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Weak asymptotic formulas for partitions free of small summands

by

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1. Introduction. A frequently occurring problem in Number Theory is the asymptotic evaluation of sums of the form

$$S_y(x) = \sum_{n \le x} h_y(n) \quad (x \to \infty), \tag{1}$$

where h_y is an arithmetical function depending on a parameter $y = y(x)$ tending to infinity.

The best known (and perhaps most important) problem of this type consists in approximating the function

$$\Psi(x, y) = \sum_{n \le x} \chi_y^*(n) = \sum_{\substack{n \le x \\ p|n \Rightarrow p \le y}} 1 \tag{2}$$

uniformly in various y-ranges. Here χ_y^* denotes the characteristic function of the positive integers free of prime divisors greater than y.

The study of $\Psi(x, y)$ has been the object of numerous articles, e.g. by de Bruijn [1], Hildebrand [11], [12] and Hensley [8] just to mention a few(¹).

De Bruijn, van Lint, Richert ([2], [14]) and others dealt with the more general problem of estimating "incomplete sums" of the form

$$\Lambda^*(x, y) = \sum_{n \le x} \chi_y^*(n)\lambda(n), \tag{3}$$

where λ is a (nonnegative) multiplicative function(²).

The main purpose of the present paper is to provide a method for deducing asymptotic formulas for the logarithms of a large class of parameter-dependent partition functions, where the result is uniform in a certain range of the parameter.

The function

$$P_y(u) = \sum_{n \le u} p_y(n)$$

(¹) Cf. Norton [15] for an extensive bibliography concerning the results before 1970.
(²) See also Wirsing's remark in [18], section 1.34, pp. 418–419.

with $p_y(n)$ denoting the number of partitions of the positive integer n into summands $\geqslant y$ may serve as a typical example(3), (4).

The tool for tackling the problem is developed in the first part of this note in the form of a uniform Tauberian theorem for families of Laplace-transforms depending on a parameter.

It turns out that the scope of this Tauberian theorem is not limited to partition functions. This is indicated in the final part of the article, where incomplete sums(5) of "fast growing" multiplicative functions are estimated.

Remark. While the present note deals only with weak asymptotic properties of parameter-dependent partition functions, i.e. formulas for the logarithm of such functions, a subsequent paper will provide asymptotic formulas for the partition functions themselves.

2. The Tauberian theorem. The idea of using Tauberian arguments in partition problems is due to Hardy and Ramanujan [6]. Using essentially their method, in 1968 W. Schwarz [16] proved a Tauberian theorem(6), (7) with remainder-term, from which he derived some rather general theorems on weak asymptotic properties of partitions(8).

Of course neither these partition results nor the underlying Tauberian theorem are useful when dealing with functions depending on a parameter, but it turns out that the proof of this Tauberian theorem can be modified in order to maintain control over the dependence of all error-terms on the involved parameters.

To simplify notation in the following theorem the subscript y is dropped in the case $y = 0$, i.e. $A(u) = A_0(u)$, $\varphi(\sigma) = \varphi_0(\sigma)$ and $\sigma_u = \sigma_u(0)$. $C_1, C_2, \ldots$ will denote positive constants throughout the rest of the paper.

THEOREM 1. *Let* $\{A_y;\ y \geqslant 0\}$ *be a set of nondecreasing functions*

$$A_y\colon [0, \infty[\to [0, \infty[$$

satisfying the following conditions:

(4) $$A(0) = 0 \leqslant A_y(u) \leqslant A(u) \quad \textit{for all } u,\ y \geqslant 0.$$

(5) *The Laplace-transform*

$$f_y(\sigma) = \sigma \int_0^\infty A_y(u)\, e^{-u\sigma}\, du$$

converges in $\sigma > 0$.

(6) *For a fixed positive real number* μ *and functions* $\varphi_y \in C^2(]0, \mu[)$ *the difference*

$$|\log f_y(\sigma) - \varphi_y(\sigma)| \leqslant C_0$$

is bounded by a positive constant C_0 *independent of* y *and* σ.

