que $k^{(2)} = H_k(i)$ et que le conjugué de $\sqrt{\lambda(\tau)}$ dans $k^{(2)}/H_k$ est $-1/\sqrt{\lambda(\tau)}$; par conséquent $\mu \in k^{(f)}$; le théorème est démontré dans ce cas.

Supposons maintenant 3 inerte dans k/Q et soit β un point de 3-division non nul de E_τ. Comme précédemment $iT(\beta)/\sqrt{\lambda(\tau)}$ appartient à $k^{(3)}$. On peut, en procédant comme dans le § 11, trouver $b \in H_k$ tel que $iT(\beta)/\sqrt{\lambda(\tau)} \equiv b$ (2). On pose alors

$$\mu = i\frac{1}{2}\left(i\frac{T(\alpha)}{\sqrt{\lambda(\tau)}} - b\right).$$

Il est alors immédiat que μ est un entier de $k^{(f)}$ et qu'il engendre avec ses puissances l'anneau des entiers de $k^{(f)}$ relativement à H_k.

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EQUIPE DE MATHÉMATIQUES DE BESANÇON
C.N.R.S. – U.A. 741
UNIVERSITÉ DE FRANCHE-COMTÉ
25030 Besançon Cedex
France

Reçu le 1.3.1988
et dans la forme modifiée le 3.10.1988 (1797)

ACTA ARITHMETICA
LIV (1990)

Hecke operators on theta series attached to lattices of arbitrary rank

by

LYNNE H. WALLING (Lewiston, Me.)

1. Preliminaries. Let K be a totally real algebraic number field of degree n over Q; let $\mathcal{O}$ denote the ring of integers of K and ∂ the different of K. Let V be a quadratic space of dimension m over K with totally positive quadratic form Q and associated bilinear form B where $B(x, x) = Q(x)$. Take L to be a lattice on V (so $KL = V$). Let $\mathcal{H}$ denote the upper half-plane; then for $\tau = (\tau_1, \ldots, \tau_n) \in \mathcal{H}^n$, define

$$\Theta(L, \tau) = \sum_{x \in L} e(Q(x)\tau)$$

where $e(\alpha) = e^{\pi i \operatorname{Tr}(\alpha)}$. Notice that $e(\alpha) = 1$ whenever $\alpha \in 2\partial^{-1}$. For $y \in V$, define

$$\Theta(L, y, \tau) = \sum_{x \in L} e(Q(x+y)\tau).$$

So when $y \in L$, $\Theta(L, y, \tau) = \Theta(L, \tau)$.

As defined in Eichler [6], let $\tilde{L}$ denote the complement of L, $N(L)$ the norm of L, and $\mathcal{N}(L) = N(L)^{-1} N(\tilde{L})^{-1} \partial^2$ the level of L. Notice that $\tilde{L} = \partial^{-1} L^\#$ where $L^\#$ is the dual of L (as defined in [12]), hence $\mathcal{N}(L) = N(L)^{-1} N(L^\#)^{-1}$, which is integral (i.e. $\mathcal{N}(L) \subseteq \mathcal{O}$; see [6]). Also, $x \in \tilde{L}$ if and only if $B(x, L) \subseteq \partial^{-1}$. For $\alpha \in K$, let L^α denote L scaled by α; that is, L^α is the lattice L together with the quadratic and bilinear forms Q^α and B^α defined by

$$Q^\alpha(x) = \alpha Q(x) \quad \text{and} \quad B^\alpha(x, y) = \alpha B(x, y)$$

(see § 89J of [12]). So $N(L^\alpha) = \alpha N(L)$ and $\mathcal{N}(L^\alpha) = \mathcal{N}(L)$.

For fractional ideals $\mathfrak{I}_1$ and $\mathfrak{I}_2$, define

$$\Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, d \in \mathcal{O},\ c \in \mathfrak{I}_1,\ b \in \mathfrak{I}_2,\ ad - bc = 1\right\}.$$

If $\mathfrak{I}_1 \mathfrak{I}_2$ is integral then $\Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)$ is a group. If $\operatorname{ord}_{\mathfrak{P}} \mathfrak{I}_2 = 0$ whenever $\mathfrak{P}$ is a prime ideal with $\operatorname{ord}_{\mathfrak{P}} \mathfrak{I}_1 \neq 0$, then we say $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are relatively prime.

The reader is referred to [12], [2] and [10] for details regarding lattices and quadratic forms.

2. The transformation formula. To prove $\Theta(L, \tau)$ is a Hilbert modular form, we begin by proving a transformation formula. For this we generalize Eichler's inversion formula (see [6]):

$$\Theta(L, y, \tau) = \frac{i^{-mn/2}}{\sqrt{\Phi(L)}} N(\tau)^{-m/2} \sum_{s \in \tilde{L}/L} e\big(2B(y, s)\big)\,\Theta(L, s, -1/\tau)$$

where L is an integral lattice of even rank and $y \in \tilde{L}$; $\Phi(L) = \det\big(\mathrm{Tr}\,(B(x_i, x_j))\big)$ where the set $\{x_1, \ldots, x_{mn}\}$ is a $\mathbf{Z}$-basis for L. Eichler's proof of this formula is independent of the parity of the rank of L. Now we choose a totally positive algebraic integer α such that L^α is an integral lattice and $\alpha y \in \tilde{L}$ (that is, $y \in (\widetilde{L^\alpha})$). Then from Eichler's formula we get that

$$\begin{aligned}\Theta(L, y, \alpha\tau) &= \Theta(L^\alpha, y, \tau)\\ &= \frac{i^{-mn/2}}{\sqrt{\Phi(L^\alpha)}} N(\tau)^{-m/2} \sum_{x \in \alpha^{-1}\tilde{L}} e\left(2\alpha B(y, x) - \alpha Q(x)\frac{1}{\tau}\right)\\ &= \frac{i^{-mn/2}}{\sqrt{\Phi(L^\alpha)}} N(\tau)^{-m/2} \sum_{x \in \tilde{L}} e\left(2B(y, x) - Q(x)\frac{1}{\alpha\tau}\right).\end{aligned}$$

It follows from the definition of Φ that $\Phi(L^\alpha) = N(\alpha)^m\,\Phi(L)$; replacing $\alpha\tau$ with τ we now get

$$(1) \qquad \Theta(L, y, \tau) = \frac{i^{-mn/2}}{\sqrt{\Phi(L)}} N(\tau)^{-m/2} \sum_{x \in \tilde{L}} e\left(2B(y, x) - Q(x)\frac{1}{\tau}\right)$$

where L is any lattice and y is any vector in V.

Using (1), we derive a transformation formula for $\Theta(L, \tau)$. We essentially follow the derivation of the transformation formula presented in [6]; however, Eichler's final formula contains some inappropriate factors, and we sometimes need to impose an additional restriction.

Take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0\big(\mathcal{N}(L) N(L)\partial,\, N(L)^{-1}\partial^{-1}\big)$ such that $d \gg 0$ and d is relatively prime to $\mathcal{N}(L)\partial$ and to $N(L)$. Following Eichler we write

$$\Theta\left(L, \frac{a\tau + b}{c\tau + d}\right) = \sum_{x \in L} e\left(\frac{b}{d}Q(x) + \frac{Q(x)}{d\left(d\frac{1}{\tau} + c\right)}\right).$$

Since $\frac{b}{d}Q(x_0 + dx) \equiv \frac{b}{d}Q(x_0) \pmod{2\partial^{-1}}$ for any $x_0, x \in L$, we have

$$\Theta\left(L, \frac{a\tau + b}{c\tau + d}\right) = \sum_{x_0 \in L/dL} e\left(\frac{b}{d}Q(x_0)\right) \sum_{x \in L} e\left(\frac{Q(dx + x_0)}{d\left(d\frac{1}{\tau} + c\right)}\right)$$

and by (1),

$$\begin{aligned}&= \frac{i^{-mn/2}}{\sqrt{\Phi(L)}} N(d)^{-m/2} N\left(d\frac{1}{\tau} + c\right)^{m/2}\\ &\quad\times \sum_{\substack{x_0 \in L/dL\\ x \in \tilde{L}}} e\left(\frac{b}{d}Q(x_0) + \frac{2}{d}B(x, x_0) - \frac{c}{d}Q(x)\right) e\left(-Q(x)\frac{1}{\tau}\right).\end{aligned}$$

For any $x_0 \in L$ and $x \in \tilde{L}$, we have

$$-\frac{c}{d}Q(bx_0 + x) \equiv \frac{b}{d}Q(x_0) + \frac{2}{d}B(x_0, x) - \frac{c}{d}Q(x) \pmod{2\partial^{-1}};$$

thus

$$\begin{aligned}\Theta\left(L, \frac{a\tau + b}{c\tau + d}\right) &= \frac{i^{-mn/2}}{\sqrt{\Phi(L)}} N(d)^{-m/2} N\left(d\frac{1}{\tau} + c\right)^{m/2}\\ &\quad\times \sum_{\substack{x_0 \in L/dL\\ x \in \tilde{L}}} e\left(-\frac{c}{d}Q(bx_0 + x)\right) e\left(-Q(x)\frac{1}{\tau}\right).\end{aligned}$$

We claim that for any $x \in \tilde{L}$, $bx_0 + x$ runs over $\tilde{L}/d\tilde{L}$ as x_0 runs over L/dL. To prove this, we need

PROPOSITION 2.1. *Let L be a lattice and let $\mathfrak{P}$ be any prime ideal. Let $L_{\mathfrak{P}}$ denote the $\mathcal{O}$-module tensor product $L \otimes \mathcal{O}_{\mathfrak{P}}$. Then $\mathfrak{P}$ divides $\mathcal{N}(L)$ if and only if $L_{\mathfrak{P}}$ is not modular. Thus $L_{\mathfrak{P}}$ is $N(L_{\mathfrak{P}})$-modular if $\mathfrak{P}$ does not divide $\mathcal{N}(L)$.*

Proof. Since $N(L)\mathcal{O}_{\mathfrak{P}} = N(L_{\mathfrak{P}})$ for any lattice L and prime ideal $\mathfrak{P}$ (see [3], p. 11), it suffices to show that $\mathfrak{P}$ divides $\mathcal{N}(L_{\mathfrak{P}}) = N(L_{\mathfrak{P}})^{-1} N(L^\#_{\mathfrak{P}})^{-1}$ if and only if $L_{\mathfrak{P}}$ is not modular.

