An analogous argument using Propositions 4.2, 6.1, 6.3 and 7.3 proves the theorem in the case that $m = 2k+2$.

Now we consider the case where $m = 2k+1$. Take e and ϱ as in Proposition 6.4. Then for $x \in \mathfrak{P}\tilde{L}$ with $Q(x) \notin \mathfrak{P}^3 N(\tilde{L})$,

$$\varrho x \in \varrho\mathfrak{P}\tilde{L}_{\mathfrak{P}} = \mathfrak{P}^e \partial\tilde{L}_{\mathfrak{P}} = \mathfrak{P}^e L^{\#}_{\mathfrak{P}} = L_{\mathfrak{P}}.$$

(Recall that by Proposition 2.1, $L_{\mathfrak{P}}$ is $N(L)\mathcal{O}_{\mathfrak{P}}$-modular.) Hence ϱx is anisotropic in the space $\mathfrak{P}^{-e} L_{\mathfrak{P}}/\mathfrak{P}^{1-e} L_{\mathfrak{P}}$, and so

$$(\operatorname{disc}(\mathfrak{P}^{-e} L)_{\mathfrak{P}}|\mathfrak{P}) = ((-1)^{(m-1)/2} Q(\varrho x)|\mathfrak{P}).$$

Now Propositions 6.4 and 7.3 yield the result of the theorem. ∎

Remark. In the case that $m = 2k+2$, we could use the methods used above to write $\Theta(L, \tau)|T(\mathfrak{P})$ as a linear combination of theta series $\{\Theta(\mathfrak{P}^{-1} L', \tau)|\ L'$ is a $\mathfrak{P}$-sublattice of $L\}$ and $\Theta(\mathfrak{P}^{-1} L, \tau)$. However, these theta series do not lie in the same space of modular forms as $\Theta(L, \tau)|T(\mathfrak{P})$.

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On Dirichlet's theorem concerning diophantine approximation

by

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1. Introduction.

(i) Let $\alpha_1, \ldots, \alpha_n$, $n \geq 2$, be given real numbers. According to Dirichlet there exist infinitely many integer points $(\xi_1, \ldots, \xi_{n+1}) \in \mathbf{Z}^{n+1}$ such that

$$|\alpha_1 \xi_1 + \ldots + \alpha_n \xi_n + \xi_{n+1}| \leq (\max_{1 \leq \nu \leq n} |\xi_\nu|)^{-n}.$$

We will show that essentially this still holds, if for the approximation of $\alpha_1, \ldots, \alpha_n$ one allows only integer points $(\xi_1, \ldots, \xi_{n+1})$ in certain subsets of $\mathbf{R}^{n+1}$. In other words, we shall prove that the effectivity in Dirichlet's theorem can be replaced by a condition concerning the position of the approximating integer points.

(ii) In what follows, an integer point is always an element of $\mathbf{R}^{n+1}$ with integer coordinates $\xi_1, \ldots, \xi_{n+1}$ and ε and δ are any positive real numbers. For $\mathscr{X} = (\xi_1, \ldots, \xi_{n+1}) \in \mathbf{R}^{n+1}$ put

$$L(\mathscr{X}) = \sum_{\nu=1}^{n} \alpha_\nu \xi_\nu + \xi_{n+1}, \quad \{\mathscr{X}\} = \max_{1 \leq \nu \leq n} |\xi_\nu|, \quad \varrho(\mathscr{X}) = (\sum_{\nu=1}^{n-1} \xi_\nu^2)^{1/2}.$$

For real w let

$$\Phi(w) = \{\mathscr{X} \in \mathbf{R}^{n+1}|\ |\xi_n| \leq (1+\varepsilon)\varrho(\mathscr{X})^w\} \cup \{\mathscr{X} \in \mathbf{R}^{n+1}|\ \varrho(\mathscr{X}) \leq 1\};$$
$$\Psi = \{\mathscr{X} \in \mathbf{R}^{n+1}|\ |\xi_n| \leq \varepsilon\varrho(\mathscr{X})\}.$$

THEOREM 1. (a) *If*

(0) $$w = w(n) = 1 + 1/n + 1/n^2,$$

then there exist infinitely many integer points $\mathscr{G}$ *such that*

$$\mathscr{G} \in \Phi(w) \quad \textit{and} \quad |L(\mathscr{G})| \leq (1+\delta)\{\mathscr{G}\}^{-n}.$$

