On Kronecker's limit formula for Dirichlet series
with periodic coefficients

by

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1. Throughout the paper, we use \( \mathbb{Q} \) for the field of all rational numbers.
Let \( f \) be an arithmetical function with period \( N \). The Dirichlet series associated
to \( f \) is defined by

\[
L(s, f) = \sum_{n=1}^{\infty} f(n)/n^s \quad (\text{Re}(s) > 1).
\]

The main aim of the paper is to show a limit formula for \( L(s, f) \) at \( s = 1 \) in
Theorem 5 of Section 3, and to give some of its applications after Section 4. We
shall study periodic arithmetical functions, in particular, Dirichlet characters in
Sections 2 and 4. Theorem 10 in Section 4 is a generalization of Hasse [10],
Addendum due to M. Newman. In Theorem 11 in Section 5, we shall give four
distinct representations of the values at \( s = 1 \) of Dirichlet \( L \) functions for every
odd character modulo \( N \), not necessarily primitive. When \( N \) is an odd prime,
it is obvious that the numbers \( \sin(2\pi r/N) \) (resp. \( \cos(2\pi r/N) \)) with
\( r = 1, \ldots, (N-1)/2 \) are linearly independent over \( \mathbb{Q} \). When \( N \) is a prime
congruent to 3 modulo 4, Chowla [7] proved that so are \( \cot(\pi r/N) \) if and only
if all Dirichlet \( L \) functions for odd characters modulo \( N \) do not vanish at \( s = 1 \).

By Hasse [10], Chowla's theorem can be rewritten in terms of \( \tan \).
Generalizations of these assertions will be established in Sections 7 and 8.
When \( N \) is any odd number, we know that the numbers \( \sin(2\pi r/N) \) (resp.
\( \cos(2\pi r/N) \)) with \( r = 1, \ldots, (N-1)/2 \) and \( (r, N) = 1 \) are linearly independent
over \( \mathbb{Q} \) if and only if \( N \) is square free. The proof, using Gaussian sums, is
elementary. On the other hand, the numbers \( \cot(\pi r/N) \) (resp. \( \tan(\pi r/N) \),
\( \sec(2\pi r/N) \)) are always linearly independent over \( \mathbb{Q} \). The proofs, which are
given in Theorem 20 of Section 7 and Theorem 24 of Section 8, are analytical.
Elementary proofs are known only in special cases. See Ayoub [3], Fujisaki
[9], Hasse [10], Okada [18], and Wang [20]. Further, the numbers
\( \csc(\pi r/N) \) are linearly independent over \( \mathbb{Q} \) if and only if the multiplicative
order of 2 mod \( N \) is even. This is a straightforward generalization of Jager and
Lenstra [12] and will be proved in Theorem 21 of Section 7. In the case of odd
prime powers \( N \), this has been proved by Bundschuh [6].
2. The Fourier transform of an arithmetical function \( f \) with period \( N \) is defined by
\[
\hat{f}(x) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} f(n) e(-nx/N),
\]
where \( e(x) = \exp(2\pi i x) \).

**Lemma 1.** Let \( f \) be any arithmetical function with period \( N \). Then \( \hat{f}(-x) = \hat{f}(x) \) holds for any \( x \).

This lemma is called the inversion formula. If \( f \) has the two properties:
(i) \( f \) induces a character \((\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*\); (ii) \( f(x) = 0 \) if and only if \( x \) is not prime to \( N \), then \( f \) is said to be a Dirichlet character modulo \( N \). The principal character \( \chi_0 \) is defined by the formula \( \chi_0(n) = 1 \) or \( 0 \) according as \( (n, N) = 1 \) or not. A nonvanishing arithmetical function \( f \) is completely multiplicative if \( f(xy) = f(x)f(y) \) for all \( x, y \). We can easily show

**Lemma 2.** Let \( f \) be any arithmetical function with period \( N \). Then \( f \) is completely multiplicative if and only if \( f \) is a Dirichlet character whose conductor is a divisor \( m \) of \( N \) prime to \( N/m \).

**Lemma 3.** Let \( f \) be any arithmetical function with period \( N \). Then the primitive period of \( \hat{f} \) is \( N \) if \( f(a) \neq 0 \) for some a prime to \( N \).

**Proof.** Let \( m \) be any period of \( f \). We set
\[
g(x) = f(x)(e(xm/N) - 1).
\]
By assumption, we get \( g^* = 0 \), which is equivalent to \( g = 0 \) by Lemma 1. Thus we have \( e(am/N) = 1 \) for \( (a, N) = 1 \) with \( f(a) \neq 0 \). Hence, \( N \) divides \( m \).

It is well known that any primitive Dirichlet character modulo \( N \) satisfies \( \chi^* = \chi^*(1) \bar{\chi} \), where \( \bar{\chi} \) is the complex-conjugate function of \( \chi \). Apostol proved in an elementary way that the converse is also true. Joris [13] gave another proof, using the functional equation of the Dirichlet \( L \) function.