Via a Jordan decomposition of $L_{\mathfrak{P}}$ we can write

$$L_{\mathfrak{P}} = J_1 \perp \ldots \perp J_h$$

where h is some positive integer and each J_k is a modular lattice. Let $e_1, \ldots, e_h$ be integers such that J_k is $\mathfrak{P}^{e_k}\mathcal{O}_{\mathfrak{P}}$-modular. If J_k and J_l are both $\mathfrak{P}^e\mathcal{O}_{\mathfrak{P}}$-modular (where $k \neq l$) then $J_k \perp J_l$ is also $\mathfrak{P}^e\mathcal{O}_{\mathfrak{P}}$-modular; thus we may assume that $e_1 < \ldots < e_h$. Then

$$N(L_{\mathfrak{P}}) = \mathfrak{P}^{e_1}\mathcal{O}_{\mathfrak{P}} \quad \text{and} \quad N(L^\#_{\mathfrak{P}}) = \mathfrak{P}^{-e_h}\mathcal{O}_{\mathfrak{P}}.$$

Hence $\mathfrak{P}$ divides $N(L_{\mathfrak{P}})^{-1} N(L^\#_{\mathfrak{P}})^{-1}$ if and only if $h > 1$; that is, $\mathfrak{P}$ divides $\mathcal{N}(L_{\mathfrak{P}})$ if and only if $L_{\mathfrak{P}}$ is not modular. ∎

Now to prove our preceding claim, we fix $x \in \tilde{L}$. Since $b \in N(L)^{-1}\partial^{-1}$, we have $bL \subseteq \tilde{L}$ and hence $bx_0 + x \in \tilde{L}$ for any $x_0 \in L$. Also, if $x_0, x_0' \in L$ such that $x_0 - x_0' \in dL$, then $(bx_0 + x) - (bx_0' + x) \in bdL \subseteq d\tilde{L}$.

Suppose $x_0, x_0' \in L$ such that $x_0 - x_0' \notin dL$. Then there exists a prime $\mathfrak{P}|d$ such that

$$\operatorname{ord}_{\mathfrak{P}} B(x_0 - x_0', d^{-1}\tilde{L}) < \operatorname{ord}_{\mathfrak{P}} \partial^{-1}.$$

Our conditions on d ensure that $\operatorname{ord}_{\mathfrak{P}} b = 0$, and these conditions together with Proposition 2.1 imply that $L_{\mathfrak{P}} = \tilde{L}_{\mathfrak{P}}$. So we get

$$\begin{aligned}\operatorname{ord}_{\mathfrak{P}} B(bx_0 - bx_0', d^{-1}L) &= \operatorname{ord}_{\mathfrak{P}} \frac{b}{d} B(x_0 - x_0', L_{\mathfrak{P}})\\ &= \operatorname{ord}_{\mathfrak{P}} \frac{1}{d} B(x_0 - x_0', \tilde{L}_{\mathfrak{P}}) < \operatorname{ord}_{\mathfrak{P}} \partial^{-1}\end{aligned}$$

proving that $(bx_0 + x) - (bx_0' + x) \notin d\tilde{L}$. Thus $bx_0 + x$ runs over $\tilde{L}/d\tilde{L}$ as x_0 runs over L/dL.

This allows us to write

$$\begin{aligned}\Theta\left(L, \frac{a\tau+b}{c\tau+d}\right) &= \frac{i^{-mn/2}}{\sqrt{\Phi(L)}} N(d)^{-m/2} N\left(d\frac{1}{\tau}+c\right)^{m/2} \sum_{x_0 \in L/dL} e\left(-\frac{c}{d} Q(bx_0)\right) \sum_{x \in \tilde{L}} e\left(-Q(x)\frac{1}{\tau}\right)\\ &= \frac{i^{-mn/2}}{\sqrt{\Phi(L)}} N(d)^{-m/2} N\left(d\frac{1}{\tau}+c\right)^{m/2} \sum_{x_0 \in L/dL} e\left(\frac{b}{d} Q(x_0)\right) \sum_{x \in \tilde{L}} e\left(-Q(x)\frac{1}{\tau}\right),\end{aligned}$$

and with (1) we get the *transformation formula*:

$$\Theta\left(L, \frac{a\tau+b}{c\tau+d}\right) = N\left(c+d\frac{1}{\tau}\right)^{m/2} N(\tau)^{m/2} N(d)^{-m/2} \sum_{x \in L/dL} e\left(\frac{b}{d} Q(x)\right) \Theta(L, \tau). \tag{2}$$

3. $\Theta(L, \tau)$ as a modular form. Let the notation be as in the preceding section. To show that $\Theta(L, \tau)$ is a modular form we need to analyze $\sum_{x \in L/dL} e\left(\frac{b}{d} Q(x)\right)$. As a first step we have

PROPOSITION 3.1. *Let $\mathfrak{I} = \mathfrak{I}_1 \mathfrak{I}_2$ where $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are relatively prime integral ideals. Then we have the $\mathcal{O}$-module isomorphism*

$$L/\mathfrak{I}L \approx \mathfrak{I}_1 L/\mathfrak{I}L \oplus \mathfrak{I}_2 L/\mathfrak{I}L$$

by means of the map from $\mathfrak{I}_1 L/\mathfrak{I}L \oplus \mathfrak{I}_2 L/\mathfrak{I}L$ onto $L/\mathfrak{I}L$ defined by $(x_1 + \mathfrak{I}L, x_2 + \mathfrak{I}L) \mapsto x_1 + x_2 + \mathfrak{I}L$. Furthermore, for $x_1 \in \mathfrak{I}_1 L$ and $x_2 \in \mathfrak{I}_2 L$, $Q(x_1 + x_2) + 2\mathfrak{I}N(L) = Q(x_1) + Q(x_2) + 2\mathfrak{I}N(L)$.

Applying this proposition repeatedly, we get

$$L/dL \approx d\mathfrak{P}_1^{-e_1} L/dL \oplus \ldots \oplus d\mathfrak{P}_g^{-e_g} L/dL$$

where $\mathfrak{P}_1, \ldots, \mathfrak{P}_g$ are distinct prime ideals such that $d\mathcal{O} = \mathfrak{P}_1^{e_1} \ldots \mathfrak{P}_g^{e_g}$.

Furthermore, we have

$$\frac{b}{d} Q(x_1 + \ldots + x_g) \equiv \frac{b}{d} Q(x_1) + \ldots + \frac{b}{d} Q(x_g) \pmod{2\partial^{-1}}$$

where $x_j \in d\mathfrak{P}_j^{-e_j} L$. Thus

$$\sum_{x \in L/dL} e\left(\frac{b}{d} Q(x)\right) = \prod_{j=1}^{g} \left(\sum_{x \in d\mathfrak{P}_j^{-e_j} L/dL} e\left(\frac{b}{d} Q(x)\right)\right).$$

We now analyze each sum in this product.

PROPOSITION 3.2. *Let L' be a lattice on V and let $\mathfrak{P}$ be a prime ideal in $\mathcal{O}$, $e \in \mathbf{Z}_+$, and $\varrho \in \mathfrak{P}^{-e} \partial^{-1} N(L')^{-1}$. If $\mathfrak{P}$ does not divide $N(L')$ then*

$$\sum_{x \in L'/P^e L'} e(\varrho Q(x)) = \begin{cases} N(\mathfrak{P})^{me/2} & \text{if } 2|e,\\ N(\mathfrak{P})^{m(e-1)/2} \sum_{x \in L'/\mathfrak{P}L'} e(v^{e-1} \varrho Q(x)) & \text{otherwise}\end{cases}$$

where v is any element of $\mathfrak{P} - \mathfrak{P}^2$.

Proof. For $e = 1$ the statement is trivial. For $e > 1$ we have

$$\sum_{x \in L'/\mathfrak{P}^e L'} e(\varrho Q(x)) = \sum_{\substack{x \in L'/\mathfrak{P}^{e-1} L'\\ y \in \mathfrak{P}^{e-1} L'/\mathfrak{P}^e L'}} e(\varrho Q(x+y)).$$

Now, $\varrho Q(y) \in 2\partial^{-1}$ for any $y \in \mathfrak{P}^{e-1} L'$, so

$$\sum_{x \in L'/\mathfrak{P}^e L'} e(\varrho Q(x)) = \sum_{x \in L'/\mathfrak{P}^{e-1} L'} e(\varrho Q(x)) \sum_{y \in \mathfrak{P}^{e-1} L'/\mathfrak{P}^e L'} e(2\varrho B(x, y)).$$

For fixed $x \in L'$, $y \mapsto e(2\varrho B(x, y))$ is a character on $\mathfrak{P}^{e-1} L'/\mathfrak{P}^e L'$; it is the trivial character only when $x \in \mathfrak{P}L'$. Hence

$$\sum_{y \in \mathfrak{P}^{e-1} L'/\mathfrak{P}^e L'} e(2\varrho B(x, y)) = \begin{cases} N(\mathfrak{P})^m & \text{if } x \in \mathfrak{P}L',\\ 0 & \text{otherwise.}\end{cases}$$

(Notice that $[\mathfrak{P}^{e-1} L' : \mathfrak{P}^e L'] = N(\mathfrak{P})^m$.) Hence

$$\begin{aligned}\sum_{x \in L'/\mathfrak{P}^e L'} e(\varrho Q(x)) &= N(\mathfrak{P})^m \sum_{x \in \mathfrak{P}L'/\mathfrak{P}^{e-1} L'} e(\varrho Q(x))\\ &= N(\mathfrak{P})^m \sum_{x \in L'/\mathfrak{P}^{e-2} L'} e(v^2 \varrho Q(x))\end{aligned}$$

where $v \in \mathfrak{P} - \mathfrak{P}^2$. Induction on e now yields the statement of the proposition. ∎

We now need to analyze the Gaussian sum

$$\sum_{x \in d\mathfrak{P}_j^{-e_j} L/d\mathfrak{P}_j^{1-e_j} L} e\left(v_j^{e_j - 1} \frac{b}{d} Q(x)\right)$$

where $v_j \in \mathfrak{P}_j - \mathfrak{P}_j^2$. Fix j and let $\mathfrak{P} = \mathfrak{P}_j$ and $L' = d\mathfrak{P}_j^{-e_j} L$; let $L'_{\mathfrak{P}} = L' \otimes \mathcal{O}_{\mathfrak{P}}$ (where the tensor product is as $\mathcal{O}$-modules). Let ϱ be any element of $\mathfrak{P}^{-1} N(L')^{-1} \partial^{-1} - N(L')^{-1} \partial^{-1}$ (i.e. we could take $\varrho = v_j^{e_j - 1} b/d$). Writing

$L' = \mathcal{O}x_1 \oplus \ldots \oplus \mathcal{O}x_{m-1} \oplus \mathfrak{A}x_m$ where $\mathfrak{A}$ is some fractional ideal, we have

$$L'_{\mathfrak{P}} = \mathcal{O}_{\mathfrak{P}} x_1 \oplus \ldots \oplus \mathcal{O}_{\mathfrak{P}} x_{m-1} \oplus \mathcal{O}_{\mathfrak{P}} \mathfrak{A} x_m.$$

The quadratic form Q extends naturally to $L'_{\mathfrak{P}}$: for $\alpha_1, \ldots, \alpha_{m-1} \in \mathcal{O}_{\mathfrak{P}}$ and $\alpha_m \in \mathcal{O}_{\mathfrak{P}} \mathfrak{A}$ we have

$$Q(\alpha_1 x_1 + \ldots + \alpha_m x_m) = \sum_{k=1}^{m} \alpha_k^2 Q(x_k) + 2 \sum_{k<l} \alpha_k \alpha_l B(x_k, x_l).$$

This allows us to evaluate the Gaussian sum over $L'/\mathfrak{P}L'$ by examining $L'_{\mathfrak{P}}/\mathfrak{P}L'_{\mathfrak{P}}$.