(b) *If*

(1) $$v = v(n) = \tfrac{1}{2}(n - 1 + \sqrt{n^2 + 2n - 3})$$

and if the numbers $1, \alpha_1, \ldots, \alpha_n$ *are* $\mathbf{Q}$*-linearly independent, then there exist infinitely many integer points* $\mathscr{G}$ *such that*

$$\mathscr{G} \in \Psi \quad \text{and} \quad |L(\mathscr{G})| \leqslant \delta \{\mathscr{G}\}^{-v}.$$

THEOREM 2. *Let* $\varkappa$ *be real,* $1/n < \varkappa \leqslant 1/(n-1)$. *Suppose the numbers* $1, \alpha_1, \ldots, \alpha_n$ *are* $\mathbf{Q}$*-linearly independent and the inequality*

$$\max_{1 \leqslant v \leqslant n} |\alpha_v y - q_v| \leqslant y^{-\varkappa} \tag{2}$$

has only finitely many solutions in integers $y, q_1, \ldots, q_n$.

If

$$u < u_0 = u_0(n, \varkappa) = \frac{1}{2n\varkappa}\{\varkappa(n-1)^2 + 1 + \sqrt{(\varkappa(n-1)^2+1)^2 + 4n^2\varkappa^2(n-1)}\}, \tag{3}$$

then there exist infinitely many integer points $\mathscr{G}$ *such that*

$$\mathscr{G} \in \Psi \quad \text{and} \quad |L(\mathscr{G})| \leqslant \{\mathscr{G}\}^{-u}. \tag{4}$$

(iii) Remarks. 1. We have $u_0(n, 1/n) = n$ and $u_0(n, 1/(n-1)) = v(n)$.

2. Theorems 1 and 2 will be proved with $\Phi(w)$ and Ψ replaced by more general sets. Let Λ: $\mathbf{R}^n \to \mathbf{R}^n$, $(\xi_1, \ldots, \xi_n) \to (x_1, \ldots, x_n)$ be any transformation of the form

$$x_v = \sum_{j=1}^{n} k_{vj}\xi_j, \quad v = 1, \ldots, n,$$

where the numbers k_{vj} are real and such that

$$\text{for } D = \det(k_{vj}) \text{ we have } D \neq 0. \tag{5}$$

Define

$$\langle \mathscr{X} \rangle = \max_{1 \leqslant v \leqslant n} |x_v|; \quad r(\mathscr{X}) = \Big(\sum_{v=1}^{n-1} x_v^2\Big)^{1/2}.$$

Then there exists a bound $d = d(\Lambda) > 0$ such that for every $\mathscr{X} \in \mathbf{R}^{n+1}$

$$\frac{1}{d}\{\mathscr{X}\} \leqslant \langle \mathscr{X} \rangle \leqslant d\{\mathscr{X}\}. \tag{6}$$

In particular $d(\mathbf{1}) = 1$, where $\mathbf{1}$ denotes the identity.

Let

$$\Phi(\Lambda, w) = \{\mathscr{X} \in \mathbf{R}^{n+1} |\ |x_n| \leqslant (1+\varepsilon) r(\mathscr{X})^w\} \cup \{\mathscr{X} \in \mathbf{R}^{n+1} |\ r(\mathscr{X}) \leqslant 1\};$$
$$\Psi(\Lambda) = \{\mathscr{X} \in \mathbf{R}^{n+1} |\ |x_n| \leqslant \varepsilon r(\mathscr{X})\}.$$

We will show more generally that the hypotheses of Theorem 1 (a) respectively 1 (b) imply that there exist infinitely many integer points $\mathscr{G}$ with

$$\text{(a)} \quad \mathscr{G} \in \Phi(\Lambda, w) \text{ and } |L(\mathscr{G})| \leqslant (1+\delta) d^{2n}\{\mathscr{G}\}^{-n}, \tag{7}$$

$$\text{(b)} \quad \mathscr{G} \in \Psi(\Lambda) \text{ and } |L(\mathscr{G})| \leqslant \delta\{G\}^{-v}. \tag{8}$$

Similarly it will be proved that

(9) Theorem 2 holds with Ψ replaced by $\Psi(\Lambda)$.