(7) $$0 \leqslant \varphi_y'' \nearrow, \quad -\sigma\varphi_y'(\sigma) \nearrow \infty \qquad \textit{for all } y \geqslant 0 \textit{ if } (9)\ \sigma \searrow 0.$$

Furthermore, it is assumed that for all sufficiently small σ *the following inequalities are satisfied:*

(8) $$-\varphi'(\sigma) \leqslant -C_1\varphi'(2\sigma),$$

(9) $$-\varphi_y'(\sigma) \leqslant -\varphi'(\sigma), \qquad \varphi_y''(\sigma) \leqslant C_2\varphi''(\sigma).$$

Suppose that for some function $0 \leqslant M(y) \nearrow \infty$ $(y \nearrow \infty)$ *the estimate*

(10) $$-\varphi'(\sigma) + \varphi_y'(\sigma) \leqslant C_3 \frac{M(y)}{\sigma^\alpha}$$

holds, where $\alpha \leqslant 1$ *is a real number.*

Now if $b\colon [0, \infty[\to [0, \infty[$ *is a function strictly increasing to infinity such that*

(11) $$M\big(b(u)\big) = o(u\sigma_u^\alpha) \qquad (u \to \infty),$$

then uniformly in $0 \leqslant y \leqslant b(u)$

(12) $$\log A_y(u) = \varphi_y\big(\sigma_u(y)\big) + u\sigma_u(y) + O\big(R(u)\big) \qquad (u \to \infty),$$

where the remainder-term in this asymptotic equation is given by

(13) $$R(u) = \sigma_u \left\{ \varphi''(\sigma_u) \log \frac{u^2}{\varphi''(\sigma_u)} \right\}^{1/2}$$

and $\sigma_u(y)$ *is uniquely determined by*

(14) $$-\varphi_y'\big(\sigma_u(y)\big) = u$$

if u *is sufficiently large.*

---

(3) The corresponding problem concerning multiplicative partitions, i.e. factorizations of n into factors $\geqslant y$, has been dealt with by Hensley [7].

(4) Recently Dixmier and Nicolas [3] obtained a sharp asymptotic formula for $p_y(n)$ uniformly in $1 \leqslant y \leqslant n^{1/4}$, which they used to improve on a result of Erdös and Szalay [5] on the number of "practical" partitions.

(5) Here the phrase "incomplete" means that the summation runs over integers free of small prime divisors.

(6) See also Kohlbecker [13].

(7) Note that there is a misprint in Theorem 1 of [16]: Formula (3.6) should read

$$\frac{\sigma\{\varphi''(\sigma)\}^{\varrho+1}}{|\varphi'(\sigma)|^{2\varrho+1}} \leqslant C.$$

(8) For applications of the partition results given in [16] see Herzog/Schwarz [9] and Herzog [10].

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(9) $\varphi'' \nearrow$ means that φ'' is nondecreasing, and $\sigma \searrow 0$, etc., should be interpreted similarly.

Remarks. (i) Applying the mean value theorem and the monotonicity of φ'' shows that

(15) $$-\varphi'(\sigma)+\varphi'(2\sigma) \geqslant \sigma\varphi''(2\sigma).$$

On the other hand,

(15') $$-\varphi'(\sigma)+\varphi'(2\sigma) \leqslant (C_1-1)(-\varphi'(2\sigma))$$

by relation (8). Therefore we have

(16) $$\sigma\varphi''(\sigma) \leqslant C_4(-\varphi'(\sigma))$$

if we set $C_4 = 2(C_1-1)$.

(ii) The inequality

(17) $$-\varphi_y'(\sigma) \leqslant \sigma\varphi_y''(\sigma)$$

is obtained from (7) via differentiation, and (16) shows that

(17') $$\varphi''(\sigma)|\varphi'(\sigma)|^{-2} \to 0 \quad (\sigma \to 0+)$$

if (7) is taken into consideration.

(iii) From (16) and (8) we deduce

$$\sigma\varphi''(\sigma) \leqslant -C_4\varphi'(\sigma) \leqslant -C_1C_4\varphi'(2\sigma) \leqslant 2C_1C_4\sigma\varphi''(2\sigma)$$

implying that

(18) $$\varphi''(\sigma) \leqslant C_5\varphi''(2\sigma).$$

(iv) The relation

(19) $$\sigma_u(y) \searrow 0 \quad (u \nearrow \infty)$$

as well as the right-hand inequality in

(20) $$\tfrac{1}{2}\sigma_u \leqslant \sigma_u(y) \leqslant \sigma_u$$

(both valid for all $y \geqslant 0$) follow immediately from (14), while the left-hand inequality (valid for $0 \leqslant y \leqslant b(u)$) will be proved below.

In view of (17') and (20) it is evident that the remainder term $R(u)$ is of smaller order than the main term in (12).

Proof of Theorem 1. The estimation from above is rather easy. The monotonicity of the function $u \mapsto A_y(u)$ implies that for all y, $T \geqslant 0$

$$f_y(\sigma) \geqslant A_y(T)\int_T^\infty \sigma e^{-u\sigma}du = A_y(T)e^{-T\sigma},$$

so by (6)

(21) $$A_y(T) \leqslant C_6\exp\{\varphi_y(\sigma)+T\sigma\}$$

if $\sigma > 0$.