When $\mathfrak{P} \nmid 2$, our initial assumptions together with Proposition 2.1 imply that $L'_{\mathfrak{P}}$ is unimodular; thus $L'_{\mathfrak{P}} \simeq \langle 1, \ldots, 1, \varepsilon_{\mathfrak{P}} \rangle$ with respect to some $\mathcal{O}_{\mathfrak{P}}$-basis $y_1, \ldots, y_m$ of $L'_{\mathfrak{P}}$ and some $\varepsilon_{\mathfrak{P}} \in \mathcal{O}_{\mathfrak{P}}$. (Notice that $\varepsilon_{\mathfrak{P}} = \operatorname{disc} L'_{\mathfrak{P}}$.) As we let $\alpha_1, \ldots, \alpha_m$ vary over $\mathcal{O}/\mathfrak{P}$, the vector $\alpha_1 y_1 + \ldots + \alpha_m y_m$ varies over $L'_{\mathfrak{P}}/\mathfrak{P}L'_{\mathfrak{P}}$. Using the Chinese Remainder Theorem (allowing congruences modulo infinite primes) we choose $\varepsilon \in \mathcal{O}$ and $\mu \in 2N(L')$ such that $\varepsilon \equiv \varepsilon_{\mathfrak{P}} \pmod{\mathfrak{P}\mathcal{O}_{\mathfrak{P}}}$ and $\mu \equiv 1 \pmod{\mathfrak{P}}$. Then

$$Q(\alpha_1 y_1 + \ldots + \alpha_m y_m) \equiv \mu(\alpha_1^2 + \ldots + \alpha_{m-1}^2 + \varepsilon\alpha_m^2) \pmod{2\mathfrak{P}N(L')\mathcal{O}_{\mathfrak{P}}}$$

and

$$\mu(\alpha_1^2 + \ldots + \alpha_{m-1}^2 + \varepsilon\alpha_m^2) \in 2N(L').$$

Since the cosets of $L'/\mathfrak{P}L'$ are in one to one correspondence with those of $L'_{\mathfrak{P}}/\mathfrak{P}L'_{\mathfrak{P}}$ via the obvious map, we find that

$$\sum_{x \in L'/\mathfrak{P}L'} e(\varrho Q(x)) = \sum_{\alpha_1, \ldots, \alpha_m \in \mathcal{O}/\mathfrak{P}} e(\varrho\mu(\alpha_1^2 + \ldots + \alpha_{m-1}^2 + \varepsilon\alpha_m^2))$$
$$= (\varepsilon|\mathfrak{P}) \Big(\sum_{\alpha \in \mathcal{O}/\mathfrak{P}} e(\varrho\mu\alpha)(\alpha|\mathfrak{P}) \Big)^m.$$

(Here $(\cdot|\cdot)$ denotes the Legendre symbol.) Utilizing standard techniques employed to evaluate Gaussian sums we get

$$\Big(\sum_{\alpha \in \mathcal{O}/\mathfrak{P}} e(\varrho\mu\alpha)(\alpha|\mathfrak{P}) \Big)^2 = \sum_{\alpha, \beta \in \mathcal{O}/\mathfrak{P}} e(\varrho\mu(\alpha+\beta))(\alpha\beta|\mathfrak{P})$$
$$= \sum_{\alpha \in \mathcal{O}/\mathfrak{P}} (\alpha|\mathfrak{P}) \Big(\sum_{\beta \in \mathcal{O}/\mathfrak{P}} e(\varrho\mu\beta(\alpha+1)) \Big) = (-1|\mathfrak{P}) N(\mathfrak{P})$$

(since $\beta \mapsto e(\varrho\mu\beta(\alpha+1))$ is a character on $\mathcal{O}/\mathfrak{P}$). This gives us

PROPOSITION 3.3. *Let L' be a lattice, $\mathfrak{P}$ a prime ideal dividing neither $2\mathcal{N}(L')\partial$ nor $N(L')$ and ϱ any element of $\mathfrak{P}^{-1}N(L')^{-1}\partial^{-1} - N(L')^{-1}\partial^{-1}$. Then*

$$\sum_{x \in L'/\mathfrak{P}L'} e(\varrho Q(x)) = \begin{cases} ((-1)^{m/2} \operatorname{disc} L'_{\mathfrak{P}}|\mathfrak{P}) N(\mathfrak{P})^{m/2} & \text{if } 2|m, \\ ((-1)^{(m-1)/2} \operatorname{disc} L'_{\mathfrak{P}}|\mathfrak{P}) N(\mathfrak{P})^{(m-1)/2} \sum_{\alpha \in \mathcal{O}/\mathfrak{P}} e(\varrho\mu\alpha^2) & \text{otherwise} \end{cases}$$

where $\mu \in 2N(L')$ with $\mu \equiv 1 \pmod{\mathfrak{P}}$.

Now consider the case where $\mathfrak{P}|2$. Since $\mathfrak{P}|d$, our conditions on d and Proposition 2.1 show that $L'_{\mathfrak{P}}$ is a unimodular lattice, so we can write

$$L'_{\mathfrak{P}} = J_1 \perp \ldots \perp J_h$$

where $J_1, \ldots, J_h$ are unary or binary unimodular lattices (see §93 of [12]). In fact, $J_1, \ldots, J_h$ must all be binary lattices, otherwise we would have $J_k \simeq \langle 2\alpha \rangle$ for some $\alpha \in N(L'_{\mathfrak{P}}) \subseteq \mathcal{O}_{\mathfrak{P}}$ and hence J_k would not be unimodular. So we must have that $\operatorname{rank} L = \operatorname{rank} L'$ is even. Thus

$$L'_{\mathfrak{P}} \simeq \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_{m/2} \end{bmatrix} \quad \text{with} \quad A_k = \begin{bmatrix} 2a_k & 1 \\ 1 & 2c_k \end{bmatrix}$$

for some $a_k, c_k \in \mathcal{O}_{\mathfrak{P}}$ (see §93 of [12]). Similarly to the case when $\mathfrak{P} \nmid 2$, we get

$$\sum_{x \in L'/\mathfrak{P}L'} e(\varrho Q(x)) = \prod_{k=1}^{m/2} \Big(\sum_{\alpha, \beta \in \mathcal{O}/\mathfrak{P}} e(2\mu\varrho(a'_k\alpha^2 + \alpha\beta + c'_k\beta^2)) \Big)$$

where $\mu \in N(L')$ with $\mu \equiv 1 \pmod{\mathfrak{P}}$, and $a'_k, c'_k \in \mathcal{O}$ with $a'_k \equiv a_k \pmod{\mathfrak{P}\mathcal{O}_{\mathfrak{P}}}$ and $c'_k \equiv c_k \pmod{\mathfrak{P}\mathcal{O}_{\mathfrak{P}}}$. Now we fix k and let $a = a'_k$, $c = c'_k$.

Suppose first that $ac \in \mathfrak{P}$; without loss of generality, suppose $c \in \mathfrak{P}$. Then $e(2\mu\varrho c\beta^2) = 1$ for any $\beta \in \mathcal{O}$, so

$$\sum_{\alpha, \beta \in \mathcal{O}/\mathfrak{P}} e(a\alpha^2 + \alpha\beta + c\beta^2) = \sum_{\alpha \in \mathcal{O}/\mathfrak{P}} e(2\mu\varrho a\alpha^2) \Big(\sum_{\beta \in \mathcal{O}/\mathfrak{P}} e(2\mu\varrho\alpha\beta) \Big) = N(\mathfrak{P})$$

(since $\beta \mapsto e(2\mu\varrho\alpha\beta)$ is a character on $\mathcal{O}/\mathfrak{P}$).

Now suppose that $ac \notin \mathfrak{P}$. Then $c\alpha$ runs over $\mathcal{O}/\mathfrak{P}$ as α does, so

$$\sum_{\alpha, \beta \in \mathcal{O}/\mathfrak{P}} e(a\alpha^2 + \alpha\beta + c\beta^2) = \sum_{\alpha \in \mathcal{O}/\mathfrak{P}} e(2\mu\varrho ac^2\alpha^2) \sum_{\beta \in \mathcal{O}/\mathfrak{P}} e(2\mu\varrho c\beta(\alpha+\beta)).$$

Since the characteristic of $\mathcal{O}/\mathfrak{P}$ is 2, $\beta \mapsto e(2\mu\varrho c\beta(\alpha+\beta))$ is a character on $\mathcal{O}/\mathfrak{P}$; it is the trivial character if and only if $\beta(\alpha+\beta) \in \mathfrak{P}$ for every $\beta \in \mathcal{O}/\mathfrak{P}$, and this happens if and only if $|\mathcal{O}/\mathfrak{P}| = 2^{f(\mathfrak{P})} = 2$ (where $f(\mathfrak{P})$ is the inertial degree of $\mathfrak{P}$) and $\alpha \in \mathcal{O} - \mathfrak{P}$. Hence we have

PROPOSITION 3.4. *Let L' be a lattice, $\mathfrak{P}$ a prime dividing 2 but dividing neither $\mathcal{N}(L')\partial$ nor $N(L')$, and $\varrho \in N(L')^{-1}\mathfrak{P}^{-1}\partial^{-1} - N(L')^{-1}\partial^{-1}$. Then $\operatorname{rank} L'$ is even and*

$$L'_{\mathfrak{P}} \simeq \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_{m/2} \end{bmatrix} \quad \text{with} \quad A_k = \begin{bmatrix} 2a_k & 1 \\ 1 & 2c_k \end{bmatrix}$$

for some $a_k, c_k \in \mathcal{O}_{\mathfrak{P}}$; furthermore,

$$\sum_{x \in L'/\mathfrak{P}L'} e(\varrho Q(x)) = \begin{cases} 0 & \text{if } f(\mathfrak{P}) > 1 \text{ and } a_k c_k \notin \mathfrak{P}\mathcal{O}_{\mathfrak{P}} \text{ for some } k, \\ N(\mathfrak{P})^{m/2} & \text{otherwise.} \end{cases}$$

Because of this possible degeneracy of the sum

$$\sum_{x \in L/dL} e\Big(\frac{b}{d} Q(x)\Big),$$

we need to strengthen our conditions on d. We could be very crude and require that d be relatively prime to 2; however the following approach, albeit a bit tedious, is more refined.

