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Trygve Nagell

Trygve Nagel (he adopted the spelling Nagell in later life) was born in Oslo 13 July 1895, the son of senior civil servant [*byråchef*] Otto Nagel and his wife Marie, *née* Olsen. According to a brief note found among his papers, the family can be traced to the middle of the 15th century in western Norway, especially Bergen, as peasants, landed gentry and members of the professions: one grandmother came from Värmland in Sweden. He studied at the University of Oslo and came under the influence of Axel Thue. Subsequently he travelled widely (Hamburg, Paris, Göttingen, Strasbourg, Berlin, Bologna, Leningrad,...) but was based on Oslo until in 1931 he was appointed to a Professorship at Uppsala, a post which he held until his retirement in 1962. For many years he was head of the department of mathematics. He remained in Uppsala, except for some extensive travels, until his death on 24 January, 1988.

His work in the theory of numbers, which will be discussed more fully below, gave him an international reputation. He had, perhaps, rather little influence on Swedish mathematical life and comparatively few pupils (some 30 doctors and licentiates), who included however, Harald Bergström, Gunnar Billing, Stig Christofferson, Adolf af Ekenstam, Lars Fjellstedt and Bengt Stolt.

During the war, the Norwegian government in exile appointed him chairman of a board for Norwegian schools outside Norway and he was directly responsible for a Norwegian *gymnasium* in Uppsala. He was awarded the Norwegian Freedom Cross in 1946 and was appointed to the Norwegian Order of St. Olaf in 1951. In 1939 he became a Knight of the Swedish Order of the North Star and in 1952 was made a Knight Commander of the Order.

He married twice. There was a son and daughter by the first marriage, both of whom predeceased him. Tall and strong, he enjoyed good health almost to the end of his life.

Nagell had a great interest in languages and in their history. He published papers in German, French, and English and also wrote and spoke Italian (his first wife was Italian). After retirement he wrote several papers on philological questions. Some of this was published [*Zur Etymologie der gotischen Stammesnamen Tervingi und Visi*, Det Kon. Norske Vid. Selsk. Skrifter 1971 No 10] and was appreciated by the experts, but his ideas became increasingly speculative as he grew older.

Almost all of Nagell's mathematical work is in the theory of numbers, and most deals with various aspects of diophantine equations.

In the early paper [14](1) Nagell considers the greatest prime factor P_N (say) of some $f(n)$, $n \leqslant N$, where $f(X)$ is a polynomial with integral coefficients. In the special case $f(X) = 1+X^2$, Chebyshev had shown that P_N/N tends to infinity. Nagell extended Chebyshev's idea to all $f(X)$ with at least one irrational root and obtains a quantitative result. Here he had to estimate the number of solutions of the congruence

$$f(X) \equiv 0 \bmod p^m$$

for primes p dividing the discriminant. His estimate (2) was first improved by G. Sándor (3) in 1952. In [18] Nagell improves the result of [14] quantitatively and also obtains the first result on the representation of squarefree numbers by polynomials: he proved that if all irreducible factors of a polynomial f are either linear or quadratic, f is squarefree (as a polynomial) and its fixed divisor is squarefree then f takes infinitely many squarefree values. This was partially rediscovered by Estermann in 1931 (cf. [48A]) but improved only by Erdös (4). Erdös also improved considerably the result of [14](5).

In another early paper, Nagell [29] obtained definitive results on the integral solution of the equation

$$f(x, y) = 1 \tag{1}$$

where f is a cubic form with integral coefficients and negative discriminant (i.e. $f(x, 1) = 0$ has a pair of complex roots). He showed that (1) has at most 3 solutions except in certain specified cases, when it can have 4 or 5. There are infinitely many inequivalent f for which there are exactly 3 solutions. The highly ingenious proofs make use of unit theory. Earlier ([16], [21], [26]), he had applied similar techniques to the equations

$$Ax^3+By^3 = C \tag{2}$$

where A and B are integers and C is 1 or 3. Here his work overlaps with the earlier and independent work of the Russian mathematician Delone (6).