The right-hand side of (21) attains its minimum at $\sigma = \sigma_T(y)$.

Hence, if $T_0 > 0$ is chosen (10) such that $\sigma_T(y)$ is well-defined for all $T \geqslant T_0$, the estimate

(22) $$A_y(T) \leqslant C_6\exp\{\varphi_y(\sigma_T(y))+T\sigma_T(y)\}$$

is valid for all $T \geqslant T_0$.

In order to find a good lower bound for $\log A_y(T)$, the integral

(23) $$f_y(\sigma_T(y)) = \sigma_T(y)\int_0^\infty A_y(u)\exp(-u\sigma_T(y))du$$

is evaluated by using the Laplace-method.

Motivated by (22) the integrand in (23) is approximated by $\exp\{\psi_y(u)\}$, where

(24) $$\psi_y(u) = \varphi_y(\sigma_u(y))+u\sigma_u(y)-u\sigma_T(y).$$

The derivatives of ψ_y are given by

(25) $$\frac{d}{du}\psi_y(u) = \sigma_u(y)-\sigma_T(y)$$

and

(26) $$\frac{d^2}{du^2}\psi_y(u) = \frac{d}{du}\sigma_u(y) = -\{\varphi_y''(\sigma_u(y))\}^{-1}.$$

This shows that $u = T$ gives the maximum-value of $\psi_y(u)$.

Therefore the integral (23) splits in the following way:

(27) $$f_y(\sigma_T(y)) = \sigma_T(y)\left\{\int_0^{T_0}+\int_{T_0}^{(1-\varepsilon)T}+\int_{(1-\varepsilon)T}^{R}+\int_R^\infty\right\}(A_y(u)e^{-u\sigma_T(y)})du$$
$$= I_y^{(0)}(T)+\ldots+I_y^{(3)}(T),$$

where $R = (1+\varepsilon)T$ and the function $\varepsilon = \varepsilon(T)$ is chosen later such that $0 < \varepsilon < 1/4$.

The main contribution will arise from $I_y^{(2)}(T)$, but at first upper bounds for the other integrals will be deduced.

The first integral is simply estimated by

(28) $$I_y^{(0)}(T) \leqslant A(T_0) = O(1).$$

Using (22) we obtain

(29) $$I_y^{(1)}(T) \leqslant C_6\sigma_T(y)\int_{T_0}^{(1-\varepsilon)T}\exp\{\psi_y(u)\}du.$$

(10) Independent of y.

The Taylor expansion near $u = T$ yields

$$\psi_y(u) = \varphi_y(\sigma_T(u)) + \tfrac{1}{2}(T-u)^2 \psi_y''(\tilde{u})$$

with some real number $\tilde{u}$ between u and T.

For $u \leqslant T$ the inequality

$$\psi_y''(\tilde{u}) = -\{\varphi_y''(\sigma_{\tilde{u}}(y))\}^{-1} \leqslant -\{\varphi_y''(\sigma_T(y))\}^{-1} = \psi_y''(T)$$

is implied by (7) and (19), and consequently we have

$$I_y^{(1)}(T) \leqslant C_6 \sigma_T(y) \exp\{\varphi_y(\sigma_T(y))\} \int_{T_0}^{(1-\varepsilon)T} \exp\left\{-\frac{(T-u)^2}{2\varphi_y''(\sigma_u(y))}\right\} du$$
$$\leqslant C_6 \sigma_T(y) \exp\left\{\varphi_y(\sigma_T(y)) - \frac{1}{2}\frac{\varepsilon^2 R^2}{\varphi_y''(\sigma_R(y))}\right\} \frac{\varphi_y''(\sigma_T(y))}{\varepsilon T}.$$

Thus the definition of R together with the monotonicity-properties of $u \mapsto \sigma_u(y)$ and $\sigma \mapsto \varphi_y''(\sigma)$ gives the estimate

(30) $$I_y^{(1)}(T) \leqslant C_6 \sigma_T(y) \exp\left\{\varphi_y(\sigma_T(y)) - \frac{1}{4}\frac{\varepsilon^2 R^2}{\varphi_y''(\sigma_R(y))}\right\} \frac{2\varphi_y''(\sigma_R(y))}{\varepsilon R}.$$

In order to obtain a similar estimate for $I_y^{(3)}$ it is helpful to approximate the difference $\sigma_T(y) - \sigma_R(y)$ in advance.