3. The case $n = 2$ of (8) is due to W. M. Schmidt [6], who was basically interested in diophantine approximation by positive integers. He also has given an example ([6], remark C), showing that Theorems 1 (b) and 2 no longer hold if the numbers $1, \alpha_1, \ldots, \alpha_n$ are allowed to be $\mathbf{Q}$-linearly dependent. For $n = 2$, results similar to Theorems 1 (a) and 2 could be proved by his method ([8]).

4. For $n \geqslant 3$ the following result of W. M. Schmidt gives a "lower bound" for the sets $\Phi(\Lambda, w)$ and $\Psi(\Lambda)$ in (7) and (8), making clear that these cannot be replaced by the set Θ of points with positive coordinates, $\Theta = \{\mathscr{X} \in \mathbf{R}^{n+1} |\ \xi_v > 0, v = 1, \ldots, n\}$, since ([6], remark F):

> given $n \geqslant 3$ and $\varepsilon > 0$ there are $\alpha_1, \ldots, \alpha_n$ and $c(\varepsilon) > 0$ such that $1, \alpha_1, \ldots, \alpha_n$ are $\mathbf{Q}$-linearly independent and $|L(\mathscr{G})| > c(\varepsilon)\{\mathscr{G}\}^{-2-\varepsilon}$ for all integer points $\mathscr{G} \in \Theta$.

5. C. A. Rogers [5] has studied the "dual" problem to the one we deal with; G. Harman [4] has investigated analogous metrical problems (see also [7]).

6. I am grateful to Professor W. M. Schmidt for encouragement.

II. Proofs of (7) and (8).

(i) Since the number δ in (8) does not depend on Λ, we may replace ε in the definition of $\Psi(\Lambda)$ by any positive number, say 3. Hence

$$\Psi(\Lambda) = \{\mathscr{X} \in \mathbf{R}^{n+1} |\ |x_n| \leqslant 3r(\mathscr{X})\}. \tag{10}$$

Let w and v be the numbers defined in (0) and (1) and let $c = c(\Lambda, n) = (1+\delta)d^{2n}$. We shall give indirect proofs for (7) and (8). Therefore in what follows, we always suppose that one of the following assumptions holds.

ASSUMPTION A. There are at most finitely many integer points $\mathscr{G}$ verifying (7).

ASSUMPTION B. The numbers $1, \alpha_1, \ldots, \alpha_n$ are $\mathbf{Q}$-linearly independent and there are at most finitely many integer points $\mathscr{G}$ verifying (8).

Then, the intersection of $\Phi(\Lambda, w)$ with any linear subspace of $\mathbf{R}^{n+1}$ being infinite, in both cases the numbers

$$1, \alpha_1, \ldots, \alpha_n \text{ are } \mathbf{Q}\text{-linearly independent.} \tag{11}$$

Let $\mathscr{G}_1, \mathscr{G}_2, \ldots$ be a sequence of minimal points, i.e. of integer points with $\langle \mathscr{G}_1 \rangle < \langle \mathscr{G}_2 \rangle < \ldots$ such that for no $j = 1, 2, \ldots$ there is an integer point $\mathscr{G} \neq \mathscr{O}$ with the properties $|L(\mathscr{G})| \leqslant |L(\mathscr{G}_j)|$ and $\langle \mathscr{G} \rangle < \langle \mathscr{G}_j \rangle$.

For $j = 1, 2, \ldots$ put

$$\xi_v(\mathscr{G}_j) = \xi_{v,j}; \quad x_v(\mathscr{G}_j) = x_{vj}, \quad v = 1, \ldots, n;$$
$$L(\mathscr{G}_j) = L_j; \quad \langle \mathscr{G}_j \rangle = N_j; \quad r(\mathscr{G}_j) = r_j.$$

