Remarks on the upper bound for $L(1, \chi)$

by

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1. Introduction. Let $D$ be a positive integer, $\chi$ a real non-principal character modulo $D$ and $L(s, \chi)$ the associated $L$-series. Several authors, such as Pólya [5], Chowla [3], Burgess [2], Stephens [6] and Pintz [4], have studied the upper bound for $L(1, \chi)$. The best known estimate is

$$(1) \quad L(1, \chi) \leq \frac{1}{2} \left( 1 - \frac{1}{\sqrt{D}} + o(1) \right) \log D$$

if $\chi$ is primitive. This inequality was first established by Stephens [6] when $D$ is a prime. Later, Pintz [4] proved the inequality (1) when $D$ is not necessarily prime. The aim of this note is to show the following theorem which enables us to improve (1) under certain conditions.

**Theorem.** Assume that $D$ is cube-free. Then for any $\epsilon > 0$, we have

$$L(1, \chi) \leq \frac{1}{4} \frac{L(1 + \epsilon, \chi)}{\zeta(1 + \epsilon)} + o(1) \log D$$

if $D > D_0(\epsilon)$, where $\zeta(s)$ denotes the Riemann zeta function and $D_0(\epsilon)$ is a constant depending only on $\epsilon$.

From this we can easily deduce the following corollaries.

**Corollary 1.** Let $D$ be as in the theorem. Then we have

$$L(1, \chi) \leq \begin{cases} 
\frac{1}{12} + o(1) \log D & \text{if } \chi(2) = -1, \\
\frac{1}{8} + o(1) \log D & \text{if } \chi(3) = -1, \\
\frac{1}{6} + o(1) \log D & \text{if } \chi(5) = -1,
\end{cases}$$

as $D \to \infty$. 
Corollary 2. Let $D$ be as in the theorem. Then we have

$$L(1, \chi) \leq \begin{cases} 
\frac{1}{24} + o(1) \log D & \text{if } \chi(2) = \chi(3) = -1, \\
\frac{1}{18} + o(1) \log D & \text{if } \chi(2) = \chi(5) = -1, \\
\frac{1}{12} + o(1) \log D & \text{if } \chi(3) = \chi(5) = -1, \\
\frac{1}{36} + o(1) \log D & \text{if } \chi(2) = \chi(3) = \chi(5) = -1. 
\end{cases}$$

as $D \to \infty$.

These results are improvements of (1) under certain conditions. Combining these corollaries with Dirichlet's class number formulae obviously gives bounds for the class number and the fundamental unit of a quadratic field.

2. Lemmas. Let $\chi$ be a real non-principal character modulo $D$ and let $\varepsilon > 0$. For any positive integer $n$, we put

$$a_\varepsilon(n) = \frac{1}{n^\varepsilon} \sum_{d | n} \chi(d)d^\varepsilon$$

and

$$b_\varepsilon(n) = \sum_{d | n} \frac{\chi(d)}{d^\varepsilon}.$$ 

Then it is easy to see that $b_\varepsilon(n) > 0$ for all $n$.

Lemma 1. For all $n$, $a_\varepsilon(n) \leq b_\varepsilon(n)$.

Proof. First, we assume that $\chi(n) \neq 0$. Since $\chi(d) = \chi(n) \chi(n/d)$ for any divisor $d$ of $n$, we get

$$a_\varepsilon(n) = \frac{\chi(n)}{n^\varepsilon} \sum_{d | n} \left( \frac{n}{d} \right)^\varepsilon = \chi(n)b_\varepsilon(n) \leq b_\varepsilon(n),$$

because $b_\varepsilon(n) > 0$.

Next, we assume that $\chi(n) = 0$. Let $n_0$ be the maximal divisor of $n$ such that $(n_0, D) = 1$. By using the same way as above, we obtain

$$a_\varepsilon(n) = \frac{1}{n^\varepsilon} \sum_{d | n_0} \chi(d)d^\varepsilon = \frac{\chi(n_0)}{n_0^\varepsilon} \sum_{d | n_0} \left( \frac{n_0}{d} \right)^\varepsilon \leq \frac{\chi(n_0)}{(n/n_0)^\varepsilon} \sum_{d | n_0} \left( \frac{n_0}{d} \right)^\varepsilon \leq b_\varepsilon(n),$$

because $b_\varepsilon(n) > 0$ and $n/n_0 \geq 1$. This completes the proof of the lemma.

Lemma 2. For $x \geq 1$ and any $\varepsilon > 0$, we have

$$\sum_{n \leq x} \frac{a_\varepsilon(n)}{n} = \zeta(1+\varepsilon) \sum_{n \leq x} \frac{\chi(n)}{n} + O(1)$$

where the $O$-constant depends only on $\varepsilon$.

Proof. By a familiar argument, we have

$$\sum_{n \leq x} \frac{\chi(n)}{n} = \sum_{n \leq x} \frac{\chi(n)}{n} \sum_{m \leq n^{1+\varepsilon}} \frac{1}{m^{1+\varepsilon}} = \sum_{n \leq x} \frac{\chi(n)}{n} \left( \zeta(1+\varepsilon) + O\left( \frac{n^{\varepsilon}}{x^{1+\varepsilon}} \right) \right)$$

$$= \zeta(1+\varepsilon) \sum_{n \leq x} \frac{\chi(n)}{n} + O\left( \frac{1}{x} \sum_{n \leq x} \frac{1}{n^{1+\varepsilon}} \right) = \zeta(1+\varepsilon) \sum_{n \leq x} \frac{\chi(n)}{n} + O(1),$$

as required.