An application of the mean value theorem shows the existence of a real number $\tilde{\sigma} \in [\sigma_R(y), \sigma_T(y)]$ such that

$$\varepsilon T = R - T = -\varphi_y'(\sigma_R(y)) + \varphi_y'(\sigma_T(y)) = (\sigma_T(y) - \sigma_R(y))\varphi_y''(\tilde{\sigma}),$$

hence by (17) we obtain

$$\sigma_T(y) - \sigma_R(y) \leqslant \frac{\varepsilon T}{\varphi_y''(\sigma_T(y))} = \frac{-\varepsilon\varphi_y'(\sigma_T(y))}{\varphi_y''(\sigma_T(y))\sigma_T(y)} \sigma_T(y) \leqslant \varepsilon\sigma_T(y).$$

The relation

$$\frac{\sigma_T(y)}{\sigma_R(y)} \leqslant \frac{R}{T} < 2,$$

which is a consequence of (7), is now used to infer that

(31) $$\sigma_T(y) - \sigma_R(y) \leqslant 2\varepsilon\sigma_R(y).$$

From below this difference is bounded by

(32) $$\sigma_T(y) - \sigma_R(y) \geqslant \frac{\varepsilon T}{\varphi_y''(\sigma_R(y))} \geqslant \frac{1}{2}\frac{\varepsilon R}{\varphi_y''(\sigma_R(y))}.$$

For $u > R$ the mean value theorem guarantees the existence of a number $u^* \in [R, u]$ such that

$$\psi_y(u) - \psi_y(R) = (u-R)\psi_y'(u^*),$$

and expanding $\psi_y(R)$ near T yields

$$\psi_y(u) \leqslant \psi_y(T) - \tfrac{1}{2}(R-T)^2|\psi_y''(R)| - (u-R)|\psi_y'(R)|$$
$$= \varphi_y(\sigma_T(y)) - \frac{1}{2}\frac{\varepsilon^2 T^2}{\varphi_y''(\sigma_R(y))} - (u-R)|\psi_y'(R)|.$$

Therefore we get

$$I_y^{(3)}(T) \leqslant C_6 \sigma_T(y) \int_R^\infty \exp\{\psi_y(u)\} du$$
$$\leqslant C_6 \frac{\sigma_T(y)}{|\psi_y'(R)|} \exp\left\{\varphi_y(\sigma_T(y)) - \frac{1}{2}\frac{\varepsilon^2 T^2}{\varphi_y''(\sigma_R(y))}\right\},$$

and by replacing $|\psi_y'(R)| = \sigma_T(y) - \sigma_R(y)$ with the right-hand side of (32) the inequality

(33) $$I_y^{(3)}(T) \leqslant C_6 \sigma_T(y) \exp\left\{\varphi_y(\sigma_T(y)) - \frac{1}{4}\frac{\varepsilon^2 R^2}{\varphi_y''(\sigma_R(y))}\right\}$$

is obtained.

From (30) and (33) it follows that

(34) $$I_y^{(1)}(T) + I_y^{(3)}(T) \leqslant 4C_6 \exp\{\varphi_y(\sigma_T(y))\} e^U,$$

where U is an abbreviation for

(35) $$U = -\frac{1}{4}\frac{\varepsilon^2 R^2}{\varphi_y''(\sigma_R(y))} + \log\left\{\frac{\sigma_T(y)\varphi_y''(\sigma_R(y))}{\varepsilon R}\right\}$$
$$\leqslant -\frac{1}{4}\frac{\varepsilon^2 R^2}{\varphi_y''(\sigma_R(y))} + \log\left\{\frac{2\sigma_R\varphi_y''(\sigma_R(y))}{\varepsilon R}\right\}.$$

The next step consists in obtaining information on the difference $\sigma_u - \sigma_u(y)$ for y in the interval $[0, b(u)]$; this information is then used to verify the relation $U \to -\infty$ uniformly in the given y-range.

Another application of the mean value theorem shows that a point $\sigma^* \in [\sigma_u(y), \sigma_u]$ can be found satisfying

$$0 = u - u = -\varphi'(\sigma_u) + \varphi_y'(\sigma_u(y))$$
$$= (\sigma_u - \sigma_u(y))(-\varphi''(\sigma^*)) + \varphi_y'(\sigma_u(y)) - \varphi'(\sigma_u(y)),$$

and this implies the inequality

$$0 < \sigma_u - \sigma_u(y) \leqslant \frac{-\varphi'(\sigma_u(y)) + \varphi_y'(\sigma_u(y))}{\varphi''(\sigma^*)}.$$

From (10) and the monotonicity of φ'' we infer that

$$\sigma_u-\sigma_u(y)\leqslant C_3\frac{M(y)}{\sigma_u(y)^\alpha\varphi''(\sigma_u)}=C_3\frac{M(y)}{\varphi''(\sigma_u)\sigma_u}\cdot\frac{\sigma_u}{\sigma_u(y)^\alpha}$$

uniformly in $y\leqslant b(u)$.