DEFINITION. Let $\mathfrak{P}$ be a prime dividing 2. We define a *normal form* q on $L/\mathfrak{P}L \simeq L_{\mathfrak{P}}/\mathfrak{P}L_{\mathfrak{P}}$ as follows: q is given by the upper triangular matrix (q_{ij}) where

$$q_{ij} = \begin{cases} \frac{1}{2}Q(x_i) & \text{if } i=j, \\ B(x_i, x_j) & \text{if } i<j, \\ 0 & \text{if } i>j \end{cases}$$

where $\{x_1, \ldots, x_m\}$ is any $\mathcal{O}/\mathfrak{P}$-basis for $L/\mathfrak{P}L$ and the q_{ij} are considered as elements of $N(L)/\mathfrak{P}N(L) \approx N(L_{\mathfrak{P}})/\mathfrak{P}N(L_{\mathfrak{P}})$. Then we define

$$q(x) = x^t (q_{ij}) x$$

where x is identified with the corresponding coordinate vector. (As discussed in [15], ch. 9, §4, this normal form is independent of choice of basis.) Following Sharlau, we say that q is *weakly metabolic* if there is a basis of $L/\mathfrak{P}L$ such that with respect to this basis q is given by a matrix of the form $\begin{bmatrix} D & I \\ 0 & 0 \end{bmatrix}$ where D is a diagonal $(m/2)\times(m/2)$ matrix and I is the $(m/2)\times(m/2)$ identity matrix.

Now we define another ideal $S(L)$ which we use to replace the ideal $\mathcal{N}(L)$ from [6].

DEFINITION. For a lattice L of even rank, we define the ideal $S(L)$, which we call the *stufe* of L, to be the ideal generated by $\mathcal{N}(L)$, the level of L, and all primes $\mathfrak{P}$ which satisfy:

1. $\mathfrak{P}|2$;
2. $f(\mathfrak{P}) > 1$; and
3. $L/\mathfrak{P}L$ is not weakly metabolic.

For a lattice L of odd rank, we define $S(L) = \mathcal{N}(L)$.

We have already seen that when $\mathfrak{P}$ divides 2 but not $\mathcal{N}(L)\partial$ or $N(L)$ then $\operatorname{rank} L$ is even. We now prove a stronger statement.

PROPOSITION 3.5. *For L a lattice of odd rank we have* $4|\mathcal{N}(L)$.

Proof. Consider the decomposition of $L_{\mathfrak{P}}$ into unary and binary lattices

$$L_{\mathfrak{P}} = J_1 \perp \ldots \perp J_{(m+1)/2}.$$

Without loss of generality, assume J_1 is unary; thus

$$J_1 \simeq (2a_1) \quad \text{and} \quad J_k \simeq \begin{bmatrix} 2a_k & b_k \\ b_k & 2c_k \end{bmatrix} \quad \text{for } k>1$$

with $a_k, b_k, c_k \in N(L_{\mathfrak{P}})$. Let $e = \operatorname{ord}_{\mathfrak{P}} N(L)$; recalling that $L_{\mathfrak{P}}^{\#}$ is given by the matrix A^{-1} where $L_{\mathfrak{P}} \simeq A$, we have

$$\tfrac{1}{4}\mathfrak{P}^{-e} \subseteq \frac{1}{4a_1}\mathcal{O}_{\mathfrak{P}} \subseteq N(L_{\mathfrak{P}}^{\#}) = N(L^{\#})\mathcal{O}_{\mathfrak{P}}$$

and hence $\mathcal{N}(L)\mathcal{O}_{\mathfrak{P}} = N(L^{\#})^{-1}N(L)^{-1}\mathcal{O}_{\mathfrak{P}} \subseteq 4\mathcal{O}_{\mathfrak{P}}$. ■

We now define an action of matrices on functions on $\mathcal{H}^n$.

DEFINITION. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathcal{O}) = \{A \in \mathrm{GL}_2(\mathcal{O}) |\ \det A \gg 0\}$. An *automorphy factor* for A is an analytic function $\Psi(\tau)$ on $\mathcal{H}^n$ such that

$$(\Psi(\tau))^2 = \pm \frac{N(c\tau+d)}{\sqrt{N(\det A)}}$$

where the sign is independent of τ. The collection of such pairs $(A, \Psi(\tau))$ forms a group with multiplication defined by

$$(A, \Psi_1(\tau))(B, \Psi_2(\tau)) = (AB, \Psi_1(B\tau)\Psi_2(\tau)).$$

For a function f on $\mathcal{H}^n$ and $k \in \frac{1}{2}\mathbf{Z}_+$, we define the *slash operator* by

$$f(\tau)|[(A, \Psi(\tau))]_k = \Psi(\tau)^{-2k} f(A\tau).$$

When $k \in \mathbf{Z}_+$ we write $f(\tau)|[A]_k$ to mean $f(\tau)\left|\left[\left(A, \frac{\sqrt{N(c\tau+d)}}{\sqrt[4]{N(\det A)}}\right)\right]_k\right.$ (where we agree to take $\sqrt[4]{N(\det A)} \in \mathbf{R}_+$). Notice that

$$f(\tau)|[(A, \Psi_1(\tau))]_k|[(B, \Psi_2(\tau))]_k = f(\tau)|[(AB, \Psi_1(B\tau)\Psi_2(\tau))]_k.$$

We also define

$$\Theta(\mathfrak{I}, \tau) = \sum_{x\in\mathfrak{I}} e(2x^2\tau)$$

where we consider the ideal $\mathfrak{I}$ to be a lattice with the quadratic form given by $x \mapsto 2x^2$. For any $A \in \Gamma_0(4\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1})$ we define

$$\tilde{A} = \left(A, \frac{\Theta(\mathfrak{I}, A\tau)}{\Theta(\mathfrak{I}, \tau)}\right),$$

and for $\mathfrak{J} \subseteq 4\mathcal{O}$ we define the group

$$\tilde{\Gamma}_0(\mathfrak{J}\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1}) = \{\tilde{A}|\ A \in \Gamma_0(\mathfrak{J}\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1})\}.$$

PROPOSITION 3.6. *Let $\mathfrak{I}$ and $\mathfrak{J}$ be fractional ideals, and let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(4\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1}) \cap \Gamma_0(4\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1}).$$

Then $\tilde{A}$ is well-defined; that is,

$$\frac{\Theta(\mathfrak{I}, A\tau)}{\Theta(\mathfrak{I}, \tau)} = \frac{\Theta(\mathfrak{J}, A\tau)}{\Theta(\mathfrak{J}, \tau)}.$$

Furthermore,

$$\left(\frac{\Theta(\mathfrak{I}, A\tau)}{\Theta(\mathfrak{I}, \tau)}\right)^2 = \pm N(c\tau+d).$$

Proof. Using the Chinese Remainder Theorem we can write any matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(4\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1})$$

as a product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

where $\beta \in \mathfrak{J}^{-2}\partial^{-1} \cap \mathfrak{J}^{-2}\partial^{-1}$, $\gamma \in 4\mathfrak{J}^2\partial \cap 4\mathfrak{J}^2\partial$, and $d' \equiv d \pmod{4\mathcal{O}}$ with d' totally positive and relatively prime to 4∂, to $\mathfrak{J}$, and to $\mathfrak{J}$. The result now follows easily from Propositions 3.1, 3.2 and 3.3. ∎

Now we prove

THEOREM 3.7. *Let L be a lattice of rank m.*

1. *If m is even, then $\Theta(L, \tau)$ is a modular form of uniform weight $m/2$ with character χ for the group $\Gamma_0(S(L)N(L)\partial, N(L)^{-1}\partial^{-1})$. The character χ is a quadratic character modulo $S(L)$; if $d \gg 0$ and d is relatively prime to $S(L)\partial$ and to $N(L)$ then*

$$\chi(d) = \prod_{\mathfrak{P}|d} ((-1)^{m/2}\operatorname{disc} L_{\mathfrak{P}} | \mathfrak{P})^{\operatorname{ord}_{\mathfrak{P}} d}.$$

2. *If m is odd, let $\mathfrak{J}$ be an ideal such that $N(L) \subseteq \mathfrak{J}^2$ and let $\mathfrak{J} = S(L)N(L)\mathfrak{J}^{-2}$; then $\Theta(L, \tau)$ is a modular form of uniform weight $m/2$ with character χ for the group $\Gamma_0(\mathfrak{J}\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1})$. Here χ is a character modulo $\mathfrak{J}$ and if $d \gg 0$ and d is relatively prime to $\mathfrak{J}\partial$ and to $\mathfrak{J}$ then*

$$\chi(d) = \prod_{\mathfrak{P}|d} (2\operatorname{disc} L_{\mathfrak{P}} | \mathfrak{P})^{\operatorname{ord}_{\mathfrak{P}} d}.$$

Proof. First we suppose that rank L is even. For $d \in \mathcal{O}$ such that $d \gg 0$ and d is relatively prime to $S(L)\partial$ and to $N(L)$, define

$$\chi(d) = \prod_{\mathfrak{P}|d} ((-1)^{m/2}\operatorname{disc} L_{\mathfrak{P}} | \mathfrak{P})^{\operatorname{ord}_{\mathfrak{P}} d}.$$