A favourite topic was the rational points on curves of genus 1. He appears to have been the first, in 1928, to show [30] that if there is a rational point on such a curve, then it is birationally equivalent over the rationals to a curve of

(1) Numbers in square brackets refer to the list of Nagell's papers.

(2) The same estimate was found independently at about the same time by Øre: *Anzahl der Wurzeln höherer Kongruenzen*, Norsk mat. tidsskr. 3 (1921), 63–66, or *Über höhere Kongruenzen*, Norsk mat. foren. skrifter I, Nr 7 (1921–22), see Satz 1 on p. 12.

(3) *Über die Anzahl der Lösungen einer Kongruenz*, Acta Math. 87 (1952), 13–16. See also Huxley, *A note on polynomial congruences*, in *Recent progress in analytic number theory* (Eds. Halberstam & Hooley) 1, 193–196, Academic Press 1981.

(4) *Arithmetical properties of polynomials*, J. London Math. Soc. 28 (1953), 416–425.

(5) *On the greatest prime factor of* $\prod f(k)$, J. London Math. Soc. 27 (1952), 379–384.

(6) He transliterates his name as Delaunay when he writes in western languages.

the special shape

$$x^3-Ax-B = y^2, \tag{3}$$

where A, B can be taken to be rational integers and the given rational point transforms into the single point at infinity on (3). The rational points on (3) have a structure of abelian group with the point at infinity as zero of the group. In 1935 he showed [56] that if the rational point (a, b) is of finite order, then a and b are necessarily integers and either $b = 0$ or b^2 divides the discriminant $4A^3-27B^2$. The kernel of the elegant proof is to show that if (a, b) is a rational, but not an integral, point on (3) and if (c, d) is obtained from it by multiplication by 2, then the denominators of c, d are strictly greater than those of a, b. Subsequently, Lutz (7) proved the corresponding p-adic result, of which this is an immediate consequence.

Nagell frequently recurred to the theme of possible torsion groups of rational points on (3). In [69] he gives what is in fact a complete set of possibilities together with descriptions of when they occur. He also disproved the possibility of some further groups occurring as torsion groups, but this was superseded by the definitive work of Mazur (8). Nagell also considered the problem for groundfields other than the rationals, where a definitive answer is not yet to hand.

We conclude by mentioning some individual results:

It is well known that every prime $p \equiv 1 \pmod 3$ is of the shape $4p = a^2+27b^2$. One of the results of [23] is that every prime divisor of a or b is a cubic residue of p. This is a quite straightforward, but elegant, application of cubic reciprocity. There is a similar, but not quite so complete, counterpart for $p \equiv 1 \pmod 4$ and quartic reciprocity. He returns to the question in [78].

In 1913 Ramanujan (9) posed the problem of finding all integer solutions of $x^2+7 = 2^n$. The equation is notable because one of the solutions is unexpectedly large. Independently, Ljunggren gave it as a problem in Norsk mat. tidsskr. and Nagell [70A] gave a solution using a p-adic method in 1948, not mentioned in Mathematical Reviews or the Zentralblatt. The same result was obtained independently in 1959 by Chowla, Lewis and Skolem (10). Nagell therefore gave an English version in [97].

(7) *Sur l'équation* $y^2 = x^3 - Ax - B$ *dans les corps p-adiques*, J.f.d. reine angew. Math. 177 (1937), 238–247.

(8) *Modular curves and the Eisenstein ideal*, IHES publ. math. 47 (1977), 33–186; *Rational isogenies of prime degree*, Invent. math. 44 (1978), 129–162.

(9) *Collected papers*, 327.

(10) *The diophantine equation* $2^{n+2}-7 = x^2$ *and related problems*, Proc. Amer. Math. Soc. 10 (1959), 663–669. In the meantime the problem had also been solved in another formulation by Browkin and Schinzel, *Sur les nombres de Mersenne qui sont triangulaires*, Comptes rendus, Paris 242 (1956), 1780–1781.

In 1967 Nagell conjectured [114] that, up to the addition of a rational integer, there are only finitely many algebraic integers of given degree and discriminant, and he gave some partial results. The conjecture was proved by Birch and Merriman[11] and in an effective way by Győry[12].

