum_{s \geqslant s_0} \frac{(\log s)^{(r-1)(2r+7)}}{s^{2r+7}}. \tag{7}$$

For $s > s_0 - 1$ we have

$$\log s > \log(s_0 - 1) > \frac{2r+7}{2r+5}(r-1) > r-1.$$

Therefore, on this halfline $(\log s)^{(r-1)(2r+7)}/s^{2r+7}$ is decreasing, since $\log s > r-1$ and

$$\frac{(\log s)^{(r-1)(2r+7)}}{s^{2r+7}} < \frac{d}{ds}\left(-\frac{(\log s)^{(r-1)(2r+7)}}{s^{2r+6}}\right),$$

since

$$\log s > \frac{2r+7}{2r+5}(r-1).$$

It follows that

$$\sum_{s\geqslant s_0}\frac{(\log s)^{(r-1)(2r+7)}}{s^{2r+7}}\leqslant\int_{s_0-1}^{\infty}\frac{(\log s)^{(r-1)(2r+7)}}{s^{2r+7}}ds$$
$$\leqslant\frac{(\log(s_0-1))^{(r-1)(2r+7)}}{(s_0-1)^{2r+6}}=\frac{\exp O(r^2\log_3 x)}{(\log x)^{r+3}}$$

and by (7)

$$N_0(x)\leqslant 2x(c_9 le^l)^{2r+7}\frac{\exp O(r^2\log_3 x)}{(\log x)^{r+3}}=o\left(\frac{x}{(\log x)^{r+2}}\right).$$

Proof of Theorem 2. For $k\leqslant x$ by (5) $f(k)$ has less than $c_{10}\log x$ prime factors. Thus we have in the notation of Lemma 4

(8) $$d\big(f(k),A(c)\big)<\binom{c_{10}\log x}{r+1}<\frac{c_{10}^{r+1}}{(r+1)!}(\log x)^{r+1}.$$

From Lemma 4 and (8) we obtain

(9) $$\sum\nolimits^{+} d\big(f(k),A(c)\big)=o\left(\frac{x}{\log x}\right),$$

where in $\sum^{+}$ k runs through all positive integers $k\leqslant x$ with

$$d\big(f(k),A(c)\big)>2lr!(2r+7)^{r+1}.$$

On the other hand, by Lemma 1

(10) $$\sum_{k=1}^{x} d\big(f(k),A(c)\big)=\sum_{a\in A(c)}\varrho_x(a)\geqslant\frac{x}{2}\sum_{a\in A(c)}\frac{\varrho(a)}{a}.$$

We evidently have

$$\sum_{a\in A(c)}\frac{\varrho(a)}{a}=\sum\nolimits_1\frac{\varrho(q_1)}{q_1}\sum\nolimits_2\frac{\varrho(q_2)}{q_2}\cdots\sum\nolimits_r\frac{\varrho(q_r)}{q_r}\sum\nolimits_{r+1}\frac{\varrho(p)}{p},$$

where the sum $\sum_i$ is taken over all primes q_i in the interval (3) $(1\leqslant i\leqslant r)$ and the sum $\sum_{r+1}$ is taken over all primes p in the interval

$$\frac{x}{2q_1\cdots q_r}\leqslant p\leqslant\frac{x}{q_1\cdots q_r}.$$

It follows from Lemma 2 that

$$\sum_i\frac{\varrho(q_i)}{q_i}>\frac{1}{4r}\log_2 x\quad(1\leqslant i\leqslant r),$$
$$\sum\nolimits_{r+1}\frac{\varrho(p)}{p}>\frac{1}{2}\log\left(1+\frac{\log 2}{\log(x/2q_1\cdots q_r)}\right)>\frac{\log 2}{2\log x}.$$