**Lemma 4.** Condition \( \chi^* = \chi^*(1) \bar{\chi} \) characterizes primitive characters with conductor \( N \) in the class of all completely multiplicative functions with period \( N \).

**Proof.** We see \( \chi^*(1) \neq 0 \) since \( \chi \) does not vanish identically. By Lemma 3, the primitive period of \( \hat{\chi} \) is \( N \). Lemma 2 combined with Lemma 1 asserts that \( \chi \) is a Dirichlet character modulo \( N \). The rest is completed by Apostol [12], Theorem 1).

We can easily rewrite properties of \( \chi^* \) as properties of \( \chi \) only in the primitive case. On the other hand, all non-principal characters modulo \( N \) are primitive if and only if \( N \) is prime. Therefore the composite cases are not parallel with the prime cases. For details, see Section 4 and later.

3. The Hurwitz zeta function for \( r/N \) is defined by
\[
\zeta(s, r/N) = \sum_{n=0}^{\infty} (n+r/N)^{-s} \quad (\text{Re}(s) > 1).
\]

Then \( L(s, f) \) can be represented as
\[
L(s, f) = \frac{1}{N^s} \sum_{n=1}^{N} f(n) \zeta(s, r/N).
\]

Since \( \zeta(s, r/N) \) can be continued analytically to the whole complex plane and is holomorphic except at \( s = 1 \), the same holds for \( L(s, f) \). We easily see that \( L(s, f) \) is holomorphic at \( s = 1 \) if and only if \( f(1) + f(2) + \ldots + f(N) = 0 \). Assuming that \( L(s, f) \) is holomorphic at \( s = 1 \), Lehmer [15] showed
\[
L(1, f) = -\frac{1}{\sqrt{N}} \sum_{r=1}^{N-1} f(r)(\log(1 - e^{i\pi r/N})),
\]
where the logarithms have their principal values between \(-\pi/2\) and \(\pi/2\).

Livingston [16] gave another formula. We show the so-called Kronecker limit formula for \( L(s, f) \) when this function is not necessarily holomorphic at \( s = 1 \).

**Theorem 5.**
\[
\lim_{s \to 1} \left( L(s, f) - \frac{f^*(N)}{\sqrt{N}(s-1)} \right) = -\frac{1}{N^s} \sum_{r=1}^{N-1} f(r) \log(\sin(\pi r/N)) + \frac{\pi}{2N} \sum_{r=1}^{N-1} f(r) \cot(\pi r/N) + \frac{f^*(N)}{\sqrt{N}} \gamma \left( \frac{f^*(N)}{\sqrt{N}} - f(N) \right) \log 2,
\]
where \( \gamma \) is Euler's constant.

**Proof.** Our proof is based on Kronecker's limit formula for the Hurwitz zeta function, that is,
\[
\lim_{s \to 1} \left( \zeta(s, a) - \frac{1}{s-1} \right) = -\psi(a),
\]
where \( \psi(a) = \Gamma'(a)/\Gamma(a) \) is called the digamma function.

Now we have
\[
L(s, f) - \frac{f^*(N)}{\sqrt{N}(s-1)} = \frac{1}{N^s} \sum_{r=1}^{N} \left( \zeta(s, r/N) - \frac{1}{s-1} \right) f(r) + \frac{N^{s-3} - N^{s-1}}{s-1} \sum_{r=1}^{N} f(r).
\]
Therefore, as \( s \to 1 \), the right-hand side tends to
\[
\frac{1}{N} \sum_{r=1}^{N} f(r) \psi(r/N) - \frac{f^*(N)}{\sqrt{N}} \log N.
\]
By Legendre’s formula, the digamma function can be expressed in the form

\[ \psi(a) = -\gamma + \frac{1}{2} \int_0^a \frac{t^{N-1} - 1}{t - 1} \, dt, \quad \text{Re}(a) > 0. \]

See Whittaker–Watson ([22], p. 260). Thus the first term of (1) reduces to

\[ -\frac{1}{N} \sum_{r=1}^N f(r) (t^{N-1} - 1) dt + \frac{N}{2} \sum_{r=1}^N f(r). \]

On writing \( t^N \) for \( t \), this is equal to

\[ -\frac{1}{N} \int F(t, f) dt + \frac{N}{2} \psi(N), \]

where

\[ F(t, f) = \sum_{r=1}^N f(r) (t^r - t^n). \]

Since \( F(t, f) \) is a polynomial in \( t \) of degree at most \( N \) such that \( F(0, f) = F(1, f) = 0 \), we can use the idea of Dirichlet–Dedekind ([8], Section 185) for the Dirichlet \( L \) functions. So we see that

\[ -\frac{1}{N} \int F(t, f) dt = -\frac{1}{N} \sum_{m=1}^{N-1} \left\{ \sum_{r=1}^N f(e(-m/N)) \log(1 - e(m/N)) \right\} \]

\[ = -\frac{1}{\sqrt{N}} \sum_{m=1}^{N-1} f^-(m) \log(1 - e(m/N)) + \frac{N}{2} \sum_{r=1}^N f(r) \sum_{m=1}^{N-1} \log(1 - e(m/N)). \]

Here

\[ \sum_{r=1}^{N-1} \log(1 - e(r/N)) = \sum_{r=1}^{N-1} \log(2 \sin(\pi r/N)) = \log N \]

since \( \prod_{r=1}^{N-1} \sin(\pi r/N) = N/2^{N-1} \).