Estimating the right-hand side above by (17) yields[11]

$$\sigma_u-\sigma_u(y)\leqslant C_3\frac{M(y)}{-\varphi'(\sigma_u)}\cdot\frac{\sigma_u}{\sigma_u(y)^\alpha}$$

$$=C_3\frac{M(y)}{-\varphi_y'(\sigma_u(y))\sigma_u(y)}\sigma_u(y)^{1-\alpha}\sigma_u,$$

and since $\sigma_u(y)\leqslant\sigma_u$ we deduce the bound

$$\sigma_u-\sigma_u(y)\leqslant C_3\frac{M(y)}{-\varphi_y'(\sigma_u)\sigma_u}\sigma_u^{1-\alpha}\sigma_u=C_3\frac{M(y)}{-\varphi_y'(\sigma_u)}\sigma_u^{1-\alpha}.$$

Here (7) was used as well as the assumption $\alpha\leqslant 1$.

Applying (10) to replace $-\varphi_y'(\sigma_u)$ by $-\varphi'(\sigma_u)$ gives[12]

$$\sigma_u-\sigma_u(y)\leqslant C_3\frac{M(b(u))\sigma_u^{1-\alpha}}{-\varphi'(\sigma_u)-C_3\sigma_u^{-\alpha}M(b(u))}$$

$$=C_3\frac{M(b(u))\sigma_u^{1-\alpha}}{u+o(u)}\leqslant 2C_3\frac{M(b(u))}{u\sigma_u^\alpha}\sigma_u$$

uniformly in $y\leqslant b(u)$.

Now the fraction occurring in the last line of the estimate above is $o(1)$ by (11), hence we obtain

(36) $$\sigma_u(y)\geqslant\tfrac{1}{2}\sigma_u\quad\text{uniformly in } 0\leqslant y\leqslant b(u)$$

for all sufficiently large u.

Therefore

$$\varphi_y''(\sigma_R(y))\leqslant C_2\varphi''(\sigma_R(y))\leqslant C_2\varphi''(\tfrac{1}{2}\sigma_R)\leqslant C_2C_5\varphi''(\sigma_R)$$

by (9) and (18), uniformly for y in the range $[0, b(R)]$.

This estimate together with (16) implies the inequalities

$$0<\frac{\sigma_R\varphi_y''(\sigma_R(y))}{R}\leqslant C_2C_5\frac{\sigma_R\varphi''(\sigma_R)}{-\varphi'(\sigma_R)}\leqslant C_2C_5C_4=\tilde{C}.$$

---

[11] Note that $-\varphi'(\sigma_u)=u=-\varphi_y'(\sigma_u(y))$ by definition of $\sigma_u(y)$.

[12] Recall that the function $y\mapsto M(y)$ is assumed to be increasing.

Inserting this result into (35) yields

$$U\leqslant-\frac{1}{4\tilde{C}}\frac{\varepsilon^2R^2}{\varphi''(\sigma_R)}+\log(2\tilde{C})+\log\frac{1}{\varepsilon},$$

hence choosing

$$\varepsilon=\varepsilon(R)=2\sqrt{\tilde{C}}\left\{\frac{\varphi''(\sigma_R)}{|\varphi'(\sigma_R)|^2}\log\frac{|\varphi'(\sigma_R)|^2}{\varphi''(\sigma_R)}\right\}^{1/2}$$

it follows that

$$U\leqslant\log(2\tilde{C})-\frac{1}{2}\log\frac{|\varphi'(\sigma_R)|^2}{\varphi''(\sigma_R)}\to-\infty\qquad(R\to\infty)$$

uniformly in $y\leqslant b(R)$.

Consequently

(37) $$I_y^{(1)}(T)+I_y^{(3)}(T)=o\big(\exp\{\varphi_y(\sigma_T(y))\}\big)\qquad(T\to\infty)$$

uniformly in $y\leqslant b(R)$.