Now choose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(S(L)N(L)\partial, N(L)^{-1}\partial^{-1})$. Using the Chinese Remainder Theorem we can write

$$A = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = A_1 A_2 A_3$$

where $A_1, A_2, A_3 \in \Gamma_0(S(L)N(L)\partial, N(L)^{-1}\partial^{-1})$, $d' \gg 0$, $d' \equiv d \pmod{S(L)}$, and d' is relatively prime to $S(L)\partial$ and to $N(L)$. Then the transformation formula (2) and the preceding propositions show that

$$\Theta(L, \tau)|[A]_{m/2} = \chi(d')\Theta(L, \tau).$$

We want to show that χ is a character modulo $S(L)$. For this we define another function ω by

$$\omega(b, d) = \chi(d + b\gamma)$$

where $d \in \mathcal{O}$, $b \in N(L)^{-1}\partial^{-1}$ such that $bN(L)\partial$ is relatively prime to d, and γ is any element of $S(L)N(L)\partial$ such that $d + b\gamma$ is totally positive and relatively prime to $S(L)\partial$ and to $N(L)$. Since $\Theta(L, \tau)|\left[\begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}\right]_{m/2} = \Theta(L, \tau)$ for any $\gamma \in S(L)N(L)\partial$, $\omega(b, d)$ is well-defined. To show ω is independent of b, we take b' to be another element of $N(L)^{-1}\partial^{-1}$ such that d is relatively prime to $b'N(L)\partial$. Then we choose $\gamma, \gamma' \in S(L)N(L)\partial$ such that $b\gamma = b'\gamma' \in bS(L)N(L)\partial \cap b'S(L)N(L)\partial$ such that $d + b\gamma \gg 0$ and $d + b\gamma$ is relatively prime to $S(L)\partial$ and to $N(L)$; thus

$$\omega(b, d) = \chi(d + b\gamma) = \chi(d + b'\gamma') = \omega(b', d).$$

Hence we can define $\chi(d) = \omega(b, d)$ where d is any element of $\mathcal{O}$ relatively prime to $S(L)$ and b is any element of $N(L)^{-1}\partial^{-1}$ such that $bN(L)\partial$ is relatively prime to d.

Now we only need to show that $\chi(d) = \chi(d')$ whenever $d \equiv d' \pmod{S(L)}$. Using the Chinese Remainder Theorem, we choose $b, b' \in N(L)^{-1}\partial^{-1}$ and $\gamma, \gamma' \in S(L)N(L)\partial$ such that d is relatively prime to $bN(L)\partial$, $d + b\gamma$ is relatively prime to $b'N(L)\partial$, and $b\gamma S(L)^{-1}$ is relatively prime to $b'\gamma'$. Then $S(L) = \mathcal{O}b\gamma + \mathcal{O}b'\gamma'$, so

$$d' = d + \alpha b\gamma + \alpha' b'\gamma'$$

for some $\alpha, \alpha' \in \mathcal{O}$. Thus

$$\chi(d) = \omega(b, d) = \omega(b, d + \alpha b\gamma) = \omega(b', d + \alpha b\gamma) = \omega(b', d') = \chi(d').$$

Now suppose that rank L is odd. Take $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(\mathfrak{J}\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1})$. If d is totally positive and relatively prime to $S(L)\partial$ and to $\mathfrak{J}$ then

$$\Theta(L, \tau)|[\tilde{A}]_{m/2} = \left(\sum_{x \in \mathfrak{J}/d\mathfrak{J}} e\left(\frac{b}{d}2x^2\right)\right)^{-m} \sum_{x \in L/dL} e\left(\frac{b}{d}Q(x)\right)\Theta(L, \tau)$$
$$= \left(\sum_{x \in L'/dL'} e\left(\frac{b}{d}Q'(x)\right)\right)^{-1} \sum_{x \in L/dL} e\left(\frac{b}{d}Q(x)\right)\Theta(L, \tau)$$

where $L' = \{(x_1, \ldots, x_m) | \; x_1, \ldots, x_m \in \mathfrak{J}\}$ is the lattice with the quadratic form Q' given by $Q'(x_1, \ldots, x_m) = 2x_1^2 + \ldots + 2x_m^2$. Propositions 3.1, 3.2 and 3.3 now give us

$$\Theta(L, \tau)|[\tilde{A}]_{m/2} = \prod_{\mathfrak{P}|d} (\operatorname{disc} L'_{\mathfrak{P}} \cdot \operatorname{disc} L_{\mathfrak{P}} | \mathfrak{P})^{\operatorname{ord}_{\mathfrak{P}} d}\,\Theta(L, \tau)$$
$$= \prod_{\mathfrak{P}|d} (2\operatorname{disc} L_{\mathfrak{P}} | \mathfrak{P})^{\operatorname{ord}_{\mathfrak{P}} d}\,\Theta(L, \tau).$$

The same procedure used in the case that rank L is even also yields the statement of the theorem in the case that rank L is odd. ∎

Remark. When $K = \mathbf{Q}$ and L is integral, Theorem 3.7 is Schoeneberg's Theorem (see Theorem 20 of [11], p. VI-22) together with Proposition 2.1 of [17] (p. 456).

4. The Hecke operators for integral weight. We begin by giving a global definition of Hecke operators. For the case of trivial character this definition is equivalent to that of Eichler [6].

Definition. Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be fractional ideals such that $\mathfrak{I}_1\mathfrak{I}_2 \subseteq \mathcal{O}$. For $k \in \mathbf{Z}_+$ and χ a character modulo $\mathfrak{I}_1\mathfrak{I}_2$, let $\mathcal{M}_k(\Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2), \chi)$ denote the space of modular forms of uniform weight k for the group $\Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)$ with character χ. Let

$$\Gamma_1(\mathfrak{I}_1, \mathfrak{I}_2) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2) \mid a \equiv d \equiv 1 \pmod{\mathfrak{I}_1\mathfrak{I}_2}\right\}.$$

For $\mathfrak{P}$ a prime ideal not dividing $\mathfrak{I}_1\mathfrak{I}_2$, choose $\varrho \in \mathfrak{P} - \mathfrak{P}^2$ such that $\varrho \gg 0$; let $\mathfrak{I}$ be the ideal such that $\varrho\mathcal{O} = \mathfrak{P}\mathfrak{I}$. Let $\{A_j\}$ be a set of right coset representatives for

$$\Gamma_1(\mathfrak{I}\mathfrak{I}_1, \mathfrak{I}^{-1}\mathfrak{I}_2) \Big/ \left(\Gamma_1(\mathfrak{I}\mathfrak{I}_1, \mathfrak{I}^{-1}\mathfrak{I}_2) \cap \begin{bmatrix} \varrho^{-1} & 0 \\ 0 & 1 \end{bmatrix} \Gamma_1(\mathfrak{I}_1, \mathfrak{I}_2) \begin{bmatrix} \varrho & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Then for k a uniform integral weight and χ a character modulo $\mathfrak{I}_1\mathfrak{I}_2$, we define the Hecke operator

$$T_k(\mathfrak{P}, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) : \mathcal{M}_k(\Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2), \chi) \to \mathcal{M}_k(\Gamma_0(\mathfrak{I}_1\mathfrak{P}^{-1}, \mathfrak{I}_2\mathfrak{P}), \chi)$$

by

$$f(\tau) \mid T_k(\mathfrak{P}, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) = N(\mathfrak{P})^{k-1} \sum_j f(\tau) \left| \left[\begin{bmatrix} \varrho & 0 \\ 0 & 1 \end{bmatrix} A_j \begin{bmatrix} \varrho^{-1} & 0 \\ 0 & 1 \end{bmatrix}\right]_k \right..$$

(Notice that this definition is independent of the choice of ϱ and of coset representatives $\{A_j\}$.)

We define the operator

$$V_k(\mathfrak{P}, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) : \mathcal{M}_k(\Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2), \chi) \to \mathcal{M}_k(\Gamma_0(\mathfrak{P}^{-2}\mathfrak{I}_1, \mathfrak{P}^2\mathfrak{I}_2), \chi)$$

by

$$f(\tau) \mid V_k(\mathfrak{P}, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) = f(\tau) \mid [A]_k$$

where A is any element of $\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1(\mathfrak{P}^{-1}\mathfrak{I}_1, \mathfrak{P}\mathfrak{I}_2) \mid d \in \mathfrak{P}\right\}$. (Notice that this operator is independent of the choice of A.) For $e > 1$, we inductively define the operators

$$V_k(\mathfrak{P}^e, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) : \mathcal{M}_k(\Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2), \chi) \to \mathcal{M}_k(\Gamma_0(\mathfrak{P}^{-2e}\mathfrak{I}_1, \mathfrak{P}^{2e}\mathfrak{I}_2), \chi)$$

by

$$f(\tau) \mid V_k(\mathfrak{P}^e, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) = f(\tau) \mid V_k(\mathfrak{P}, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) \mid V_k(\mathfrak{P}^{e-1}, \Gamma_0(\mathfrak{P}^{-2}\mathfrak{I}_1, \mathfrak{P}^2\mathfrak{I}_2)).$$

We now inductively define the Hecke operators

$$T_k(\mathfrak{P}^e, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) : \mathcal{M}_k(\Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2), \chi) \to \mathcal{M}_k(\Gamma_0(\mathfrak{P}^{-e}\mathfrak{I}_1, \mathfrak{P}^e\mathfrak{I}_2), \chi)$$

by

$$\begin{aligned} f(\tau) &\mid T_k(\mathfrak{P}^e, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) \mid T_k(\mathfrak{P}, \Gamma_0(\mathfrak{P}^{-e}\mathfrak{I}_1, \mathfrak{P}^e\mathfrak{I}_2)) \\ &= f(t) \mid T_k(\mathfrak{P}^{e+1}, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) \\ &\quad + N(\mathfrak{P})^{k-1} f(\tau) \mid V_k(\mathfrak{P}, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) \mid T_k(\mathfrak{P}^{e-1}, \Gamma_0(\mathfrak{P}^{-2}\mathfrak{I}_1, \mathfrak{P}^2\mathfrak{I}_2)). \end{aligned}$$

(It is understood here that $T(\mathfrak{P}^0)$ is the identity operator.)

Finally, for $\mathfrak{I}_1$ and $\mathfrak{I}_2$ relatively prime integral ideals which are also relatively prime to $\mathfrak{I}_1\mathfrak{I}_2$, we define

$$f(\tau) \mid T_k(\mathfrak{I}_1\mathfrak{I}_2, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) = f(\tau) \mid T_k(\mathfrak{I}_1, \Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2)) \mid T_k(\mathfrak{I}_2, \Gamma_0(\mathfrak{I}_1^{-1}\mathfrak{I}_1, \mathfrak{I}_1\mathfrak{I}_2)).$$

For simplicity, we shall often refer to the operators defined above as $T(\mathfrak{I})$ and $V(\mathfrak{I})$.

We now prove the following useful proposition.