Since any minimal point $\mathcal{G}_j$ can be replaced by $-\mathcal{G}_j$, we may suppose $x_{nj} \geqslant 0$, $j = 1, 2, \ldots$ By Dirichlet's theorem we have with (6)

$$|L_j| \leqslant d^n N_{j+1}^{-n}, \quad j = 1, 2, \ldots \tag{12}$$

Henceforth let m be a sufficiently large subscript. It then follows from (12) that $\mathcal{G}_m \notin \Phi(\Lambda, w)$ if A holds and that $\mathcal{G} \notin \Psi(\Lambda)$ if B holds. Therefore

$$1 < r_m < x_{nm}^{1/w} \leqslant N_m^{1/w}, \quad \text{if A holds} \tag{13}$$

and in any case

$$r_m < \tfrac{1}{3} x_{nm} \leqslant \tfrac{1}{3} N_m \tag{14}$$

by (10). Hence

$$\langle \mathcal{G}_m \rangle = x_{nm} = N_m. \tag{15}$$

(ii) If A holds, then

$$N_m \geqslant N_{m+1}^{1/(n+1)} r_m. \tag{16}$$

Proof. Let $[x]$ denote the integral part of the real number x. For the integer point

$$\mathcal{H} = \mathcal{H}(m) = \left[r_m^{1/(n+1)} \left(\frac{N_m}{r_m^w} \right)^{1/(w-1)} \right] \mathcal{G}_m$$

we have by (13)

$$r(\mathcal{H}) \geqslant \left(r_m^{1/(n+1)} \left(\frac{N_m}{r_m^w} \right)^{1/(w-1)} - 1 \right) r_m = (1 + o(1))\, r_m^{1/(n+1)} \left(\frac{N_m}{r_m} \right)^{1/(w-1)}, \quad m \to \infty,$$

and

$$|x_n(\mathcal{H})| = \langle \mathcal{H} \rangle \leqslant x_{nm} r_m^{1/(n+1)} \left(\frac{N_m}{r_m^w} \right)^{1/(w-1)} \leqslant r_m^{1/(n+1)} \left(\frac{N_m}{r_m} \right)^{w/(w-1)}. \tag{17}$$

Hence

$$\mathcal{H} \in \Phi(\Lambda, w). \tag{18}$$

It follows from (12) that

$$|L(\mathcal{H})| \leqslant d^n r_m^{1/(n+1)} \left(\frac{N_m}{r_m^w} \right)^{1/(w-1)} N_{m+1}^{-n} = d^n \left(\frac{N_m}{r_m} \right)^{1/(w-1)} r_m^{-n/(n+1)} N_{m+1}^{-n}. \tag{19}$$

By the definition (0) of w we have $n(n+1) = (1+nw)/(w-1)$. Now assume indirectly that

$$N_{m+1} > \left(\frac{N_m}{r_m} \right)^{n+1},$$

and hence that

$$N_{m+1}^{-n} < \left(\frac{N_m}{r_m} \right)^{-(1+nw)/(w-1)}.$$

By (19) and (17) we then have

$$|L(\mathcal{H})| \leqslant d^n \left(\frac{N_m}{r_m} \right)^{-nw/(w-1)} r_m^{-n/(n+1)} \leqslant d^n \langle \mathcal{H} \rangle^{-n}.$$

Combined with (18) this contradicts A, which proves (16). In what follows, the constant implied by $\ll$ does not depend on m.

(iii) If B holds, then

$$|L_m| \ll N_{m+1}^{-v/(v+1-n)}. \tag{20}$$

Proof ([6], Lemma 1). Since $n-1 < v$ we can pick σ so large that $D\sigma^{n-1} < \delta(\sigma/d)^v$, where D is defined in (5). Let $N > 1$. According to Minkowski, the set

$$\left\{ \mathcal{X} \in \mathbf{R}^{n+1} \middle| \; |x_\nu| \leqslant \frac{1}{\sigma} N^{1/(v+1-n)}, \; \nu = 1, \ldots, n-1; \; |x_n| \leqslant N; \; |L(\mathcal{X})| \leqslant D\sigma^{n-1} N^{-v/(v+1-n)} \right\}$$

contains an integer point $\mathcal{G} \neq \mathcal{O}$. Since $n > v$ we have $\{\mathcal{G}\} \leqslant d\langle \mathcal{G} \rangle \leqslant N^{1/(v+1-n)} d/\sigma$ and hence

$$|L(\mathcal{G})| \leqslant D\sigma^{n-1} N^{-v/(v+1-n)} < \delta (N^{1/(v+1-n)} d/\sigma)^{-v} \leqslant \delta \{\mathcal{G}\}^{-v}.$$