We denote by \( g \) and \( h \) the even and odd parts of \( f \) respectively. Namely,

\( g(N-x) = g(x) \) and \( h(N-x) = -h(x) \) for \( x = 1, 2, \ldots, N \), and \( f \) is represented as \( g + h \) uniquely. Then

\[ \sum_{r=1}^{N-1} g^-(r) \log(1 - e(r/N)) = \sum_{r=1}^{N-1} g^-(r) \log(\sin(\pi r/N)) + \sum_{r=1}^{N-1} g^+(r) \log 2. \]

Here

\[ \sum_{r=1}^{N-1} g^+(r) = g(N) \sqrt{N} - g^+(N). \]

Now we have

\[ \sum_{r=1}^{N-1} h^+(r) \log(1 - e(r/N)) = \frac{\pi i}{N} \sum_{r=1}^{N-1} h^+(r) r \]

\[ = \frac{\pi i}{N \sqrt{N}} \sum_{r=1}^{N-1} h(r) \sum_{r=1}^{N-1} e(-rt/N) r \]

\[ = -\frac{\pi i}{2} \sum_{r=1}^{N-1} h(r) \cot(\pi r/N). \]

Further we have

\[ \sum_{r=1}^{N-1} f^-(r) \log(\sin(\pi r/N)) = \sum_{r=1}^{N-1} g^-(r) \log(\sin(\pi r/N)), \]

\[ \sum_{r=1}^{N-1} f(r) \cot(\pi r/N) = \sum_{r=1}^{N-1} h(r) \cot(\pi r/N), \]

and \( f^+(N) = g^+(N), f(N) = g(N) \). Summing up, we establish the theorem.

Theorem 5 immediately implies

**Corollary 6.** Let \( \chi \) be any non-principal Dirichlet character modulo \( N \). Then

\[ (1 - \chi(-1)) L(1, \chi) = \frac{\pi}{N} \sum_{r=1}^{N-1} \chi(r) \cot(\pi r/N), \]

\[ (1 + \chi(-1)) L(1, \chi^*) = -\frac{2}{\sqrt{N}} \sum_{r=1}^{N} \chi(r) \log(\sin(\pi r/N)). \]

4. In order to give applications of Theorem 5, we need to prepare some equalities involving trigonometrical functions.

**Lemma 7.**

\[ \sum_{r=1}^{N-1} f^-(r)(2r - N) = i \sqrt{N} \sum_{r=1}^{N-1} f(r) \cot(r\pi/N). \]

**Proof.** The left-hand side can be written as

\[ \frac{1}{\sqrt{N}} \sum_{m=1}^{N-1} \sum_{r=1}^{N-1} e(-rm/N)(2r - N), \]

which is equal to the right-hand side.

**Lemma 8.** Whenever \( N \) is even, assume that \( f(N/2) = 0 \) and the term with \( N/2 \) is excepted from the right sum below. Then

\[ \sum_{r=1}^{N-1} f^-(r) \{ (-1)^r f(-1)^{N/2}(r - N) \} = \sqrt{N} \sum_{r=1}^{N-1} f(r) \tan(r\pi/N). \]
Proof. It is similar to Lemma 7.

**Lemma 9.** Let \( \chi \) be any odd Dirichlet character modulo \( N \). Then the following identities hold:
\[
A_x \sum_{r=1}^{N-1} \chi(r) \cot\left(\frac{\pi r}{N}\right) = B_x \sum_{r=1}^{N-1} \chi(r) \tan\left(\frac{\pi r}{N}\right),
\]
\[
\bar{A}_x \sum_{r=1}^{N-1} \chi^*(r) \cot\left(\frac{\pi r}{N}\right) = \bar{B}_x \sum_{r=1}^{N-1} \chi^*(r) \tan\left(\frac{\pi r}{N}\right),
\]
where \( A_x \) and \( B_x \) are as follows:
\[
N \equiv 1 \mod 2 \quad N \equiv 0 \mod 4 \quad N \equiv 2 \mod 4
\]
\[
A_x = \begin{cases} 2\bar{x}(2)-1 & \text{if } N \equiv 1 \mod 2 \\ -1 & \text{if } N \equiv 0 \mod 4 \\ 2\bar{x}(2+N-2) & \text{if } N \equiv 2 \mod 4 \\ \end{cases}
\]
\[
B_x = \begin{cases} \chi(1+N/2) & \text{if } N \equiv 1 \mod 2 \\ -1 & \text{if } N \equiv 0 \mod 4 \\ \end{cases}
\]

Proof. The first case follows from \( 2\cot 2x = \cot x - \tan x \). The second case follows from \( \cot(\pi/2-x) = \tan x \). The third case reduces to the first case, since there exists a Dirichlet character \( \xi \mod N/2 \) such that \( \xi(r) = \chi(r) \) for all \( r \).

