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Remarks on the upper bound for $L(1, \chi)$

by

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1. Introduction. Let D be a positive integer, χ a real non-principal character modulo D and $L(s, \chi)$ the associated L-series. Several authors, such as Pólya [5], Chowla [3], Burgess [2], Stephens [6] and Pintz [4], have studied the upper bound for $L(1, \chi)$. The best known estimate is

$$L(1, \chi) \leqslant \frac{1}{2}\left(1-\frac{1}{\sqrt{e}}+o(1)\right)\log D \tag{1}$$

if χ is primitive. This inequality was first established by Stephens [6] when D is a prime. Later, Pintz [4] proved the inequality (1) when D is not necessarily prime. The aim of this note is to show the following theorem which enables us to improve (1) under certain conditions.

THEOREM. *Assume that D is cube-free. Then for any $\varepsilon > 0$, we have*

$$L(1, \chi) \leqslant \left(\frac{1}{4}\frac{L(1+\varepsilon, \chi)}{\zeta(1+\varepsilon)}+\varepsilon\right)\log D$$

if $D > D_0(\varepsilon)$, where $\zeta(s)$ denotes the Riemann zeta function and $D_0(\varepsilon)$ is a constant depending only on ε.

From this we can easily deduce the following corollaries.

COROLLARY 1. *Let D be as in the theorem. Then we have*

$$L(1, \chi) \leqslant \begin{cases} \left(\frac{1}{12}+o(1)\right)\log D & \text{if } \chi(2) = -1, \\ \left(\frac{1}{8}+o(1)\right)\log D & \text{if } \chi(3) = -1, \\ \left(\frac{1}{6}+o(1)\right)\log D & \text{if } \chi(5) = -1, \end{cases}$$

as $D \to \infty$.

COROLLARY 2. *Let D be as in the theorem. Then we have*

$$L(1,\chi) \leqslant \begin{cases} \left(\frac{1}{24}+o(1)\right)\log D & \text{if } \chi(2)=\chi(3)=-1,\\ \left(\frac{1}{18}+o(1)\right)\log D & \text{if } \chi(2)=\chi(5)=-1,\\ \left(\frac{1}{12}+o(1)\right)\log D & \text{if } \chi(3)=\chi(5)=-1,\\ \left(\frac{1}{36}+o(1)\right)\log D & \text{if } \chi(2)=\chi(3)=\chi(5)=-1,\end{cases}$$

as $D\to\infty$.

These results are improvements of (1) under certain conditions. Combining these corollaries with Dirichlet's class number formulae obviously gives bounds for the class number and the fundamental unit of a quadratic field.

2. Lemmas. Let χ be a real non-principal character modulo D and let $\varepsilon>0$. For any positive integer n, we put

$$a_\varepsilon(n)=\frac{1}{n^\varepsilon}\sum_{d\mid n}\chi(d)d^\varepsilon$$

and

$$b_\varepsilon(n)=\sum_{d\mid n}\frac{\chi(d)}{d^\varepsilon}.$$

Then it is easy to see that $b_\varepsilon(n)>0$ for all n.

LEMMA 1. *For all n, $a_\varepsilon(n)\leqslant b_\varepsilon(n)$.*

Proof. First, we assume that $\chi(n)\neq 0$. Since $\chi(d)=\chi(n)\chi(n/d)$ for any divisor d of n, we get

$$a_\varepsilon(n)=\frac{\chi(n)}{n^\varepsilon}\sum_{d\mid n}\chi\left(\frac{n}{d}\right)d^\varepsilon=\chi(n)b_\varepsilon(n)\leqslant b_\varepsilon(n),$$

because $b_\varepsilon(n)>0$.

Next, we assume that $\chi(n)=0$. Let n_0 be the maximal divisor of n such that $(n_0,D)=1$. By using the same way as above, we obtain

$$a_\varepsilon(n)=\frac{1}{n^\varepsilon}\sum_{d\mid n_0}\chi(d)d^\varepsilon=\frac{\chi(n_0)}{n^\varepsilon}\sum_{d\mid n_0}\chi\left(\frac{n_0}{d}\right)d^\varepsilon$$
$$=\frac{\chi(n_0)}{(n/n_0)^\varepsilon}\sum_{d\mid n_0}\frac{\chi(d)}{d^\varepsilon}=\frac{\chi(n_0)}{(n/n_0)^\varepsilon}b_\varepsilon(n)\leqslant b_\varepsilon(n),$$

because $b_\varepsilon(n)>0$ and $n/n_0\geqslant 1$. This completes the proof of the lemma.

LEMMA 2. *For $x\geqslant 1$ and any $\varepsilon>0$, we have*

$$\sum_{n\leqslant x}\frac{a_\varepsilon(n)}{n}=\zeta(1+\varepsilon)\sum_{n\leqslant x}\frac{\chi(n)}{n}+O(1)$$

where the O-constant depends only on ε.

Proof. By a familiar argument, we have

$$\sum_{n\leqslant x}\frac{a_\varepsilon(n)}{n}=\sum_{n\leqslant x}\frac{\chi(n)}{n}\sum_{m\leqslant x/n}\frac{1}{m^{1+\varepsilon}}=\sum_{n\leqslant x}\frac{\chi(n)}{n}\left\{\zeta(1+\varepsilon)+O\left(\frac{n^\varepsilon}{x^\varepsilon}\right)\right\}$$
$$=\zeta(1+\varepsilon)\sum_{n\leqslant x}\frac{\chi(n)}{n}+O\left(\frac{1}{x^\varepsilon}\sum_{n\leqslant x}\frac{1}{n^{1-\varepsilon}}\right)=\zeta(1+\varepsilon)\sum_{n\leqslant x}\frac{\chi(n)}{n}+O(1),$$

as required.