Now, if N is large then $|L(\mathcal{G})|$ is small and $\langle \mathcal{G} \rangle$ is large by (11). Therefore, since B is supposed to hold, $\mathcal{G} \notin \Psi(\Lambda)$ for large N. It follows that $r(\mathcal{G}) < |x_n(\mathcal{G})|/3 < N/3$ and $\langle \mathcal{G} \rangle \leqslant N$. Hence, given N large enough, there exists an integer point $\mathcal{G} \neq \mathcal{O}$ such that $\langle \mathcal{G} \rangle \leqslant N$ and $|L(\mathcal{G})| \ll N^{-v/(v+1-n)}$. Combined with the definition of minimal points, this implies (20).

(iv) We have

$$|L_{m-1}| \leqslant 2 \frac{N_{m+1}}{N_m} |L_m| + |L_{m+1}|. \tag{21}$$

Proof. Consider the integer point

$$\mathcal{K} = \mathcal{K}(m) = \mathcal{G}_{m+1} - \left(\left[\frac{x_{nm+1}}{x_{nm}} \right] + \mu \right) \mathcal{G}_m,$$

where $\mu \in \{0, 1\}$ is chosen such that

$$|x_n(\mathcal{K})| \leqslant x_{nm}/2 \leqslant N_m/2. \tag{22}$$

By (15) $x_{nm+1}/x_{nm} = N_{m+1}/N_m$. Hence

$$(23)\qquad |L(\mathcal{K})| \leq |L_{m+1}| + \left(\frac{N_{m+1}}{N_m}+1\right)|L_m|$$

and

$$(24)\qquad r(\mathcal{K}) \leq r_{m+1} + \left(\frac{N_{m+1}}{N_m}+1\right)r_m.$$

We next show that

$$(25)\qquad |L(\mathcal{K})| \leq c\{\mathcal{K}\}^{-n}, \quad \text{if A holds,}$$

$$(26)\qquad \leq \delta\{\mathcal{K}\}^{-v}, \quad \text{if B holds.}$$

Indeed, if A holds, it follows from (24), (13), (16) and (0) that for $\nu = 1, \ldots, n-1$

$$(27)\qquad |x_\nu(\mathcal{K})| \leq r(\mathcal{K}) < N_{m+1}^{1/w} + N_{m+1}^{n/(n+1)} + N_m^{1/w}.$$

Using (23), (12) and (16) we find

$$(28)\qquad |L(\mathcal{K})| \leq d^n\left(N_{m+2}^{-n} + \frac{N_{m+1}^{1-n}}{N_m} + N_{m+1}^{-n}\right) \leq (1+o(1))\,d^n N_{m+1}^{-n^2/(n+1)}, \quad m \to \infty.$$

Now, by (22) and (27) we have either $\{\mathcal{K}\} \leq N_m d/2$ — which implies (25) in view of the first estimate in (28) — or $\{\mathcal{K}\} \leq d(1+o(1))\,N_{m+1}^{n/(n+1)}$, $m \to \infty$, from which we get (25) using the second estimate in (28).

If B holds, we have by (23), (20) and (1)

$$|L(\mathcal{K})| \ll N_{m+2}^{-v/(v+1-n)} + \frac{N_{m+1}}{N_m} N_{m+1}^{-v/(v+1-n)} \ll N_m^{-1} N_{m+1}^{-v}.$$

Since $\{\mathcal{K}\} \ll N_{m+1}$, this proves (26).

Now $\mathcal{K}(m) \neq \mathcal{O}$ for all m. If m is large, $|L(\mathcal{K})|$ is small by (23), hence $\{\mathcal{K}\}$ is large by (11). In view of (25), (26) this implies $\mathcal{K} \notin \Phi(\Lambda, w)$ if A holds and $\mathcal{K} \notin \Psi(\Lambda)$ if B holds, such that in both cases

$$r(\mathcal{K}) < |x_n(\mathcal{K})|/3 \leq N_m/6$$

by (22). Hence $\langle\mathcal{K}\rangle < N_m$. It follows from the definition of minimal points that $|L_{m-1}| < |L(\mathcal{K})|$. This, combined with (23), proves (21).