Many of Nagell's papers deal with the solution of special equations: some, having served to point the way to more general theories, are now subsumed in them as illustrative examples. As much of his work appeared in comparatively little-read journals (some in little-read languages), it was not as influential as it might have been. In several of his later papers Nagell draws attention to and repeats some of his earlier work which had been overlooked. Nagell's work, particularly that in his earlier papers, has had a significant rôle in shaping our present knowledge.

J. W. S. Cassels[13])

(11) *Finiteness theorems for binary forms with given discriminant*, Proc. London Math. Soc. 24 (1972), 385–394.

(12) *Sur les polynômes à coefficients entiers et de discriminant donné*, Acta Arith. 23 (1973), 419–426.

(13) I am grateful to Dr Stig Christofferson, Dr Bengt Stolt and Professor A. Schinzel for information and for commenting on earlier drafts.

ACTA ARITHMETICA
LV (1990)

Publications of Trygve Nagell[1]

Books

L'analyse indéterminée de degré supérieur, Mémorial des sciences mathématiques 39, 1929, 63 pp. 2nd ed., 1946. [Jbuch 55, 712][2].

Lärobok i algebra, 303 pp., Stockholm–Uppsala. [MR 10–500].

Elementär talteori, 271 pp., Stockholm–Uppsala. [MR 11–640].

Introduction to number theory, 309 pp., John Wiley & Sons, New York; Almqvist & Wiksell, Stockholm, 1951. 2nd ed. (Chelsea Pub. Co.), 1964. Bulgarian ed., 1971. [MR 13–207, 30 # 4714].

Papers

1. *Über einige Sinus- und Cosinus-Produkte*, Nyt tidsskr. f. matem. 28B (1917), 33–45. [Jbuch 46.568].
2. *Einige Sätze über die ganzen, rationalen Funktionen*, Nyt tidsskr. f. matem. 29B (1918), 53–62 [Jbuch 46.241].
3. *Über zahlentheoretische Polynome*, Norsk mat. tidsskr. 1 (1919), 14–23. [Jbuch 47.122].
4. *Note sur l'application d'une formule d'inversion de la théorie des nombres*, Norsk mat. tidsskr. 1 (1919), 40–44. [Jbuch 47.120].
5. *Über höhere Kongruenzen nach einer Primzahlpotenz als Modulus*, Norsk mat. tidsskr. 1 (1919), 95–98. [Jbuch 47.122].
6. *Le discriminant de l'équation de la division du cercle*, Norsk mat. tidsskr. 1 (1919), 99–101. [Jbuch 47.122].
7. *Sur l'impossibilité de l'équation indéterminée* $(x^5-y^5)(x-y)^{-1}=5z^2$, Norsk mat. tidsskr. 2 (1920), 51–54. [Jbuch 47.121].
8. *Sur l'impossibilité de quelques équations biquadratiques à trois indéterminées*, Norsk mat. tidsskr. 2 (1920), 55–57. [Jbuch 47.121].
9. *Note sur l'équation indéterminée* $(x^n-1)(x-1)^{-1}=y^q$, Norsk mat. tidsskr. 2 (1920), 75–78. [Jbuch 47.121].
10. *Des équations indéterminées* $x^2+x+1=y^n$ *et* $x^2+x+1=3y^n$, Norsk matem. forenings skrifter. I, No 2 (1921), 14 pp. [Jbuch 48.138].
11. *Sur l'équation indéterminée* $(x^n-1)(x-1)^{-1}=y^2$, Norsk matem. forenings skrifter. I, No 3 (1921), 17 pp. [Jbuch 48.138, Skolem VI.2].
12. *Sur l'impossibilité de l'équation indéterminée* $z^p+1=y^2$, Norsk matem. forenings skrifter. I, No 4 (1921), 10 pp. [Jbuch 48.138].
13. *Fermats problem. En oversigt*, Norsk mat. tidsskr. 3 (1921), 7–21. [Jbuch 48.130].

(1) In earlier publications the surname is given as: Nagel.

(2) References are to the Jahrbuch [Jbuch], the Zentralblatt [Zblatt] or to Mathematical Reviews [MR]. In addition there are references to Skolem's Ergebnisbericht *Diophantische Gleichungen* [Skolem].