LEMMA 3. *For $x\geqslant 1$ and any $\varepsilon>0$, we have*

$$\sum_{n\leqslant x}\frac{b_\varepsilon(n)}{n}=L(1+\varepsilon,\chi)\log x+O\left(\frac{\log x}{x^\varepsilon}\right)+O(1),$$

where the O-constants depend only on ε.

Proof. By a familiar argument, we get

$$\sum_{n\leqslant x}\frac{b_\varepsilon(n)}{n}=\sum_{n\leqslant x}\frac{\chi(n)}{n^{1+\varepsilon}}\sum_{m\leqslant x/n}\frac{1}{m}=\sum_{n\leqslant x}\frac{\chi(n)}{n^{1+\varepsilon}}\left\{\log\left(\frac{x}{n}\right)+O(1)\right\}$$
$$=(\log x)\sum_{n\leqslant x}\frac{\chi(n)}{n^{1+\varepsilon}}+O\left(\sum_{n\leqslant x}\frac{\log n}{n^{1+\varepsilon}}\right)+O\left(\sum_{n\leqslant x}\frac{1}{n^{1+\varepsilon}}\right)$$
$$=L(1+\varepsilon,\chi)\log x+O\left(\log x\sum_{n>x}\frac{1}{n^{1+\varepsilon}}\right)+O\left(\sum_{n=1}^{\infty}\frac{\log n}{n^{1+\varepsilon}}\right)$$
$$=L(1+\varepsilon,\chi)\log x+O\left(\frac{\log x}{x^\varepsilon}\right)+O(1),$$

as required.

3. Proof of the theorem. By Lemmas 1, 2 and 3, we see that for $x\geqslant 1$ and any $\varepsilon>0$,

$$\sum_{n\leqslant x}\frac{\chi(n)}{n}\leqslant\frac{L(1+\varepsilon,\chi)}{\zeta(1+\varepsilon)}\log x+O\left(\frac{\log x}{x^\varepsilon}\right)+O(1). \tag{2}$$

We may assume that $0<\varepsilon<1$. Putting $x=D^{(1+\varepsilon)/4}$ in (2), we get

$$\sum_{n\leqslant D^{(1+\varepsilon)/4}}\frac{\chi(n)}{n}\leqslant\frac{1}{4}\frac{L(1+\varepsilon,\chi)}{\zeta(1+\varepsilon)}\log D+\frac{\varepsilon}{4}\log D+O(1) \tag{3}$$

if $D > D_1(\varepsilon)$. It follows from Burgess's inequality [1] that if $D > D_2(\varepsilon)$, then

$$\Big|\sum_{n=N+1}^{N+H} \chi(n)\Big| \leqslant \frac{1}{4}\varepsilon H$$

for $H > D^{(1+\varepsilon)/4}$. Thus, by partial summation, we get

$$\sum_{n > D^{(1+\varepsilon)/4}} \frac{\chi(n)}{n} = \sum_{D^{(1+\varepsilon)/4} < n \leqslant D} \frac{\chi(n)}{n} + \sum_{D<n} \frac{\chi(n)}{n} \leqslant \frac{1}{4}\varepsilon \log D + 1. \tag{4}$$

Taking

$$D_0(\varepsilon) = \max\{D_1(\varepsilon), D_2(\varepsilon)\},$$

our assertion follows immediately from (3) and (4).

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On the Kummer–Mirimanoff congruences

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1. Introduction. Let p be an odd prime, $\mathbf{Z}$ the ring of integers, $\mathbf{Z}_p$ the ring of all rational numbers which are p-integral, B_m the mth Bernoulli number defined by

$$v/(e^v - 1) = \sum_{k=0}^{\infty} (B_k/k!)v^k$$

and $\varphi_n(v)$ the Mirimanoff polynomial, i.e.,

$$\varphi_n(v) = \sum_{i=1}^{p-1} i^{n-1} v^i \qquad (n \in \mathbf{Z}).$$

We denote by $[f(v)]_0^{(m)}$ the value of $d^m\{f(v)\}/dv^m$ at $v = 0$ for the m-times differentiable function $f(v)$ of v.

In 1857 Kummer [7] showed that if ν and ϱ are units in $\mathbf{Z}_p$ such that $\nu^p + \varrho^p - 1 = 0$, then the following congruences hold for $t = \nu, \varrho$:

$$(K_0) \qquad [U_t(v)]_0^{(p-2)} \equiv 0 \pmod p,$$

$$(K_m) \qquad B_{2m}[U_t(v)]_0^{(p-1-2m)} \equiv 0 \pmod p, \qquad m = 1, 2, \ldots, g,$$

where $U_t(v) = 1/(1-te^v)$ and $g = (p-3)/2$.

In [5] Hasse gives the proof of this result by using the reciprocity law for the power residue symbol (see also the proof of Inkeri [6]).

Here we should note that $[U_t(v)]_0^{(i)}$ may be replaced by $\varphi_{i+1}(t)$ if $i \geqslant 1$ and $t \not\equiv 0, 1 \pmod p$ (see Lemma 4 in § 2). Thus, in the above congruences we shall treat $[U_t(v)]_0^{(i)}$ and $\varphi_{i+1}(t)$ without distinction.

On the other hand, Mirimanoff [10] made the full observation for the above result and proved that the congruences (K_m), $m = 0, 1, \ldots, g$, hold for $t = t'$ with $t' \not\equiv 0, 1 \pmod p$ if and only if the following congruences hold for $t = t'$:

$$(M_0) \qquad \varphi_{p-1}(t) \equiv 0 \pmod p,$$

$$(M_m) \qquad \varphi_{m+1}(t)\,\varphi_{p-1-m}(t) \equiv 0 \pmod p, \qquad m = 1, 2, \ldots, g.$$