Proposition 4.1. *Let $\mathfrak{P}$ be a prime ideal not dividing $\mathfrak{I}_1\mathfrak{I}_2$, and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix such that $a \in \mathfrak{P}^{-1}$, $b \in \mathfrak{P}\mathfrak{I}_2$, $c \in \mathfrak{P}^{-1}\mathfrak{I}_1$, $d \in \mathfrak{P}$, $d \equiv 1 \pmod{\mathfrak{I}_1\mathfrak{I}_2}$, and $\det A = 1$. Then $f(\tau) \mid [A]_k = f(\tau) \mid V(\mathfrak{P})$.*

Proof. Let B be a matrix giving the action of $V(\mathfrak{P})$ (i.e. B is a matrix such that $f(\tau) \mid V(\mathfrak{P}) = f(\tau) \mid [B]_k$). Then $AB^{-1} \in \Gamma_1(\mathfrak{I}_1, \mathfrak{I}_2)$ and hence

$$f(\tau) \mid [AB^{-1}]_k = f(\tau). \ ∎$$

Now we prove some multiplicative identities among the operators defined in this section; these identities imply that the operators $T(\mathfrak{I})$ are well-defined.

Proposition 4.2. *Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be distinct prime ideals not dividing $\mathfrak{I}_1\mathfrak{I}_2$. Then:*

1. $f(\tau) \mid V(\mathfrak{P}_1) \mid V(\mathfrak{P}_2) = f(\tau) \mid V(\mathfrak{P}_2) \mid V(\mathfrak{P}_1)$;
2. $f(\tau) \mid T(\mathfrak{P}_1) \mid V(\mathfrak{P}_2) = f(\tau) \mid V(\mathfrak{P}_2) \mid T(\mathfrak{P}_1)$;
3. $f(\tau) \mid T(\mathfrak{P}_1) \mid T(\mathfrak{P}_2) = f(\tau) \mid T(\mathfrak{P}_2) \mid T(\mathfrak{P}_1)$;
4. $f(\tau) \mid T(\mathfrak{P}_1) \mid V(\mathfrak{P}_1) = f(\tau) \mid V(\mathfrak{P}_1) \mid T(\mathfrak{P}_1)$;
5. $f(\tau) \mid T(\mathfrak{P}_1^a) \mid T(\mathfrak{P}_1^b) = \sum_{c=0}^{\min(a,b)} N(\mathfrak{P}_1)^{c(k-1)} f(\tau) \mid V(\mathfrak{P}_1^c) \mid T(\mathfrak{P}_1^{a+b-2c})$

where $a, b \in \mathbf{Z}_+$.

Proof. To prove the first four identities we make prudent choices for our coset representatives; the final identity then follows from the fourth.

First, taking $A \in \Gamma_1(\mathfrak{P}_1^{-1}\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{P}_2^2\mathfrak{I}_2)$, $A' \in \Gamma_1(\mathfrak{P}_2^{-1}\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{P}_2\mathfrak{I}_2)$, $B \in \Gamma_1(\mathfrak{P}^{-1}\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{P}_2\mathfrak{I}_2)$, and $B' \in \Gamma_1(\mathfrak{P}_2^{-1}\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{P}_2\mathfrak{I}_2)$ such that the lower right entries of A and B are elements of $\mathfrak{P}_1$ and those of A' and B' are elements of $\mathfrak{P}_2$, we get

$$f(\tau)|V(\mathfrak{P}_2)|V(\mathfrak{P}_1) = f(\tau)|[A']_k|[A]_k = f(\tau)|[B]_k|[B']_k = f(\tau)|V(\mathfrak{P}_1)|V(\mathfrak{P}_2).$$

Next, let $\{A_j\}$ be a set of right coset representatives for

$$\Gamma_1(\mathfrak{P}_1^{-1}\mathfrak{P}_2^{-2}\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{P}_2^2\mathfrak{I}_2)/\Gamma_1(\mathfrak{P}_2^{-2}\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{P}_2^2\mathfrak{I}_2).$$

With A' as above, $\{A'A_j(A')^{-1}\}$ is a set of right coset representatives for

$$\Gamma_1(\mathfrak{P}_1^{-1}\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{I}_2)/\Gamma_1(\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{I}_2);$$

hence

$$\begin{aligned} f(\tau)|V(\mathfrak{P}_2)|T(\mathfrak{P}_1) &= f(\tau)|[A']_k|\sum_j [A_j]_k \\ &= f(\tau)|\sum_j [A'A_j(A')^{-1}]_k|[A']_k = f(\tau)|T(\mathfrak{P}_1)|V(\mathfrak{P}_2). \end{aligned}$$

Now let $\{A_j\}$ be a set of right coset representatives for

$$\Gamma_1(\mathfrak{P}_1^{-1}\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{P}_2\mathfrak{I}_2)/\Gamma_1(\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{P}_2\mathfrak{I}_2),$$

and let $\{B_l\}$ be a set of right coset representatives for

$$\Gamma_1(\mathfrak{P}_2^{-1}\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{P}_2\mathfrak{I}_2)/\Gamma_1(\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{P}_2\mathfrak{I}_2).$$

Then $\{A_jB_l\}$ and $\{B_lA_j\}$ are both complete sets of right coset representatives for

$$\Gamma_1(\mathfrak{P}_1^{-1}\mathfrak{P}_2^{-1}\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{P}_2\mathfrak{I}_2)/\Gamma_1(\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{P}_2\mathfrak{I}_2),$$

and so $f(\tau)|T(\mathfrak{P}_1)|T(\mathfrak{P}_2) = f(\tau)|T(\mathfrak{P}_2)|T(\mathfrak{P}_1)$.

Finally, take $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a \in \mathfrak{P}_1^{-1}$, $b \in \mathfrak{P}_1^2\mathfrak{I}_2$, $c \in \mathfrak{P}_1^{-1}\mathfrak{I}_1$, $d \in \mathfrak{P}_1$, $d \equiv 1 \pmod{\mathfrak{I}_1\mathfrak{I}_2}$, $\det A = 1$; let $\{A_j\}$ be a set of right coset representatives for

$$\Gamma_1(\mathfrak{P}_1^{-3}\mathfrak{I}_1, \mathfrak{P}_1^3\mathfrak{I}_2)/\Gamma_1(\mathfrak{P}_1^{-2}\mathfrak{I}_1, \mathfrak{P}_1^3\mathfrak{I}_2),$$

and let $B \in \Gamma_1(\mathfrak{P}_1^{-2}\mathfrak{I}_1, \mathfrak{P}_1^2\mathfrak{I}_2)$ be a matrix whose lower right entry is in $\mathfrak{P}_1^2$. Then $\{AA_jB^{-1}\}$ is a complete set of right coset representatives for $\Gamma_1(\mathfrak{P}_1^{-1}\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{I}_2)/\Gamma_1(\mathfrak{I}_1, \mathfrak{P}_1\mathfrak{I}_2)$, giving us the fourth identity of the proposition. ∎

Remark. It is really an abuse of the language to call the map $T(\mathfrak{J})$ an operator since its range is not contained in its domain. However, we can view $T(\mathfrak{J})$ as an operator via the following procedure. We say two fractional ideals $\mathfrak{I}$ and $\mathfrak{J}$ are *equivalent* (written $\mathfrak{I} \sim \mathfrak{J}$) if $\mathfrak{I}\mathfrak{J}^{-1} = \alpha\mathcal{O}$ for some $\alpha \in \boldsymbol{K}$ with $\alpha \gg 0$. There are a finite number of equivalence classes (see Cor. 1.6 of [8], p. 112); we let $\mathfrak{I}_1, \ldots, \mathfrak{I}_h$ represent the distinct classes. Fixing $\mathfrak{I}$ and integral ideal, χ a character modulo $\mathfrak{I}$, and k a uniform integral weight, we form the direct product

$$\mathcal{M}_k(\mathfrak{I}, \chi) = \bigoplus_{l=1}^{h} \mathcal{M}_k(\Gamma_0(\mathfrak{I}\mathfrak{I}_l, \mathfrak{I}_l^{-1}), \chi).$$

Whenever $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are fractional ideals with $\mathfrak{I}_1\mathfrak{I}_2 = \mathfrak{I}$, we can find $\alpha \in \boldsymbol{K}$ with $\alpha \gg 0$ and some l $(1 \leqslant l \leqslant h)$ such that the map

$$f(\tau) \mapsto f(\tau)\left|\left[\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}\right]\right._k$$

defines an isomorphism from $\mathcal{M}_k(\Gamma_0(\mathfrak{I}_1, \mathfrak{I}_2), \chi)$ onto $\mathcal{M}_k(\Gamma_0(\mathfrak{I}\mathfrak{I}_l, \mathfrak{I}_l^{-1}), \chi)$. Identifying such isomorphic spaces, $\mathcal{M}_k(\mathfrak{I}, \chi)$ becomes a space which is invariant under the action of $T(\mathfrak{J})$ (where $T(\mathfrak{J})$ acts on each summand of $\mathcal{M}_k(\mathfrak{I}, \chi)$ and $\mathfrak{J}$ is relatively prime to $\mathfrak{I}$). Notice that if $\boldsymbol{K} = \boldsymbol{Q}$, these operators are the usual Hecke operators on $\mathcal{M}_k(\mathfrak{I}, \chi)$.

5. The Hecke operators for half-integral weight. To define these Hecke operators, we mimic as much as possible the definition given by Shimura in the case $\boldsymbol{K} = \boldsymbol{Q}$ (see [17]).