Lemma 3. For $x \geq 1$ and any $\varepsilon > 0$, we have

$$\sum_{n \leq x} \frac{b_\varepsilon(n)}{n} = L(1+\varepsilon, \chi) \log x + O\left( \frac{\log x}{x^\varepsilon} \right) + O(1),$$

where the $O$-constants depend only on $\varepsilon$.

Proof. By a familiar argument, we get

$$\sum_{n \leq x} \frac{b_\varepsilon(n)}{n} = \sum_{n \leq x} \frac{\chi(n)}{n^{1+\varepsilon}} \sum_{m \leq n^{1+\varepsilon}} \frac{1}{m} = \sum_{n \leq x} \frac{\chi(n)}{n^{1+\varepsilon}} \left( \log \frac{x}{n} + O(1) \right)$$

$$= (\log x) \sum_{n \leq x} \frac{\chi(n)}{n^{1+\varepsilon}} + O\left( \sum_{n \leq x} \frac{\log n}{n^{1+\varepsilon}} \right) + O\left( \sum_{n \leq x} \frac{1}{n^{1+\varepsilon}} \right)$$

$$= L(1+\varepsilon, \chi) \log x + O\left( \log x \sum_{n \leq x} \frac{1}{n^{1+\varepsilon}} \right) + O\left( \frac{\log n}{n^{1+\varepsilon}} \right)$$

$$= L(1+\varepsilon, \chi) \log x + O\left( \frac{\log x}{x^\varepsilon} \right) + O(1),$$

as required.

3. Proof of the theorem. By Lemmas 1, 2 and 3, we see that for $x \geq 1$ and any $\varepsilon > 0$,

$$\sum_{n \leq x} \frac{\chi(n)}{n} = \frac{L(1+\varepsilon, \chi)}{\zeta(1+\varepsilon)} \log x + O\left( \frac{\log x}{x^\varepsilon} \right) + O(1).$$

We may assume that $0 < \varepsilon < 1$. Putting $x = D^{1+\varepsilon/4}$ in (2), we get

$$\sum_{n \leq D^{1+\varepsilon/4}} \frac{\chi(n)}{n} \leq \frac{L(1+\varepsilon, \chi)}{\zeta(1+\varepsilon)} \log D + \frac{\varepsilon}{4} \log D + O(1).$$
if $D > D_1(e)$. It follows from Burgess's inequality [1] that if $D > D_2(e)$, then

$$\left| \sum_{n=N+1}^{N+H} \chi(n) \right| \leq \frac{1}{4} eH$$

for $H > D^{(1+e)/4}$. Thus, by partial summation, we get

$$\sum_{n>D^{1+e/4}} \frac{\chi(n)}{n} = \sum_{D^{1+e/4} < n \leq D} \frac{\chi(n)}{n} + \sum_{D < n} \frac{\chi(n)}{n} \leq \frac{1}{4} e \log D + 1.$$

Taking

$$D_0(e) = \max \{ D_1(e), D_2(e) \},$$

our assertion follows immediately from (3) and (4).

References


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On the Kummer–Mirimanoff congruences

by

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1. Introduction. Let $p$ be an odd prime, $Z$ the ring of integers, $Z_p$ the ring of all rational numbers which are $p$-integral, $B_m$ the $m$th Bernoulli number defined by

$$v/(e^v - 1) = \sum_{k=0}^{\infty} (B_k/k!) v^k$$

and $\varphi_n(v)$ the Mirimanoff polynomial, i.e.,

$$\varphi_n(v) = \sum_{i=1}^{p^n-1} v^i \quad (n \in Z).$$

We denote by $[f(v)]_0^m$ the value of $d^m \{ f(v) \}/dv^m$ at $v = 0$ for the $m$-times differentiable function $f(v)$ of $v$.

In 1857 Kummer [7] showed that if $v$ and $e$ are units in $Z_p$ such that $v^p + e^p - 1 = 0$, then the following congruences hold for $t = v, e$:

$$[U_t(v)]_{t^{p-1}} \equiv 0 \pmod{p},$$

$$B_{2m} [U_t(v)]_{t^{p-1-2m}} \equiv 0 \pmod{p}, \quad m = 1, 2, \ldots, g,$$

where $U_t(v) = 1/(1-tv^p)$ and $g = (p-3)/2$.

In [5] Hasse gives the proof of this result by using the reciprocity law for the power residue symbol (see also the proof of Inkeri [6]).

Here we should note that $[U_t(v)]_0^m$ may be replaced by $\varphi_{i+1}(t)$ if $i \geq 1$ and $t \equiv 0, 1 \pmod{p}$ (see Lemma 4 in §2). Thus, in the above congruences we shall treat $[U_t(v)]_0^m$ and $\varphi_{i+1}(t)$ without distinction.

On the other hand, Mirimanoff [10] made the full observation for the above result and proved that the congruences $(K_m)$, $m = 0, 1, \ldots, g$, hold for $t = t'$ with $t' \not\equiv 0, 1 \pmod{p}$ if and only if the following congruences hold for $t = t'$:

$$\varphi_{t-1} = 0 \pmod{p},$$

$$\varphi_{t+1} \varphi_{t-1-m} = 0 \pmod{p}, \quad m = 1, 2, \ldots, g.$$