The estimate

$$A_y(R)\exp\{-\sigma_T(y)(1-\varepsilon)T\}\geqslant\sigma_T(y)\int_{(1-\varepsilon)T}^{R}A_y(u)\exp\{-u\sigma_T(y)\}\,du=I_y^{(2)}(T)$$

holds by monotonicity of the function $u\mapsto A_y(u)$, and since

$$I_y^{(2)}(T)=f_y(\sigma_T(y))+o\big(\exp\{\varphi_y(\sigma_T(y))\}\big)$$

by (28) and (37), we get the lower bound[13]

$$\log A_y(R)\geqslant\varphi_y(\sigma_T(y))+T\sigma_T(y)+\log C_7-\varepsilon T\sigma_T(y)$$

$$\geqslant\varphi_y(\sigma_T(y))+T\sigma_T(y)+\log C_7-\varepsilon R\sigma_R.$$

In order to complete the proof we replace $\varphi_y(\sigma_T(y))$ resp. $T\sigma_T(y)$ by $\varphi_y(\sigma_R(y))$ resp. $R\sigma_R(y)$ in the formula above.

In view of (31) we obtain the following inequalities:

$$0<R\sigma_R(y)-T\sigma_T(y)=T\cdot(\sigma_R(y)-\sigma_T(y))+\varepsilon T\sigma_T(y)\leqslant 3\varepsilon R\sigma_R$$

and

$$0<\varphi_y(\sigma_R(y))-\varphi_y(\sigma_T(y))\leqslant(\sigma_T(y)-\sigma_R(y))|\varphi_y'(\sigma_R(y))|\leqslant 2\varepsilon R\sigma_R,$$

where the φ_y-difference was estimated by means of the mean value theorem.

Therefore

$$\log A_y(R)\geqslant\varphi_y(\sigma_R(y))+R\sigma_R(y)-6\varepsilon R\sigma_R+\log C_7$$

---

[13] Observe that $T\sigma_T(y)=-\varphi'(\sigma_T)\sigma_T(y)\leqslant-\varphi'(\sigma_T)\sigma_T\leqslant-\varphi'(\sigma_R)\sigma_R=R\sigma_R$ by (20) and (7).

uniformly for y in the interval $[0, b(R)]$, and this finishes the proof of Theorem 1.

3. Partitions. The present section is devoted to the study of partitions which are free of summands smaller than a parameter y. As an application of Theorem 1 we shall derive asymptotic formulas for the logarithm of such partition functions which are valid uniformly in a certain range for y. Since our approach to the problem is rather general and works for a large class of partition functions it is possible to find much sharper estimates when dealing with special partitions ([14]).

Let (λ_ν) be a strictly increasing unbounded sequence of positive real numbers such that the counting function

$$N(u) = \sum_{\lambda_\nu < u} 1$$

satisfies

$$N(u) \ll_\varepsilon \exp(\varepsilon u) \tag{38}$$

for all $\varepsilon > 0$ and let $k = k(y) = \min\{\nu \in N;\ \lambda_{\nu-1} < y \leqslant \lambda_\nu\}$.

Denote by $\Lambda(y)$ the (countable) set of all real numbers of the form $l = \sum_{\nu \geqslant k} r_\nu \lambda_\nu$, where the r_ν's run through the set of nonnegative integers.

If $l \in \Lambda(y)$, then $p_y(l)$ denotes the number of solutions of the Diophantine equation

$$l = \sum_{\nu \geqslant k} r_\nu \lambda_\nu \tag{39}$$

in nonnegative integers r_ν, i.e. $p_y(l)$ is the number of partitions of l into summands $\geqslant y$.

We are looking for an asymptotic formula for $\log P_y(u)$ with remainder term, where

$$P_y(u) = \sum_{\substack{l \in \Lambda(y) \\ l < u}} p_y(l).$$

The generating function of the sequence $(p_y(l))_{l \in \Lambda(y)}$ is given by

$$g_y(s) = \prod_{\lambda_\nu \geqslant y} \{1 - \exp(-\lambda_\nu s)\}^{-1} = \sum_{l \in \Lambda(y)} p_y(l) e^{-ls},$$

where in view of (38) both the product and the series are convergent in $\operatorname{Re} s > 0$.

([14]) Cf. Erdös and Szalay [4], p. 432 ff. and, in particular, Dixmier and Nicolas [3].

The logarithm of $g_y(s)$

$$\varphi_y(s) = \log g_y(s) = -\sum_{\lambda_\nu \geqslant y} \log\{1 - \exp(-\lambda_\nu s)\}$$

is defined in $\operatorname{Re} s > 0$.

Using only weak assumptions on the enumerating function $N(u)$ the following theorem ([15]) gives the desired uniform estimate for $\log P_y(u)$.