DEFINITION. Let $\mathfrak{I}$ and $\mathfrak{J}$ be fractional ideals such that $\mathfrak{J} \subseteq 4\mathcal{O}$; let $\mathfrak{P}$ be a prime not dividing $\mathfrak{J}$. Choose $\varrho \in \mathfrak{P} - \mathfrak{P}^2$ such that $\varrho \gg 0$ and ϱ is relatively prime to 2; write $\varrho\mathfrak{I} = \mathfrak{P}\mathfrak{I}'$. Set

$$\xi = \left(\begin{bmatrix} \varrho & 0 \\ 0 & 1 \end{bmatrix}, \frac{1}{\sqrt[4]{N(\varrho)}}\right),$$

and let $\{\tilde{A}_j\}$ be a set of right coset representatives for

$$\tilde{\Gamma}_1(\mathfrak{I}'\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1})/[\tilde{\Gamma}_1(\mathfrak{I}'\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1}) \cap \xi^{-1}\tilde{\Gamma}_1(\mathfrak{J}\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1})\xi].$$

For $k \in \frac{1}{2}\boldsymbol{Z}_+ - \boldsymbol{Z}_+$, χ a character modulo $\mathfrak{J}$, and $f(\tau) \in \mathcal{M}_k(\tilde{\Gamma}_0(\mathfrak{J}\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1}), \chi)$, we define

$$f(\tau)|T_k(\mathfrak{P}, \tilde{\Gamma}_0(\mathfrak{J}\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1})) = N(\mathfrak{P})^{k-1}\sum_j f(\tau)|[\xi\tilde{A}_j\xi^{-1}]_k.$$

PROPOSITION 5.1. *With the notation as above,*

$$f(\tau)|T_k(\mathfrak{P}, \tilde{\Gamma}_0(\mathfrak{J}\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1})) = 0.$$

Proof. Notice that

$$\begin{aligned} &\tilde{\Gamma}_1(\mathfrak{I}'\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1}) \cap \xi^{-1}\tilde{\Gamma}_1(\mathfrak{J}\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1})\xi \\ &\quad = \left\{\tilde{A} \in \tilde{\Gamma}_1(\mathfrak{I}'\mathfrak{I}^2\partial, \mathfrak{I}^{-2}\partial^{-1}) \,\Big|\, \frac{\Theta\left(\mathfrak{I}, A\frac{\tau}{\varrho}\right)}{\Theta(\mathfrak{I}, \tau/\varrho)} = \frac{\Theta(\mathfrak{I}, A^{\varrho}\tau)}{\Theta(\mathfrak{I}, \tau)}\right\} \end{aligned}$$

where $A^{\varrho} = \begin{bmatrix} \varrho & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} \varrho^{-1} & 0 \\ 0 & 1 \end{bmatrix}$. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $d \gg 0$ and d relatively prime to 4∂ and to $\mathfrak{J}$, (2) shows that

$$\frac{\Theta\left(\mathfrak{J}, A\frac{\tau}{\varrho}\right)}{\Theta(\mathfrak{J}, \tau/\varrho)} = \frac{\Theta(\mathfrak{J}, A^{\varrho}\tau)}{\Theta(\mathfrak{J}, \tau)} \quad \text{if and only if } (\varrho|d) = 1.$$

Since we can find such a matrix A with $(\varrho|d) = -1$, we have that

$$[\tilde{\Gamma}: \tilde{\Gamma}_1(\mathfrak{J}'\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1}) \cap \xi^{-1}\tilde{\Gamma}_1(\mathfrak{J}\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1})\xi] = 2$$

where

$$\Gamma = \Gamma_1(\mathfrak{J}'\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1}) \cap \begin{bmatrix} \varrho^{-1} & 0 \\ 0 & 1 \end{bmatrix} \Gamma_1(\mathfrak{J}\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1}) \begin{bmatrix} \varrho & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $\tilde{A}_1$ and $\tilde{A}_2$ represent these two right cosets. Letting $\{\tilde{B}_j\}$ represent the right cosets of $\tilde{\Gamma}$ in $\tilde{\Gamma}_1(\mathfrak{J}'\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1})$, we get

$$N(\mathfrak{P})^{1-k} f(\tau)|T(\mathfrak{P}) = \sum_j f(\tau)|[\xi\tilde{A}_1\tilde{B}_j\xi^{-1}]_k + \sum_j f(\tau)|[\xi\tilde{A}_2\tilde{B}_j\xi^{-1}]_k$$
$$= \sum_j f(\tau)|[\xi\tilde{B}_j\xi^{-1}]_k - \sum_j f(\tau)|[\xi\tilde{B}_j\xi^{-1}]_k = 0. \quad \blacksquare$$

This proposition motivates the following

Definition. Let $\mathfrak{J}$ and $\mathfrak{J}$ be fractional ideals with $\mathfrak{J} \subseteq 4\mathcal{O}$; let $\mathfrak{J}_1$ be an integral ideal which is relatively prime to $\mathfrak{J}$. Choose $\varrho \in \mathfrak{J}_1$ such that $\varrho\mathfrak{J}_1^{-1}$ is relatively prime to $2\mathfrak{J}_1$; write $\varrho\mathcal{O} = \mathfrak{J}_1\mathfrak{J}'$. Set

$$\xi = \left(\begin{bmatrix} \varrho^2 & 0 \\ 0 & 1 \end{bmatrix}, \frac{1}{\sqrt[4]{N(\varrho^2)}}\right),$$

and let $\{\tilde{A}_j\}$ be a set of right coset representatives for

$$\tilde{\Gamma}_1(\mathfrak{J}(\mathfrak{J}'\mathfrak{J})^2\partial, (\mathfrak{J}'\mathfrak{J})^{-2}\partial^{-1})$$
$$/[\tilde{\Gamma}_1(\mathfrak{J}(\mathfrak{J}'\mathfrak{J})^2\partial, (\mathfrak{J}'\mathfrak{J})^{-2}\partial^{-1}) \cap \xi^{-1}\tilde{\Gamma}_1(\mathfrak{J}\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1})\xi].$$

Then for $k \in \frac{1}{2}\mathbf{Z}_+ - \mathbf{Z}_+$, we define the Hecke operator

$$T_k(\mathfrak{J}_1^2, \tilde{\Gamma}_0(\mathfrak{J}\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1})): \mathcal{M}_k(\tilde{\Gamma}_0(\mathfrak{J}\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1}), \chi)$$
$$\to \mathcal{M}_k(\tilde{\Gamma}_0(\mathfrak{J}(\mathfrak{J}')^{-2}\mathfrak{J}^2\partial, (\mathfrak{J}')^2\mathfrak{J}^{-2}\partial^{-1}), \chi)$$

by

$$f(\tau)|T_k(\mathfrak{J}_1^2, \tilde{\Gamma}_0(\mathfrak{J}\mathfrak{J}^2\partial, \mathfrak{J}^{-2}\partial^{-1})) = N(\mathfrak{J}_1)^{k-2} f(\tau)|\sum_j [\xi\tilde{A}_j\xi^{-1}]_k.$$

One can verify that this definition is independent of the choice of ϱ and of the choice of coset representatives $\{\tilde{A}_j\}$.

Although these maps $T(\mathfrak{J}_1^2)$ are not truly operators, the procedure used in the previous section can be used here as well, where the map

$$f(\tau) \mapsto f(\tau)\left|\left[\left(\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \frac{1}{\sqrt[4]{N(\alpha)}}\right)\right]_k\right.$$

defines the appropriate isomorphism of spaces of modular forms.

6. The action of the Hecke operators on $\Theta(L, \tau)$. We now study the effect of the operators $T(\mathfrak{P})$ and $T(\mathfrak{P}^2)$ on the theta series $\Theta(L, \tau)$ and relate these effects to the structure of $L/\mathfrak{P}L$. First we prove

Proposition 6.1. *Suppose L has even rank m. Then for $\mathfrak{P}$ a prime ideal not dividing $2S(L)$, $\Theta(L, \tau)|V(\mathfrak{P}) = \varepsilon N(\mathfrak{P})^{-m/2}\Theta(\mathfrak{P}^{-1}L, \tau)$ where $\varepsilon = +1$ if $L/\mathfrak{P}L$ is hyperbolic as a quadratic space over $\mathcal{O}/\mathfrak{P}$, and $\varepsilon = -1$ otherwise.*

Remark. We view $L/\mathfrak{P}L$ as a quadratic space over $\mathcal{O}/\mathfrak{P}$ by choosing $\alpha \gg 0$ such that $N(L^{\alpha}) = \alpha N(L) \subseteq \mathcal{O}$ but $\mathfrak{P} \nmid N(L^{\alpha})$; we then identify $L/\mathfrak{P}L$ with $L^{\alpha}/\mathfrak{P}L^{\alpha}$ via the identity map on $L/\mathfrak{P}L$.

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix such that $a \in \mathfrak{P}$, $b \in \mathfrak{P}^{-1}N(L)^{-1}\partial^{-1}$, $c \in \mathfrak{P}S(L)N(L)\partial$, $d \in \mathcal{O}$ with $d \gg 0$, $d \equiv 1 \pmod{S(L)}$ and $\det A = 1$. Then

$$\Theta(\mathfrak{P}L, \tau)|[A^{-1}]_{m/2} = \Theta(\mathfrak{P}L, \tau)|V(\mathfrak{P});$$

we show that $\Theta(L, \tau)|[A]_{m/2} = \varepsilon N(\mathfrak{P})^{m/2}\Theta(\mathfrak{P}L, \tau)$.

Calculations similar to those used to prove (2) give us

$$\Theta\left(L, \frac{a\tau+b}{c\tau+d}\right) = \sum_{x_0 \in L/d\mathfrak{P}L} e\left(\frac{b}{d}Q(x_0)\right) \sum_{x \in \mathfrak{P}L} e\left(d\frac{Q(x+x_0/d)}{d\frac{1}{\tau}+c}\right)$$

and using (1),

$$= \frac{i^{-mn/2}}{\sqrt{\Phi(\mathfrak{P}L)}} N(d)^{-m/2} N\left(d\frac{1}{\tau}+c\right)^{m/2}$$
$$\times \sum_{\substack{x \in (\widetilde{\mathfrak{P}L}) \\ x_0 \in L/d\mathfrak{P}L}} e\left(-\frac{c}{d}Q(bx_0+x)\right) e\left(-Q(x)\frac{1}{\tau}\right).$$

For any $x \in (\widetilde{\mathfrak{P}L}) = \mathfrak{P}^{-1}\tilde{L}$, $bx_0 + x$ runs over $\mathfrak{P}^{-1}\tilde{L}/d\tilde{L}$ as x_0 runs over $L/d\mathfrak{P}L$. Also, for $x \in \mathfrak{P}^{-1}\tilde{L}$ and $y \in d\tilde{L}$ we have $\frac{c}{d}Q(x+y) \equiv \frac{c}{d}Q(x) \pmod{2\partial^{-1}}$; thus

$$\Theta\left(L, \frac{a\tau+b}{c\tau+d}\right)$$
$$= \frac{i^{-mn/2}}{\sqrt{\Phi(\mathfrak{P}L)}} N(d)^{-m/2} N\left(d\frac{1}{\tau}+c\right)^{m/2} \sum_{x_0 \in L/d\mathfrak{P}L} e\left(\frac{b}{d}Q(x_0)\right) \sum_{x \in (\widetilde{\mathfrak{P}L})} e\left(-Q(x)\frac{1}{\tau}\right)$$

and by (1),

$$= N(d)^{-m/2} N\left(\frac{d}{\tau}\right)^{m/2} N(\tau)^{m/2} \sum_{x\in L/d\mathfrak{P}L} e\left(\frac{b}{d}Q(x)\right)\Theta(\mathfrak{P}L,\tau)$$

and by Proposition 3.1,

$$= N(d)^{-m/2} N\left(\frac{d}{\tau}\right)^{m/2} N(\tau)^{m/2} \sum_{x\in \mathfrak{P}L/d\mathfrak{P}L} e\left(\frac{b}{d}Q(x)\right)\sum_{x\in dL/d\mathfrak{P}L} e\left(\frac{b}{d}Q(x)\right)\Theta(\mathfrak{P}L,\tau).$$