This lemma leads to a generalization of the equality in Hasse (10), Addendum due to M. Newman.

**Theorem 10.** Let \( N \) be any odd number and let \( \chi \) be any Dirichlet character modulo \( N \), not necessarily primitive. Then
\[
(1 - 2\chi(2)) \sum_{r=1}^{N-1} \chi(r) r = \chi(2) N \sum_{r=1}^{(N-1)/2} \chi(r).
\]

Proof. If \( \chi \) is even then both sides are equal to 0. Assuming that \( \chi \) is odd, we easily see that
\[
\sum_{r=1}^{N-1} \chi(r) (-1)^r = 2\chi(2) \sum_{r=1}^{(N-1)/2} \chi(r).
\]

Therefore the theorem follows from the first case of Lemma 9 and Lemmas 7 and 8.

When \( N \) is even, the two sums in Theorem 10 vanish for a primitive character \( \chi \). For, we can assume \( N \equiv 0 \mod 4 \), so that it is obvious that the right sum vanishes, and the left sum is equal to
\[
(1 + \chi(1+N/2)) \sum_{r=1}^{N/2} \chi(r) r = 0,
\]
since the primitiveness of \( \chi \) implies \( \chi(1+N/2) = -1 \).

5. We give some known and some not-so-known representations of the values of Dirichlet \( L \)-functions at \( s = 1 \) for odd Dirichlet characters.

**Theorem 11.** Let \( \chi \) be any odd Dirichlet character modulo \( N \), not necessarily primitive. Then
\[
L(1, \chi) = \frac{\pi i^{N/2}}{N} \sum_{r=1}^{N-1} \chi(r) \cot\left(\frac{n r}{N}\right)
= \frac{iN \sqrt{\pi}}{N} \sum_{r=1}^{N-1} \chi(r) r
= \frac{\pi B_x}{A_x} \sum_{r=1}^{(N/2)} \chi(r) \tan\left(\frac{\pi r}{N}\right)
= \frac{\pi B_x}{2iA_x} (\sum_{r=1}^{N-1} \chi(r) (-1)^r (-1)^{-N} (r-N)).
\]

Proof. The first formula is just Corollary 6. The second formula follows from Lemma 7. These formulas are well known. The third and fourth formulas are consequences of Lemma 9.

**Corollary 12.** For any odd character \( \chi \),
\[
\frac{A_x + B_x}{B_x} L(1, \chi) = \frac{2\pi i^{N/2}}{N} \sum_{r=1}^{(N/2)} \chi(r) \cos\left(\frac{2\pi r}{N}\right).
\]

Proof. Since \( 2\cos 2x = \tan x + \cot x \), we just add the first and third formulas in Theorem 11 to obtain the corollary.

6. Okada (19), Lemma) showed the following lemma on the Frobenius determinant.

**Lemma 13.** Let \( G \) be a finite abelian group and let \( H \) be a subgroup of \( G \). Let \( \lambda \) be a character of \( H \) and let \( \Delta \) be the set of all characters of \( G \) whose restriction to \( H \) is equal to \( \lambda \). Then for any complex-valued function \( f \) on \( G \) with \( f(gh) = \lambda(h) f(g) \) for all \( g, h \in G \), we have
\[
\det f(a-1) = \prod \left( \sum_{\chi \in \Delta} \chi(a) f(a) \right),
\]
where \( T \) is a complete representative system of \( G \) by \( H \).

We denote by \( \chi^* \) the primitive character corresponding to a Dirichlet character \( \chi \). Let \( p \) be any prime and let \( N' = p^m \) (\( m \geq 0 \)) and \( (N, p) = 1 \). We define \( M(p) \) and \( L(p) \) as follows: if \( N' \neq 1 \) then
\[
M(p) = \min \{ m > 0 \mid p^m \equiv 1 \mod N' \}
\]
and \( L(p) = \phi(N')/M(p) \); if \( N' = 1 \) then both are 0. We note that if \( N' \neq 1 \) then \( M(p) \) is called the multiplicative order of \( p \) modulo \( N' \). This coincides with the residue class degree of \( p \) as an ideal in the \( N \)th cyclotomic field \( K \). The number \( L(p) \) coincides with the number of prime ideals lying above \( p \) in \( K \).
Lemma 14.

\[ \prod_{x \text{ odd}} (1 - \chi^*(p) X) \prod_{x \text{ even}} (1 - \chi^*(p) X) = (1 + X^{M(p)/2} \bar{L}(p)) \quad \text{if } M(p) \text{ is even}, \]
\[ (1 - X^{M(p)/2} \bar{L}(p)) \quad \text{if } M(p) \text{ is odd}. \]

Proof. The lemma follows from

\[ \prod_{x \text{ odd}} (1 - \chi^*(p) X) \prod_{x \text{ even}} (1 - \chi^*(p) X) = (1 - X^{M(p)/2} \bar{L}(p)). \]

For the product over even characters is \((1 - X^{M(p)/2} \bar{L}(p))\) or \((1 - X^{M(p)/2} \bar{L}(p)/2)\) according as \(M(p)\) is even or odd.