(v) We have

$$(29)\qquad N_{m+1} \geq N_m + N_{m-1}.$$

Proof. Consider the integer points $\mathcal{M}_\nu = \mathcal{M}_\nu(m)$, $\nu = 1, 2, 3$;

$$\mathcal{M}_1 = \mathcal{G}_{m+1} - \mathcal{G}_m; \quad \mathcal{M}_2 = \mathcal{G}_m - \mathcal{G}_{m-1}; \quad \mathcal{M}_3 = \mathcal{G}_{m+1} - \mathcal{G}_{m-1}.$$

Assume indirectly that $N_{m+1} < N_m + N_{m-1}$. Then by (15) and (14)

$$|x_n(\mathcal{M}_\nu)| < N_m; \quad r(\mathcal{M}_\nu) < (N_{m+1} + N_m)/3 < N_m, \quad \nu = 1, 2, 3.$$

Hence $\langle\mathcal{M}_\nu\rangle < N_m$, $\nu = 1, 2, 3$. Since $\mathcal{M}_\nu \neq \mathcal{O}$ this implies $|L(\mathcal{M}_\nu)| \geq |L_{m-1}|$, $\nu = 1, 2, 3$. Noting that $0 < |L_\mu| \leq |L_{m-1}|$ for $\mu = m-1, m, m+1$, we get

$$L_{m+1}L_m < 0; \quad L_m L_{m-1} < 0; \quad L_{m+1}L_{m-1} < 0.$$

This contradiction proves (29).

(vi) Define the sequences $a_0, a_1, \ldots; b_0, b_1, \ldots$ by

$$a_0 = a_1 = 1; \quad a_k = a_{k-1} + a_{k-2} \quad \text{for } k \geq 2;$$

$$b_0 = 1, \quad b_1 = 2; \quad b_k = 2b_{k-1} + b_{k-2} \quad \text{for } k \geq 2.$$

Then $a_0, a_1, \ldots$ is the Fibonacci sequence and for $k = 1, 2, \ldots$ we have according to [3], § 10.13,

$$(30)\qquad a_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right),$$

$$(31)\qquad b_k = \frac{1}{2\sqrt{2}}\left((1+\sqrt{2})^k - (1-\sqrt{2})^k\right).$$

(vii) For any positive integer k we have

$$(32)\qquad \text{(a)}\quad N_{m+k} \geq a_{k+1} N_{m-1}.$$

$$(33)\qquad \text{(b)}\quad |L_{m-1}| \leq b_k \frac{N_{m+k}}{N_m}|L_{m+k-1}| + b_{k-1}\frac{N_{m+k-1}}{N_m}|L_{m+k}|.$$

$$(34)\qquad \text{(c)}\quad N_{m+k} \leq b_k^{1/(n-1)}\left(2d^n N_m^{-1}|L_{m-1}|^{-1}\right)^{1/(n-1)}.$$

Proof. Formula (32) is an immediate consequence of (29). To prove (33) we note that for $k = 1$ (33) holds by (21). Assume now (33) to hold for k. Then, from (21) with m replaced by $m+k$ we find

$$|L_{m-1}| \leq b_k \frac{N_{m+k}}{N_m}\left(2\frac{N_{m+k+1}}{N_{m+k}}|L_{m+k}| + |L_{m+k+1}|\right) + b_{k-1}\frac{N_{m+k-1}}{N_m}|L_{m+k}|$$

$$\leq (2b_k + b_{k-1})\frac{N_{m+k+1}}{N_m}|L_{m+k}| + b_k\frac{N_{m+k}}{N_m}|L_{m+k+1}|,$$

which proves (33), since $2b_k + b_{k-1} = b_{k+1}$.

From (33) we get (34) using (12).

(viii) The case $n \geq 3$. By (32) and (30) respectively (34) and (31) there are positive bounds d_1, d_2, which do not depend on k and such that

$$d_1\left(\frac{1+\sqrt{5}}{2}\right)^k \leq d_2\left((1+\sqrt{2})^k\right)^{1/(n-1)} \leq d_2\left((1+\sqrt{2})^{1/2}\right)^k, \quad n \geq 3.$$

Since $(1+\sqrt{5})/2 > (1+\sqrt{2})^{1/2}$, this cannot hold for large k. This contradiction proves (7) and (8) for $n \geq 3$.

