Daher folgt für diese $K$ mit $\eta := z^{-1} \log y$ aus (8.2) und (8.5)

$$S \ll K^{-1} MN + (K^2 \eta^2 M^3 N^7)^{1/6} + (K^2 \eta^3 MN^3)^{1/10} + M^{1/2} N \log y + (K\eta N)^{1/2}.$$ 

Setzt man $K := (\eta^{-2} M^2 N)^{1/7}$, so ist $0 < K < M^{3/7} < y$ und $K^{-1} MN = (K^2 \eta^2 M^3 N^7)^{1/5}$, und mit $A := (\eta^2 M^5 N^8)^{1/14}$, $B := M^{1/2} N \log y$ und $C := (\eta^2 M^5 N^8)^{1/14}$ gilt

$$S \ll A^2 + A \eta^{1/2} + B + C$$

für $K > 1$.

Ferner gilt

$$S \ll A^2$$

für $K \leq 1$,

da $S \ll MN$ und $A^2 = K^{-1} MN$ ist. Ich zeige nun

$$A \eta^{1/2} \gg C, \quad A^2 \gg B, \quad A \gg \eta^{1/2}$$

für $K > 1$.

$A \eta^{1/2} \gg C$ ist zu $\eta^4 M^2 \gg N^2$ äquivalent und daher erfüllt. $A^2 \gg B$ ist zu $(y/z)^4 (M/N)^3 M \gg \log ^{10} y$ äquivalent. Aus (8.1) folgt aber $(y/z)^4 (M/N)^3 M > (y/z)^{1/2} \cdot 1.2 \cdot z^{1/2} = y^{1/2} \log ^{10} y$. Schließlich ist $A \gg \eta^{1/2}$ zu $M^2 N^5 \gg \eta^5$ äquivalent. Wegen $K > 1$ ist $M^2 N^5 > \eta^5$, und daher $M^2 N^6 \gg (M^2 N^5)^{1/7} > \eta^2$.

(8.6)–(8.8) ergeben

$$S \ll A^2 = \{y^2 z^{1/2} - 2 (N/M)^6 - 10 \log ^{2} y\}^{1/7},$$

und Hilfssatz 5 ist bewiesen.

**Weitere Literatur**


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**On linear recurrence relations satisfied by Pisot sequences**

**Addenda and errata**

by

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The proof of Theorem 4 of [1] contains the incorrect inequality $t! < t^{1 + 1/t}$. If this is replaced by the correct $t! < t^{1 + 1/t}$ the method of proof yields a weaker theorem, namely that $\theta$ is recurrent if $\theta > 11 d \log d$. We present here a corrected proof of a stronger assertion, giving more details than presented in [1]. I would like to thank Mustapha Ben Amri Bettaieb for pointing out this error.

**Theorem 4.** Let $\theta$ be a Pisot or Salem number of degree $d$. If $\theta > 5 d \log d$ then $\theta$ is recurrent. In fact, as $d \to \infty$, the assumption that $\theta > (2 + o(1)) d \log d$ implies that $\theta$ is recurrent. Furthermore the set of $K$ admissible for $\theta$ is finite and effectively computable.

**Proof.** By Theorem 2 of [1], $\theta$ is recurrent if $\theta > 2^{d-2}$. Since $2^{d-2} < 5 d \log d$ for $d \leq 8$ this implies Theorem 4 for these values of $d$, so we may assume $d > 8$. By Lemma 2 and Theorem 1 (a), $\theta$ will be recurrent if we can determine positive integers $t$ and $L$ so that $E^{-t} > t! 3^3 \theta^t$ and $L < (\theta - 1)^2$.

Choosing $L = (\theta - 1)^2$ the second inequality holds and, since $L < (\theta - 1)^2 - 1 = \theta(\theta - 2)$, it suffices for the first inequality that $t! 3^3 \theta^t$ Taking logarithms, it suffices to have $F(t, \theta, t, d > 0)$, where

$$F(t, \theta, t, d) = (t - d) \log \theta(t - d) - d \log \log(t - d) - d \log 3.$$ 

Differentiating $F$ with respect to $\theta$ we find that $F_\theta > 0$ if $\theta > 2$ and $t \geq 2d$. Thus, for each $t \geq 2d$ there is a unique solution $\theta_0$ of $F(t, \theta, t, 0 = 0$ and $\theta > \theta_0$ implies $F(t, \theta, t, 0 = 0$. For given $d$, it is natural to choose $t$ so as to minimize $\theta_0$. For large $d$, $\theta$ and $t$, $F \approx (t - 2d) \log(t - 2)$ so that $\theta = t \log(t - 2d)$. An elementary calculation shows that the minimum of $\theta$ occurs for $t = (2 + o(1)) d \log d$ as $d \to \infty$ with the corresponding value of $\theta_0$ $= (2 + o(1)) d \log d$. This suggests the second assertion of the Theorem. A more precise analysis is given below.

To obtain results valid for all $d \geq 9$, we first examine $F(t, \theta, t, 4)$ numerically for small $d$. For example, we find that, for $d = 9$, the minimum value of $\theta_0$ occurs for $t = 95$ giving $\theta_0 = 97.2978 = 4.920 d \log d$. For $d = 10$, the minimum
occurs at \( t = 108 \), giving \( \theta_0 = 110.0895 = 4.781 \, d \log d \). These prove the first statement of the Theorem for \( d = 9 \) and 10, and suggest that a reasonable choice of \( t \) is \( t = kd \log d \), for some \( k \approx 5 \) and that \( \theta_0 > t \) for this choice of \( t \).

Assuming then that \( t = kd \log d \), where \( k \) will be specified shortly, it suffices to prove that \( F(t, t, d) > 0 \). We assume that \( t \geq 95 \) so that \( \log ((t-2)/t) > \log (93/95) \) and use the inequality \( t^t e^{-t} > t! \). We then have

\[
F(t, t, d) \geq -(2d+1) \log t + t - d \log 3 + (t-d) \log (93/95),
\]

which is an increasing function of \( t \) for the set of \( t \) and \( d \) under consideration. Thus \( F(t, t, d) > g(d) \) where

\[
g(d) = (Ak - 2) d \log d - (2d+1) \log \log d - (2 \log k + B) d - \log d - \log k,
\]

where \( A = 1 + \log (93/95) \) and \( B = \log 3 + \log (93/95) \). If we choose \( k = 4.94 \), we find that \( g \) is increasing for \( d \geq 9 \) and that \( g(10) > 0 \), and thus that \( F(kd \log d, kd \log d, d) > 0 \) if \( d \geq 10 \) and \( k \geq 4.94 \). To insure that \( t \) is an integer, we take

\[
t = \lfloor 5d \log d \rfloor = kd \log d \]

with \( k = 5 - 1/d \log d \geq 4.94 \), provided \( d \log d \geq 50/3 \) which is true if \( d \geq 9 \). This proves the first claim of the Theorem for \( d \geq 10 \).

To prove the second statement, we again examine \( F(t, t, d) \) for \( t = kd \log d \), where now \( k = 2 + c(\log \log d/d \log d) \). Since now \( A = 1 + O(1/\log d) \), it is easy to verify that \( F(t, t, d) > 0 \) for a suitable choice of the constant \( c \). This proves the second assertion of the Theorem.

The final statement of the Theorem follows as in [1].

References


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