Using Lemma 14, we explicitly calculate the determinants of trigonometric functions partially shown by Ayoub [3], Okada [18] and others. From now on, we use \(\varphi\) for the Euler totient function.

Lemma 15. Set

\[ D_N = \prod_{x \text{ odd}} L(1, \chi), \quad n = \varphi(N)/2. \]

Then we have

\[ \det(\cot(abx/N)) = \pm (N/n) \nu D_N, \]
\[ \det(\tan(abx/N)) = \pm (N/n) \nu D_N E_N, \]
\[ \det(\cosec(2abx/N)) = \pm (N/n) \nu D_N F_N, \]

where \(a\) as row and \(b\) as column run over all positive integers prime to \(N\) and less than \(N/2\). The constants \(E_N\) and \(F_N\) are given as follows:

\[ E_N = \begin{cases} 1 & \text{if } N \equiv 0 \mod 4 \\ 2^k & \text{if } N \equiv 1 \mod 2, M \equiv 0 \mod 2 \\ 2^{k+1} & \text{if } N \equiv 1 \mod 2, M \equiv 1 \mod 2 \\ 2^{k+2} & \text{if } N \equiv 2 \mod 4, M \equiv 0 \mod 2 \\ 2^{k+1} & \text{if } N \equiv 2 \mod 4, M \equiv 1 \mod 2, \end{cases} \]

\[ F_N = \begin{cases} 2^k & \text{if } N \equiv 0 \mod 4 \\ 2^{k+1} & \text{if } N \equiv 1 \mod 2, M \equiv 0 \mod 2 \\ 2^{k+2} & \text{if } N \equiv 1 \mod 2, M \equiv 1 \mod 2 \\ 2^{k+3} & \text{if } N \equiv 2 \mod 4, M \equiv 0 \mod 2 \\ 2^{k+2} & \text{if } N \equiv 2 \mod 4, M \equiv 1 \mod 2, \end{cases} \]

where \(M = M(2)\) and \(L = L(2)\).

Proof. It follows from Theorem 11, Corollary 12 and Lemma 13 that

\[ E_N = \pm \prod_{x \text{ odd}} A_x/B_x, \quad F_N = \pm \prod_{x \text{ odd}} (A_x + B_x)/B_x. \]

Since \(A_x = A_x^*\) and \(B_x = B_x^*\) in any case, the number \(E_N\) is immediately determined from Lemma 14. Taking a prime \(p\) such that 

\[ \text{prime } p \text{ such that } p = 1 + N/2 \mod N, \]

we easily see \(M(p) = 2\) and \(L(p) = \varphi(N)\) if \(N \equiv 0 \mod 4\), so that we also determine \(F_N\).

7. Let \(\chi^*\) be the primitive character with conductor \(f_\chi\) corresponding to a Dirichlet character modulo \(N\). Then from Hasse [11], Kimura [14] and Washington [21] for example, we know the following formula:

\[ L(s, \chi) = L(s, \chi^*) \prod_{p|N} (1 - \chi^*(p) p^{-s}). \]

Let \(K^+\) be the maximal real subfield of the \(N\)th cyclotomic field \(K\). Let \(h^\prime\) be the quotient of their class numbers, \(h(K)/h(K^+)\). The discriminant \(d(K)\) of \(K\) is

\[ \pm N^{w(N)} \prod_{p|N} p^{p(N)/2(c/p - 1)}. \]

By the conductor-discriminant formula, we get

\[ \prod_{x \text{ odd}} f_x = \begin{cases} \sqrt{d_K} & \text{if } N \text{ is not a prime power}, \\ \sqrt{d_K/p} & \text{if } N \text{ is an odd prime power}, \\ \sqrt{d_K/4} & \text{if } N \text{ is an even prime power}, \end{cases} \]

which is denoted by \(d^-\). By the class number formula, we have

\[ \prod_{x \text{ odd}} L(1, \chi^*) = \frac{(2N)^{w(N)/2} h^-}{Q^w \sqrt{d^-}}, \]

where \(Q = 1\) or \(2\) according as \(N\) is a prime power or not and where \(w = N\) or \(2N\) according as \(N\) is even or odd.