THEOREM 2. *Suppose there is a constant $C > 0$ such that*

$$N(2u) \leqslant CN(u) \tag{40}$$

for all $u > \lambda_1$.

Further let b: $[0, \infty[\ \to [0, \infty[$ be a strictly increasing unbounded function satisfying

$$N(b(u)) = o(u\sigma_u) \quad (u \to \infty). \tag{41}$$

Then uniformly in $0 \leqslant y \leqslant b(u)$ the formula ([16])

$$\log P_y(u) = \varphi_y(\sigma_u(y)) + u\sigma_u(y) + O(R(u)) \tag{42}$$

holds, where $\sigma_u(y)$ is defined by

$$-\varphi_y'(\sigma_u(y)) = u$$

and the error-term is given by

$$R(u) = \sigma_u \cdot \left\{\varphi''(\sigma_u) \log \frac{u^2}{\varphi''(\sigma_u)}\right\}^{1/2}.$$

Sketch of proof. Partial summation shows that in $\operatorname{Re} s > 0$

$$g_y(s) = s\int_0^\infty P_y(u) e^{-us} du,$$

$$\varphi_y'(s) = -\sum_{\lambda_\nu \geqslant y} \frac{\lambda_\nu}{\exp(\lambda_\nu s) - 1} = \int_y^\infty (N(u) - N(y)) \frac{d}{du}(u(e^{us} - 1)^{-1}) du$$

and

$$\varphi_y''(s) = \int_y^\infty \frac{u^2 e^{us}}{(e^{us} - 1)^2} d(N(u) - N(y)).$$

Since

$$-\sigma\varphi_y'(\sigma) = \sum_{\lambda_\nu \geqslant y} \lambda_\nu \sigma \cdot \{e^{\lambda_\nu \sigma} - 1\}^{-1},$$

([15]) Some of the calculations concerned with $\varphi(s)$ may be found in more detail in the paper [16] of Schwarz.

([16]) Recall that $\sigma_u = \sigma_u(0)$ and $\varphi(s) = \varphi_0(s)$.

the monotonicity

$$-\sigma\varphi_y'(\sigma)\nearrow \qquad (\sigma\searrow 0)$$

is obvious.

In order to show that $-\sigma\varphi_y'(\sigma)$ is unbounded as σ decreases to zero, take an arbitrarily large constant K and choose $M > y$ such that $N(u)-N(y)\geqslant K$ for all $u\geqslant M$. Then

$$-\varphi_y'(\sigma)\geqslant -K\int_M^\infty \frac{d}{du}\big(u(e^{u\sigma}-1)^{-1}\big)=\frac{K}{\sigma}\cdot\frac{M\sigma}{e^{M\sigma}-1}\geqslant\frac{K}{2}\sigma^{-1}$$

if σ is sufficiently small.

From

$$-\varphi'(\sigma)=\int_{\lambda_1}^\infty N(u)h(u\sigma)du$$

where $h(w)=-\frac{d}{dw}\cdot\frac{w}{e^w-1}$, we deduce

$$-\varphi'(\sigma)=2\int_{\lambda_1/2}^\infty N(2v)h(2v\sigma)dv$$

$$\leqslant 2C\int_{\lambda_1}^\infty N(v)h(v\cdot 2\sigma)dv+O(1)\leqslant C'\cdot\big(-\varphi'(2\sigma)\big).$$

Condition (10) is verified with $\alpha=1$ and $M(y)=N(y)$ by observing that

$$0\leqslant-\varphi'(\sigma)+\varphi_y'(\sigma)=-\int_{\lambda_1}^y N(u)\frac{d}{du}\big(u(e^{u\sigma}-1)^{-1}\big)du-N(y)\int_y^\infty\frac{d}{du}\big(u(e^{u\sigma}-1)^{-1}\big)du$$

$$\leqslant N(y)\left\{\frac{\lambda_1}{e^{\lambda_1\sigma}-1}-\frac{y}{e^{y\sigma}-1}\right\}+N(y)\frac{y}{e^{y\sigma}-1}\leqslant N(y)\sigma^{-1}.$$

The remaining conditions in Theorem 1 obviously hold, and so the assertion follows.

Example. Consider the sequence $\lambda_\nu=\nu$, i.e. partitions into positive integers. After some simple but lengthy calculations the following results concerning the corresponding partition function $P_y(u)$ are obtained:

If $b(u)$ is a strictly increasing unbounded function satisfying $b(u)=o(u^{1/2})$, then uniformly in $y\leqslant b(u)$

$$\log P_y(u)=\sqrt{\tfrac{2}{3}}\pi u^{1/2}-\tfrac12 y\log u+y\log y-y\{1+\log(\sqrt{6}/\pi)\}+$$

$$+O\big(b^2(u)u^{-1/2}+u^{1/4}\sqrt{\log u}\big).$$

If, in particular, $b(u)=O\big(u^{3/8}(\log u)^{1/4}\big)$, then the same asymptotic formula holds with remainder $O\big(u^{1/4}\sqrt{\log u}\big)$ uniformly in $0\leqslant y\leqslant b(u)$.