(Observing that if B holds, (20) and (33) yield a better estimate for N_{m+k} than (34), one could prove (8) even for $n = 2$ in the same way.)

(ix) The case $n = 2$. Following the arguments in [2], Lemma 3, and using (12), we see that

(35) the minimal points $\mathscr{G}_{m-1}$, $\mathscr{G}_m$, $\mathscr{G}_{m+1}$ are linearly independent for infinitely many m.

For such an m we have

$$(36)\quad 1 \leqslant \left|\det\begin{bmatrix} \xi_{m-1,1} & \xi_{m-1,2} & \xi_{m-1,3} \\ \xi_{m,1} & \xi_{m,2} & \xi_{m,3} \\ \xi_{m+1,1} & \xi_{m+1,2} & \xi_{m+1,3} \end{bmatrix}\right|$$

$$= \left|\det\begin{bmatrix} \xi_{m-1,1} & \xi_{m-1,2} & L_{m-1} \\ \xi_{m,1} & \xi_{m,2} & L_m \\ \xi_{m+1,1} & \xi_{m+1,2} & L_{m+1} \end{bmatrix}\right| \ll \left|\det\begin{bmatrix} x_{m-1,1} & x_{m-1,2} & L_{m-1} \\ x_{m,1} & x_{m,2} & L_m \\ x_{m+1,1} & x_{m+1,2} & L_{m+1} \end{bmatrix}\right|.$$

Let D_ν, $\nu = m-1, m, m+1$, denote the algebraic complement of L_ν in this last matrix. Then we have by (13)

$$|D_\nu| \leqslant 2N_m^{1/w} N_{m+1}, \quad \text{if A holds,}$$
$$\leqslant 2N_m N_{m+1}, \quad \text{if B holds,}$$

and by (21), (12) and (20)

$$|L_\nu| \leqslant |L_{m-1}| \leqslant 3\frac{N_{m+1}}{N_m}|L_m| \ll (N_m N_{m+1})^{-1}, \qquad \text{if A holds,}$$
$$\leqslant N_m^{-1} N_{m+1}^{-1/(\nu-1)}, \qquad \text{if B holds,}$$

such that in both cases

$$L_\nu D_\nu = o(1), \quad m \to \infty, \quad \nu = m-1, m, m+1.$$

This contradicts (35), (36) and hence proves (7) and (8) for $n = 2$.

III. On the proof of (9).

(i) We can prove (9) by similar arguments. Assume again that the numbers $1, \alpha_1, \ldots, \alpha_n$ are $\boldsymbol{Q}$-linearly independent and that there are at most finitely many integer points $\mathscr{G}$ verifying (4) with Ψ replaced by $\Psi(\Lambda)$. Define the sequence $\mathscr{G}_1, \mathscr{G}_2, \ldots$ of minimal points as in II, (i). Then (15) holds and by the arguments in [6], Lemma 1 (see II, (ii)) we have

$$(37)\qquad |L_m| \ll N_{m+1}^{-u/(u+1-n)}.$$

(ii) Let

$$\omega_0 = \omega_0(n, \varkappa) = \frac{n^2(n\varkappa-1)}{1-(n-1)(n\varkappa-1)}$$

and let $\omega > \omega_0$. According to Khintchin's transference principle ([1], Chap. V, Theorem IV), the assumption (2) implies

$$|L_m| > N_m^{-n-\omega}.$$

It follows from (37) that

$$(38)\qquad N_m \gg N_{m+1}^{u/(u+1-n)(n+\omega)}.$$

(iii) We have

$$(39)\qquad 1-\frac{u_0}{u_0+1-n}-\frac{u_0}{(u_0+1-n)(n+\omega_0)} = -u_0.$$

Proof. Equation (39) is equivalent to

$$(1+u_0)(u_0+1-n)(n+\omega_0)-u_0(n+\omega_0)-u_0 = 0$$

and hence to

$$u_0^2-u_0\left(n-1+\frac{1}{n+\omega_0}\right)-n+1 = 0.$$

Noting that

$$n-1+\frac{1}{n+\omega_0} = \frac{n^2\varkappa-2n\varkappa+\varkappa+1}{n\varkappa},$$

we find that (39) is equivalent to the equation

$$u_0^2 n\varkappa - u_0(\varkappa(n-1)^2+1) - n\varkappa(n-1) = 0,$$

which holds in view of the definition (3) of u_0.