Lemma 16. Let \(\chi\) be any odd character modulo \(N\). Then

\[ \sum_{m=1}^{(N/2)} \chi(m) (2m - N) = \frac{c(\chi) N}{i \pi} \prod_{p|N} (1 - \chi^*(p)) L(1, \chi^*), \]

where

\[ c(\chi) = \sum_{r=1}^{f_\chi} \chi^*(r) e(-r/f_\chi). \]

Proof. Let \(p, q, \ldots\) be the primes dividing \(N\) but not dividing \(f_\chi\). The left-hand side is equal to

\[ \sum_{m=1}^{N} \chi(m) m = \sum_{m=1}^{N} \chi^*(m) m \]

\[ = \sum_{m=1}^{N} \chi^*(m) m - \sum_{p|N} \chi^*(p) p \sum_{m=1}^{N/p} \chi^*(m) m \]

\[ + \sum_{p 
eq q} \chi^*(pq) pq \sum_{m=1}^{N/pq} \chi^*(m) m - \ldots \]
\[
\frac{N}{f_x} \sum_{m=1}^{f_x} \chi^*(m)m - \sum_p \chi^*(p)p \frac{N}{f_x} \sum_{m=1}^{f_x} \chi^*(m)m \\
+ \sum_p \chi^*(p) \frac{N}{p} \sum_{m=1}^{f_x} \chi^*(m)m - \ldots \\
= \frac{N}{f_x} \prod_{p \neq q} (1 - \chi^*(p)) \frac{f_x}{\zeta^*(m)m}.
\]

By the second formula of Theorem 11, we get
\[
\frac{f_x}{\zeta^*(m)m} = \frac{c(\chi)}{\pi^2} L(1, \chi^*).
\]

This completes the proof.

**Lemma 17.** Let \( \chi \) be an odd Dirichlet character modulo \( N \). If \( N \) is odd, then
\[
\sum_{m=1}^{(N-1)/2} \chi(m) (-1)^m = \frac{c(\chi)}{\pi^2} A_\chi \prod_{p \in \mathcal{P}_N} (1 - \chi^*(p)) L(1, \chi^*),
\]
where \( c(\chi) \) is as in Lemma 16.

**Proof.** It is similar to Lemma 16.

**Lemma 18.** For positive integers \( a, b \) prime to \( N \) and less than \( N/2 \), we define an integer \( c(a, b) \) by \( a = c(a, b) b \mod N \) and \( 1 \leq c(a, b) \leq N - 1 \). Set
\[
\delta(N) = \prod_{p \in \mathcal{P}_N} (1 + (-1)^m(p)) \zeta^2(p).
\]
Denote \( \varphi(N)/2 \) by \( n \). Then we have
\[
\det(2c(a, b) - N) = \pm \delta(N)(2\pi)^{\chi}/Q_w,
\]
where \( a \) as row and \( b \) as column run over all positive integers prime to \( N \) and less than \( N/2 \). If \( N \) is odd, then we further have
\[
\det((-1)^{a,b}) = \pm \delta(N)(2\pi)^{\chi} E_n/Q_w.
\]

**Proof.** This follows from Lemmas 16 and 17.

Under the assumptions and the notations of Lemma 13, we can show

**Lemma 19.** Assume that for every \( b \in \mathcal{T} \) there exists an automorphism \( \sigma \) of the field of all algebraic numbers over \( \mathbb{Q} \) such that \( \sigma(f(a)) = f(\sigma(a)) \) for all \( a \in \mathcal{T} \). Then the determinant given in Lemma 13 does not vanish if and only if the values \( f(a) (a \in \mathcal{T}) \) are linearly independent over \( \mathbb{Q} \).

Here we give a generalization of Fujisaki [9].

**Theorem 20-I.** For positive integers \( N \), the following assertions are equivalent:

(I-0) \( L(1, \chi) \neq 0 \) for all odd Dirichlet characters.

(I-1) The numbers \( \cot(r\pi/N) \) with \( r = 1, \ldots, [N/2] \) and \( (r, N) = 1 \) are linearly independent over \( \mathbb{Q} \).

(I-2) The numbers \( \tan(r\pi/N) \) with \( r = 1, \ldots, [N/2] \) and \( (r, N) = 1 \) are linearly independent over \( \mathbb{Q} \).

(I-3) For every divisor \( d \) of \( N \), the number \( \sin(2\pi/d) \) can be expressed as a linear combination with rational coefficients of the numbers \( \cot(r\pi/N) \) with \( r = 1, \ldots, [N/2] \) and \( (r, N) = 1 \).

(I-4) For every divisor \( d \) of \( N \), the number \( \sin(2\pi/d) \) can be expressed as a linear combination with rational coefficients of the numbers \( \tan(r\pi/N) \) with \( r = 1, \ldots, [N/2] \) and \( (r, N) = 1 \).