Since $n\mapsto p_y(n)$ is a monotonic arithmetical function, the formula remains valid if $P_y(u)$ is replaced by $p_y(N)$, $N\in\mathbf{N}$.

This is an immediate consequence of the inequalities

$$\frac{1}{N}\sum_{n\leqslant N}p_y(n)\leqslant p_y(N)\leqslant\sum_{n\leqslant N}p_y(n).$$

4. Multiplicative functions. The following theorem shows another application of our Tauberian theorem and is a generalization of a result on fast growing multiplicative functions obtained by W. Schwarz ([17], Satz 1).

Theorem 3. *Let λ be a nonnegative multiplicative arithmetical function such that*

(43) $$\sum_p\frac{\lambda^2(p)}{p^2}<\infty$$

and

(44) $$\sum_p\sum_{k\geqslant 2}\frac{\lambda(p^k)}{p^k}<\infty.$$

The function

$$t(x)=\sum_{p<x}\frac{\lambda(p)}{p}\log p$$

is assumed to satisfy

(45) $$t(x)(\log x)^{-1}\to\infty\qquad(x\to\infty)$$

and

(46) $$t(e^{2u})\leqslant Ct(e^u)\qquad(u\geqslant u_0).$$

Further we suppose that(17)

(47) $$\sum_p\frac{\lambda(p)}{p}\log p\cdot p^{-\sigma}\{1-\sigma\log p\}\leqslant 0$$

if $\sigma>0$ is sufficiently small.

Define

$$\varphi_y(\sigma)=\sum_{p>y}\frac{\lambda(p)}{p}p^{-\sigma}\qquad(\sigma>0)$$

(17) In [17], p. 357, Schwarz shows how condition (47) can be replaced by simpler assumptions on $t(x)$.

and

$$\Lambda_y(x) = \sum_{n<x} \chi_y(n)\frac{\lambda(n)}{n},$$

where χ_y is the characteristic function of the positive integers free of prime divisors smaller than y.

If $b(u)$ is an unbounded strictly increasing function satisfying

(48) $$t(b(u)) = o(u) \quad (u \to \infty),$$

then uniformly for y in the range $[0, b(u)]$ the asymptotic formula

(49) $$\log \Lambda_y(e^u) = \varphi_y(\sigma_u(y)) + u\sigma_u(y) + O(R(u))$$

holds true, $\sigma_u(y)$ resp. $R(u)$ again defined by

$$-\varphi'_y(\sigma_u(y)) = u$$

respectively

$$R(u) = \sigma_u \cdot \left\{\varphi''(\sigma_u)\log\frac{u^2}{\varphi''(\sigma_u)}\right\}^{1/2}.$$

Sketch of proof. In $\sigma > 0$ the generating Dirichlet-series is given by

$$f_y(\sigma) = \sum_{n\geq 1} \chi_y(n)\frac{\lambda(n)}{n}n^{-\sigma} = \prod_{p\geq y}\left\{1 + \frac{\lambda(p)}{p}p^{-\sigma} + \frac{\lambda(p^2)}{p^2}p^{-2\sigma} + \ldots\right\},$$

hence

$$f_y(\sigma) = \sigma \int_{\log y}^{\infty} \Lambda_y(e^u)e^{-u\sigma}\,du \quad (\sigma > 0)$$

by partial summation.

The relation

$$\log f_y(\sigma) = \varphi_y(\sigma) + O(1)$$

is derived from the Euler-product representation and (44).

The monotonicity

$$-\sigma\varphi'_y(\sigma) \nearrow \infty \quad (\sigma \searrow 0)$$

follows from (47).

Since

$$-\varphi'_y(\sigma) = \sigma \int_{\log y}^{\infty} \{t(e^u) - t(y)\} e^{-u\sigma}\,du,$$

a short calculation shows that

$$0 \leq -\varphi'(\sigma) + \varphi'_y(\sigma) \leq t(y).$$

Therefore condition (10) of Theorem 1 is satisfied by choosing $\alpha = 0$ and $M(y) = t(y)$.

The remaining conditions are easily verified by using some of the results given in [17].

So the assertion follows from Theorem 1 again.

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