(iv) We have

$$(21)\qquad |L_{m-1}| \leqslant 2\frac{N_{m+1}}{N_m}|L_m|+|L_{m+1}|.$$

Proof. Define the integer point $\mathscr{K} = \mathscr{K}(m)$ as in II, (iv). Then (23) holds and using (15), (23), (37) and (38) we get

$$(40)\qquad |L(\mathscr{K})| \leqslant N_{m+2}^{-u/(u+1-n)}+\frac{N_{m+1}}{N_m}N_{m+1}^{-u/(u+1-n)}$$
$$\leqslant 2N_{m+1}^{1-u/(u+1-n)-u/(u+1-n)(n+\omega)}.$$

Since $u < u_0$, by (39) we can choose $\omega > \omega_0$ such that with (40) and $\langle\mathscr{K}\rangle \ll N_{m+1}$ we have

$$|L(\mathscr{K})| \leqslant o(1) N_{m+1}^{-u} < \langle\mathscr{K}\rangle^{-u}, \quad m \to \infty.$$

Hence, in view of the indirect assumption, $\mathscr{K}(m) \notin \Psi(\Lambda)$ for large m. By the arguments in II, (iv) this implies (21).

(v) The rest of the proof is the same as for (7) and (8).

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Über die Anzahl quadratvoller Zahlen in kurzen Intervallen und ein verwandtes Gitterpunktproblem Corrigendum zu Acta Arithmetica L(1988), 195–201 (1)

von

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6. Vorbemerkungen. Wie ich in §7 darlegen werde, ist der auf B. R. Srinivasan zurückgehende Hilfssatz 2 nicht gesichert. Darauf beruhen aber die Sätze 1 und 2 und (1.1)–(1.2). Statt dessen werde ich in §8 das etwas schwächere Resultat $\Delta_{2,3,6}(x) \ll (x\log^{16} x)^{1/7}$ erzielen, also (1.1) für $1/7 < \theta < 1/2$ beweisen. Herr H. Menzer (Jena) hat mir in einem Brief vom 27.3.88 mitgeteilt, daß er $\Delta_{2,3,6}(x) \ll x^{7/48}\log^3 x$ zeigen kann.

7. Srinivasansche Exponentenpaare. B. R. Srinivasan [8] verallgemeinerte die Phillipssche Theorie der Exponentenpaare zur Abschätzung eindimensionaler Exponentialsummen [4] auf den mehrdimensionalen Fall.

W. G. Nowak [13] äußerte, ohne jedoch konkrete Fehlerhinweise zu geben, Zweifel an der Korrektheit der Srinivasanschen Theorie. Ich teile seine Auffassung:

Srinivasan übersieht, daß die in [15] auf Seite 333 auftretenden Größen ξ_1' im allgemeinen von x_2 abhängen: $\xi_1' = \xi_1'(x_2)$. Statt $|(\partial/\partial x_2)^2 f(x_1, x_2)| \geqslant r_2$ bräuchte er $|(d/dx_2)^2 f(\xi_1'(x_2), x_2)| \gg r_2$. Gravierender dürfte diese Ungenauigkeit in den Beweisen der Lemmata 4, 5 und 6 in [16] sein (siehe etwa [16], Seite 181, Zeilen 10–17). Sie wird sich meines Erachtens, wenn überhaupt, nur mit großem Aufwand beheben lassen. Auf den Lemmata 4, 5 und 6 in [16] basieren aber die dortigen Theoreme 1 und 2 und darauf die in [8] entwickelte Theorie der Exponentenpaare. Daher ist auch [9], Theorem 5 und folglich Hilfssatz 2 nicht gesichert.

8. Corrigendum.

SATZ 4. *Für* $x \to \infty$ *gilt* $\Delta_{2,3,6}(x) \ll x^{1/7}(\log x)^{2+2/7}$.

Satz 4 folgt unmittelbar aus Hilfssatz 3 und dem Korollar zu

(1) In dieser Arbeit finden sich die §§1–5, Sätze 1–3, Hilfssätze 1–4, Formeln (1.1)–(5.3) und Literaturhinweise [1]–[11].