**Proof.** The equivalence of (I-0), (I-1), and (I-2) follows from Theorem 11 and Lemmas 13, 15, and 19.

Let \( K \) be the \( N \)th cyclotomic field. Put \( K^\times = \{x \in K; \, \bar{x} = \pm x\} \). We note that \( K^\times \) is the maximal real subfield but \( K^- \) is not a field. \( K^+ + K^- = K \) as vector spaces over \( \mathbb{Q} \) and the dimensions of \( K^+ \) and \( K^- \) are \( \varphi(N)/2 \). If (I-1) holds then the numbers
\[
\frac{\cot(r\pi/N)}{N} = \frac{1 + e(r/N)}{1 - e(r/N)} \quad \text{for} \quad 1 \leq r \leq [N/2], \quad (r, N) = 1,
\]
are a basis of \( K^- \). Since \( i \sin(2\pi/d) \) belongs to \( K^- \), (I-3) holds. For every positive integer \( t \) prime to \( N \) and less than \( N/2 \), there exists an automorphism \( \sigma \) of \( K \) such that \( \sigma(e(t/N)) = e(t/N) \) for all \( t \). Conversely, if (I-3) holds then for all \( r \), the numbers \( \sin(2\pi/r) \) can be expressed as a linear combination of the numbers \( \cot(r\pi/N) \). Since the numbers \( i \sin(2\pi/N) \) span \( K^- \), (I-3) holds. Hence (I-3) is equivalent to (I-1). Similarly, (I-4) is equivalent to (I-2). Now the proof is complete.

**Theorem 20-II.** Assume that one of the assertions in Theorem 20-I holds. Then the following assertions are equivalent:

(I-1) All \( M(p) \) with \( p \mid N \) are even, where
\[
M(p) = \begin{cases} 0 & \text{if } N \text{ is a power of } p, \\
\min \{m > 0 \mid p^m \equiv 1 \mod N \} & \text{otherwise},
\end{cases}
\]
for \( N' = N/p^\nu \) with \( p^\nu \parallel N \).

(I-2) For any odd Dirichlet character \( \chi \) modulo \( N \),
\[
\sum_{a=1}^{N-1} \chi(a)a \neq 0.
\]

(I-3) The square matrix \( (2c(a, b) - N) \) with \( \varphi(N)/2 \) rows is regular, where notations are the same as in Lemma 18.

Assume further that \( N \) is odd. Then the above assertions are also equivalent to either of the following assertions:
(II-4) For any odd Dirichlet character \( \chi \) modulo \( N \),
\[
\sum_{a=1}^{N-1} \chi(a)(-1)^a \neq 0.
\]

(II-5) The square matrix \((-1)^{a,b}\) with \( \varphi(N)/2 \) rows is regular.

Proof. This follows from Theorem 20-I and Lemmas 13-19.

Theorem 20-III. Assume that (II-1) in Theorem 20-II holds. Then the remaining claims (II-2)-(II-5) are equivalent to the assertions in Theorem 20-I.

Proof. This follows from Theorem 20-II and Lemma 18.

We remark that if \( N \) is a prime power then (II-1) is true and that if \( N \) has two distinct prime divisors \( p \) and \( q \) with \( q \equiv 1 \mod p \) then (II-1) is false. We have
\[
\sqrt{N} \chi^* (1) = \begin{cases} 
\sum_{r=1}^{N-1} \chi(r) \cos \frac{2\pi r}{N} & \text{if } \chi \text{ is even}, \\
\sum_{r=1}^{N-1} \chi(r) \sin \frac{2\pi r}{N} & \text{if } \chi \text{ is odd}.
\end{cases}
\]

All Dirichlet characters \( \chi \) modulo \( N \) satisfy \( \chi^* (1) \neq 0 \) if and only if \( N \) is square free. Therefore the numbers \( \sin(2\pi r/N) \) (resp. \( \cos(2\pi r/N) \)) with \( r = 1, \ldots, [N/2] \) and \( (r, N) = 1 \) are linearly independent over \( \mathbb{Q} \) if and only if \( N \) is square free.

Therefore we cannot replace “For every divisor \( d \) of \( N \), the number \( \sin(2\pi/d) \)” by “the number \( \sin(2\pi/N) \)” in (I-3) and (I-4) of Theorem 20-I. As a generalization of Jager and Lenstra [12], we state

Theorem 21. Assume that one of the assertions in Theorem 20-I holds. In order that the numbers \( \cos(2\pi r/N) \) with \( r = 1, \ldots, [N/2] \) and \( (r, N) = 1 \) are linearly independent over \( \mathbb{Q} \), it is necessary and sufficient that one of the following conditions hold: (i) \( N \equiv 0 \mod 4 \); (ii) \( N \equiv 2 \mod 4 \) and the multiplicative order of 2 modulo \( N/2 \) is even; (iii) \( N \equiv 1, 3 \mod 4 \) and the multiplicative order of 2 modulo \( N \) is even.

Proof. This follows from Corollary 12 and Lemma 15.

8. Finally, we shall consider the linear independence of the values of “sec”.

Lemma 22. Let \( N \) be odd and let \( \chi \) be any even Dirichlet character modulo \( N \). Let \( \chi^* \) be the primitive character with conductor \( f_2 \) corresponding to \( \chi \). Then
\[
\sum_{r=1}^{[N/2]} (-1)^r \chi(r) = \pm \prod_{p \nmid N} \left( \chi^*(p) - (-1)^{(p-1)/2} \right) \sum_{r=1}^{[N/2]} (-1)^r \chi^*(r),
\]
\[
\sum_{r=1}^{[N/2]} (-1)^r \chi^*(r) = \frac{\chi(2)}{\sqrt{N}} \sum_{r=1}^{[N/2]} \chi(r) \sec \frac{2\pi r}{N}.
\]

Proof. The lemma is proved by modifying the proof of Lemmas 7 and 16.