Therefore,

$$\sum_{a\in A(c)}\frac{\varrho(a)}{a}>\left(\frac{\log_2 x}{4r}\right)^r\frac{\log 2}{2\log x}$$

and by (10)

$$\sum_{k=1}^{x} d\big(f(k),A(c)\big)>\frac{\log 2}{4}\frac{x}{\log x}\left(\frac{\log_2 x}{4r}\right)^r.$$

Since $r=o(\log_2 x)$, it follows from (9) that

(11) $$\sum\nolimits^{-} d\big(f(k),A(c)\big)>\frac{x}{6\log x}\left(\frac{\log_2 x}{4r}\right)^r,$$

where $\sum^{-}$ is taken over all positive integers $k\leqslant x$ such that

$$d\big(f(k),A(c)\big)\leqslant 2lr!(2r+7)^{r+1}.$$

From (11) we obtain

$$N(x)>\frac{1}{12lr!(2r+7)^{r+1}}\frac{x}{\log x}\left(\frac{\log_2 x}{4r}\right)^r$$
$$>\frac{x}{\log x}\exp\left(r\left(\log_3 x-3\log r+1-\log 8+O\left(\frac{\log r}{r}\right)\right)\right)$$
$$>\frac{x}{\log x}\exp\big(c(\log_2 x)^{1/3}(-3\log c+1-\log 8)+O(\log_3 x)\big).$$

Choosing $c<\sqrt[3]{e/8}$ (the choice $c=\sqrt[3]{1/(8e^2)}$ is optimal) we obtain the theorem.

Remark. If instead of the theorem of Erdős and Rádo we use their conjecture $r!(2r+7)^{r+1}$ is replaced throughout by $(2r+7)^{r+1}$ and the above proof for $r=\left[c'\left(\frac{\log_2 x}{\log_3 x}\right)^{1/2}\right]$ gives

$$N(x)>\frac{x}{\log x}\exp\left(c_{11}\left(\frac{\log_2 x}{\log_3 x}\right)^{1/2}\log_4 x\right)\quad\text{for } x>x_4,$$

where c_{11} is an absolute constant.

We proceed to the proof of Theorem 3. Denote by U the set of all integers u of the interval $(x/\log x,x]$ for which $f(u)$ has no prime factor satisfying

$$x<p\leqslant c_{12}x,\quad\text{where } c_{12}^{l-1}=2c_8.$$

LEMMA 5. $\operatorname{card} U>x-c_{13}\dfrac{x}{\log x}$.

Proof. Clearly

$$\operatorname{card} U = [x] - \left[\frac{x}{\log x}\right] - \sum_{x<p\leqslant c_{12}x} \left(\varrho_x(p) - \varrho_{\frac{x}{\log x}}(p)\right)$$

$$> x - \frac{x}{\log x} - 1 - l\pi(c_{12}x) > x - c_{13}\frac{x}{\log x}.$$

For $k \leqslant x$ put

(12) $$|f(k)| = A_k B_k, \quad \text{where } A_k = \prod_{\substack{p^\alpha \| f(k) \\ p\leqslant x}} p^\alpha,\ B_k = |f(k)|/A_k$$

and let

(13) $$P\Big(\prod_{k=1}^{x} f(k)\Big) = P_x.$$

Lemma 6. *For all* $u\in U$

$$A_k > \frac{x^l}{2(\log x)^l P_x^{l-1}}.$$

Proof. Since by the definition of U: $x/\log x < u \leqslant x$ we have for $x > x_5$