Lemma 23. Let \( N \) be odd. For any even Dirichlet character \( \chi \) modulo \( N \), there exists an odd Dirichlet character \( \xi \) modulo \( 4N \) such that
\[
2 \sum_{r=1}^{N-1} \chi(r) \sec \frac{2\pi r}{N} = \sum_{r=1}^{4N} \xi(r) \cot \frac{2\pi r}{4N}.
\]

Proof. Since \( 2 \sec 2x = \cot(x + \pi/4) - \cot(x - \pi/4) \), the left-hand side is equal to
\[
\sum_{1 \leq r \leq 4N} \chi(r) \cot \frac{2\pi r}{4N} - \sum_{1 \leq r \leq 4N} \xi(r) \cot \frac{2\pi r}{4N}.
\]

We define \( \xi \) by \( \xi(x) = \chi(x) \) if \( x \equiv 1 \mod 4 \), \(-\chi(x) \) if \( x \equiv 3 \mod 4 \), and 0 if \( x \equiv 0 \mod 4 \). It is easily proved that \( \xi \) is an odd Dirichlet character modulo \( 4N \).

Theorem 24. Assume that one of the assertions in Theorem 20-I holds. Then the numbers \( \sec(2\pi r/N) \) with \( r = 1, \ldots, [N/2] \) and \( (r, N) = 1 \) are linearly independent over \( \mathbb{Q} \).

Proof. Lemma 23 leads to the theorem in the case of \( N \) odd. For \( N \) even, set \( N' = N/2 \). If \( N' \) is odd then the numbers \( \sec(2\pi r/N) \) with \( r = 1, \ldots, [N/2] \) and \( (r, N') = 1 \) are linearly independent over \( \mathbb{Q} \), so that the theorem follows from \( \sec(4\pi r/N) = -\sec(2(N'-2)\pi r/N) \). If \( N' \) is even then by \( \sec(2\pi r/N) = \csc(2(N'-2)\pi r/N) \), the theorem reduces to Theorem 21.

Note that by Theorem 24, the numbers \( \sec(2\pi r/N) \) are always a normal basis of the maximal real subfield \( K^+ \) of the \( N \)th cyclotomic field \( K \) over \( \mathbb{Q} \), while their reciprocal numbers \( \cos(2\pi r/N) \) are a normal basis if and only if \( N \) is square free.

Compare the assertion (II-5) in Theorem 20-II with the following theorem.

Theorem 25. Assume that one of the assertions in Theorem 20-I holds and further that \( N \) is odd. In order that the square matrix \((-1)^{a,b}\) with \( \varphi(N)/2 \) rows is regular, where \( a \) as row and \( b \) as column run over all positive integers prime to \( N \) and less than \( N/2 \), and where \( d(a, b) \) means the unique integer with \( a = d(a, b) b \mod N \) and \(-N/2 < d(a, b) < N/2 \), it is necessary and sufficient that all prime divisors of \( N \) are congruent to 3 mod 4.

Proof. Necessity. We see from Lemma 13 that the left-hand side of the first formula of Lemma 22 does not vanish. Therefore taking the principal character for \( \chi \), we obtain
\[
\prod_{p \nmid N} \left( 1 - (-1)^{(p-1)/2} \right) \neq 0,
\]
and so \( p \equiv 3 \mod 4 \).
Sufficiency. By Theorem 11 and Lemmas 22 and 23, there exists an odd Dirichlet character $\chi$ modulo $4f_2$ such that
\[
\sum_{r=1}^{(N-1)/2} (-1)^r \chi(r) = \pm \prod_{p|N} (\chi^*(p)+1)c(\chi)L(1, \chi),
\]
where
\[
c(\chi) = \frac{2}{\pi} \sum_{r=1}^{\infty} \chi^*(r)e(-2rf_2).
\]
From $L(1, \chi) \neq 0$, it is sufficient to show that $\chi^*(p) \neq -1$ when all prime divisors are congruent to 3 modulo 4. Then the multiplicative order $M(p)$ of $p$ modulo $N'$ for $N' = N/p^s$ with $p^s|N$ is either odd or twice an odd number since $\varphi(N') \equiv 2 \mod 4$ for every prime $q$ with $q \equiv 3 \mod 4$. If $M(p)$ is odd then $\chi^*(p) \neq -1$ by $\chi^*(p)^{M(p)} = 1$. If $M(p)$ is twice an odd number, then $\chi^*(p)^{M(p)/2} = -1$. Therefore $\chi^*(p)^{M(p)/2} = \chi^*(-1) = 1$, since $\chi$ is even, so that $\chi^*(p) \neq -1$. This completes the proof of the sufficiency.

References