(14) $$\frac{1}{2}\left(\frac{x}{\log x}\right)^l < |f(u)| < c_8 x^l.$$

Further, $f(u)$ has no prime factor in the interval $(x, c_{12}x]$. Therefore by (12) and the choice of c_{12} B_u can have at most $l-1$ prime factors, multiple factors counted multiply. By (13) all prime factors of $f(u)$ are at most P_x, thus

$$B_u \leqslant P_x^{l-1}.$$

Hence

$$A_u = \frac{|f(u)|}{B_u} > \frac{x^l}{2(\log x)^l P_x^{l-1}}.$$

Lemma 7. *Let* $u\in U$ *be such that* $f(u)$ *has a divisor in* $[x/2, x]$. *Then*

$$A_u > \frac{x^l}{2(\log x)^l P_x^{l-2}}.$$

Proof. By the definition of U all prime factors of B_u are greater than $c_{12}x$. Since $f(u) \equiv 0 \pmod d$ for some $d\in[x/2, x]$ we have by (12), (14) and the choice of c_{12}

$$B_u < 2c_8 x^{l-1} = (c_{12}x)^{l-1}.$$

Thus B_u can have at most $l-2$ prime factors, multiple factors counted multiply. Thus by (12) and (13)

$$A_u = \frac{|f(u)|}{B_u} > \frac{x^l}{2(\log x)^l P_x^{l-2}}.$$

Lemma 8.

$$\sum_{k=1}^{x} \log A_k < x\log x + c_{14}x.$$

Proof, see Nagell [6], pp. 180–182.

Proof of Theorem 3. The number of $u\in U$ for which $f(u)$ has a divisor in $[x/2, x]$ is at least equal to $N(x) - (x - \operatorname{card} U)$, hence by Lemma 5 is at least $N(x) - c_9 x/\log x$. From Lemmata 5, 6, 7, and (8) we now obtain

$$x\log x + c_{14}x$$

$$\geqslant \sum_{u\in U} \log A_u > \left(x - c_{13}\frac{x}{\log x}\right)(l\log x - l\log_2 x - (l-1)\log P_x - \log 2)$$

$$+ \left(N(x) - c_{13}\frac{x}{\log x}\right)\log P_x$$

$$> lx\log x - lx\log_2 x - (l-1)x\log P_x - x - c_{13}lx$$

$$+ c_{13}(l-1)\frac{x}{\log x}\log P_x + N(x)\log P_x - c_{13}\frac{x}{\log x}\log P_x$$

$$> lx\log x - lx\log_2 x - (l-1)x\log P_x - (c_{13}l+1)x + N(x)\log P_x.$$

Hence

(15) $$(l-1)x\log\frac{P_x}{x} > N(x)\log P_x - lx\log_2 x - (c_{13}l + c_{14} + 1)x.$$

By Lemma 2 for $x > x_6$ there is at least one prime $p\in[x/2, x]$ with $\varrho(p) > 0$, hence $P_x \geqslant x/2$. On the other hand, by Theorem 1 $x\log_2 x = o(N(x)\log x)$. Thus for $x > x_7$

$$lx\log_2 x + (c_{13}l + c_{14} + 1)x < \frac{1}{l}N(x)\log x - N(x)\log 2$$

and the inequality (15) gives

$$(l-1)x\log\frac{P_x}{x} > \frac{l-1}{l}N(x)\log x,$$

i.e.

$$P_x > x\exp\left(\frac{\log x}{lx}N(x)\right),$$

which was to be proved.

References

[1] P. Erdős, *On the greatest prime factor of* $\prod_{k=1}^{x} f(k)$, J. London Math. Soc. 27 (1952), 379–384.
[2] P. Erdős and R. Rádo, *Intersection theorems for systems of sets*, ibid. 35 (1960), 85–90.
[3] G. H. Hardy and S. Ramanujan, *The normal number of prime factors of a number n*, Quarterly J. Math. 48 (1917), 76–92.
[4] E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, Chelsea 1949.
[5] T. Nagell, *Généralisation d'un théorème de Tchebycheff*, J. Math. Pur. Appl. (8) 4 (1921), 343–356.
[6] — *Zur Arithmetik der Polynome*, Abh. Math. Seminar Univ. Hamburg 1 (1922), 179–194.
[7] G. Tenenbaum, *Sur une question d'Erdős et Schinzel*, the volume: *A tribute to Paul Erdős* to be published by the Cambridge University Press.

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