Theorem 3.7 shows that

$$N(d)^{-m/2}\sum_{x\in\mathfrak{P}L/d\mathfrak{P}L} e\left(\frac{b}{d}Q(x)\right) = (\Theta(\mathfrak{P}L,\tau)|[A]_{m/2})/\Theta(\mathfrak{P}L,\tau) = 1$$

(since $d\equiv 1 \pmod{S(L)}$ and $S(L)=S(\mathfrak{P}L)$) while Proposition 3.3 shows that

$$\begin{aligned}\sum_{x\in dL/d\mathfrak{P}L} e\left(\frac{b}{d}Q(x)\right) &= \sum_{x\in L/\mathfrak{P}L} e(bdQ(x))\\ &= ((-1)^{m/2}\operatorname{disc} L^{bd}_{\mathfrak{P}}|\mathfrak{P})\, N(\mathfrak{P})^{m/2}\\ &= \begin{cases} +N(\mathfrak{P})^{m/2} & \text{if } L^{bd}/\mathfrak{P}L^{bd} \text{ is hyperbolic,}\\ -N(\mathfrak{P})^{m/2} & \text{otherwise.}\end{cases}\end{aligned}$$

(Recall that a regular quadratic space over a finite field is completely determined by its dimension and its discriminant, see [12], § 62.) Since $L^{bd}/\mathfrak{P}L^{bd}$ is hyperbolic if and only if $L/\mathfrak{P}L$ is, the proposition follows. ∎

We use this in proving

Proposition 6.2. *Let L be a lattice of even rank m and let $\mathfrak{P}$ be a prime ideal with $\mathfrak{P}\nmid 2S(L)$. Then*

$$\Theta(L,\tau)|T(\mathfrak{P}) = \frac{i^{-mn/2}}{\sqrt{\Phi(L)}} N(\tau)^{-m/2}\left(N(\mathfrak{P})^{m/2}\sum_{\substack{x\in\tilde L\\ Q(x)\in\mathfrak{P}N(\tilde L)}} e\left(-Q(x)\frac{1}{\tau}\right) + \varepsilon N(\mathfrak{P})^{m-1}\sum_{x\in\mathfrak{P}\tilde L} e\left(-Q(x)\frac{1}{\tau}\right)\right).$$

Proof. Let $\{c_j\}$ be a set of coset representatives for the quotient group $\mathfrak{P}^{-1}S(L)N(L)\partial/S(L)N(L)\partial$; then

$$\Theta(L,\tau)|T(\mathfrak{P}) = N(\mathfrak{P})^{(m/2)-1}\left(\sum_j \Theta(L,\tau)\left|\left[\begin{bmatrix}1&0\\c_j&1\end{bmatrix}\right]_{m/2}\right. + \Theta(L,\tau)|V(\mathfrak{P})\right).$$

Using (1), we get

$$\sum_j \Theta(L,\tau)\left|\left[\begin{bmatrix}1&0\\c_j&1\end{bmatrix}\right]_{m/2}\right. = \frac{i^{-mn/2}}{\sqrt{\Phi(L)}}N(\tau)^{-m/2}\sum_{x\in\tilde L}\left(\sum_j e(-c_jQ(x))\right)e\left(-Q(x)\frac{1}{\tau}\right).$$

The proposition now follows easily from this last equation and Proposition 6.1. ∎

Similarly, we obtain

Proposition 6.3. *Let L be a lattice of even rank m, and let $\mathfrak{P}$ be a prime ideal with $\mathfrak{P}\nmid 2S(L)$. Then*

$$\begin{aligned}\Theta(L,\tau)|T(\mathfrak{P})|T(\mathfrak{P}) = \frac{i^{-mn/2}}{\sqrt{\Phi(L)}}N(\tau)^{-m/2}\Bigg(& N(\mathfrak{P})^{m}\sum_{\substack{x\in L\\ Q(x)\in\mathfrak{P}^2N(\tilde L)}} e\left(-Q(x)\frac{1}{\tau}\right)\\ &+\varepsilon N(\mathfrak{P})^{(3m/2)-1}\sum_{\substack{x\in\mathfrak{P}\tilde L\\ Q(x)\in\mathfrak{P}^3N(\tilde L)}} e\left(-Q(x)\frac{1}{\tau}\right)\\ &+\varepsilon N(\mathfrak{P})^{(3m/2)-1}\sum_{x\in\mathfrak{P}\tilde L} e\left(-Q(x)\frac{1}{\tau}\right)\\ &+N(\mathfrak{P})^{2m-2}\sum_{x\in\mathfrak{P}^2\tilde L} e\left(-Q(x)\frac{1}{\tau}\right)\Bigg)\end{aligned}$$

where ε is as in Proposition 6.1.

Now we consider the case when rank L is odd.

Proposition 6.4. *Let L be a lattice of odd rank m. Choose $\mathfrak{I}$ to be the smallest fractional ideal such that $N(L)\subseteq\mathfrak{I}^2$, and set $\mathfrak{J}=S(L)N(L)\mathfrak{I}^{-2}$. For $\mathfrak{P}$ a prime ideal not dividing $\mathfrak{J}$,*

$$\begin{aligned}\Theta(L,\tau)|T(\mathfrak{P}^2) = \frac{i^{-mn/2}}{\sqrt{\Phi(L)}}N(\tau)^{-m/2}\Bigg(& N(\mathfrak{P})^{m/2}\sum_{\substack{x\in L\\ Q(x)\in\mathfrak{P}^2N(\tilde L)}} e\left(-Q(x)\frac{1}{\tau}\right)\\ +N(\mathfrak{P})^{m-(3/2)}((-1)^{(m-1)/2}\operatorname{disc}(\mathfrak{P}^{-e}L)_{\mathfrak{P}}|\mathfrak{P}) &\sum_{\substack{x\in\mathfrak{P}\tilde L\\ Q(x)\notin\mathfrak{P}^3N(\tilde L)}} (Q(x)\varrho^2|\mathfrak{P})\,e\left(-Q(x)\frac{1}{\tau}\right)\\ &+N(\mathfrak{P})^{(3m/2)-2}\sum_{x\in\mathfrak{P}^2\tilde L} e\left(-Q(x)\frac{1}{\tau}\right)\Bigg)\end{aligned}$$

where $e=\operatorname{ord}_{\mathfrak{P}}\mathfrak{J}$ and $\varrho\in K$ such that $\operatorname{ord}_{\mathfrak{P}}\varrho=-1+\operatorname{ord}_{\mathfrak{P}}\mathfrak{J}\partial$. (To evaluate $(Q(x)\varrho^2|\mathfrak{P})$, we identify $Q(x)\varrho^2+\mathfrak{P}\mathcal{O}_{\mathfrak{P}}\in\mathcal{O}_{\mathfrak{P}}/\mathfrak{P}\mathcal{O}_{\mathfrak{P}}$ with its canonical image in $\mathcal{O}/\mathfrak{P}$.)

Proof. Let $\{\gamma_j\}$ be a set of coset representatives for $\mathfrak{J}\mathfrak{P}^{-2}\mathfrak{I}^2\partial/\mathfrak{J}\mathfrak{I}^2\partial$, and set $A_j=\begin{bmatrix}1&0\\\gamma_j&1\end{bmatrix}$. Using Theorem 10.3 of [8] (p. 182) we choose $d\in\mathcal{O}$ such that $d\gg 0$, $d\equiv 1 \pmod{\mathfrak{J}}$ and $d\mathcal{O}=\mathfrak{P}_0\mathfrak{P}$ with $\mathfrak{P}_0$ a prime ideal not dividing $\partial\mathfrak{P}$ nor $\mathfrak{J}$. Let $\{c_k\}$ be a set of coset representatives for $(\mathfrak{J}\mathfrak{P}^{-2}\mathfrak{I}^2\partial/\mathfrak{J}\mathfrak{P}^{-1}\mathfrak{I}^2\partial)^\times$ such that each c_k is relatively prime to $\mathfrak{P}_0$; then by the Chinese Remainder Theorem we can find $a_k\in\mathcal{O}$ and $b_k\in\mathfrak{P}^2\mathfrak{I}^{-2}\partial^{-1}$ such that the determinant of $B_k=\begin{bmatrix}a_k&b_k\\c_k&d\end{bmatrix}$ is equal to 1. Now we choose $\delta\in\mathfrak{P}^2$ such that $\delta\gg 0$, $\delta\equiv 1 \pmod{\mathfrak{J}}$ and $\delta\mathfrak{P}^{-2}$ is relatively prime to ∂ and to $\mathfrak{J}$; we again use the Chinese Remainder Theorem to choose $\alpha\in\mathcal{O}$, $\beta\in\mathfrak{P}^2\mathfrak{I}^{-2}\partial^{-1}$ and

$\gamma \in \mathfrak{J}\mathfrak{P}^{-2}\mathfrak{J}^2\partial$ such that $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ has determinant 1. Then

$$\Theta(L, \tau)|T(\mathfrak{P}^2)$$
$$= N(\mathfrak{P})^{(m/2)-2}\Big(\sum_j \Theta(L, \tau)|[\tilde{A}_j]_{m/2} + \sum_k \Theta(L, \tau)|[\tilde{B}_k]_{m/2} + \Theta(L, \tau)|[\tilde{B}]_{m/2}\Big).$$

Using the techniques previously employed, we get

$$\sum_j \Theta(L, \tau)|[\tilde{A}_j]_{m/2} = \frac{i^{-mn/2}}{\sqrt{\Phi(L)}} N(\mathfrak{P})^2 \sum_{\substack{x \in L \\ Q(x) \in \mathfrak{P}^2 N(\tilde{L})}} e\left(-Q(x)\frac{1}{\tau}\right)$$

and

$$\Theta(L, \tau)|[\tilde{B}]_{m/2} = N(\mathfrak{P})^m \frac{i^{-mn/2}}{\sqrt{\Phi(L)}} N(\tau)^{-m/2} \sum_{x \in \mathfrak{P}^2\tilde{L}} e\left(-Q(x)\frac{1}{\tau}\right).$$

Tedious calculations using our previous techniques yield

$$\Theta(L, \tau)|[\tilde{B}_k]_{m/2}$$
$$= N(\mathfrak{P})^{(m-1)/2} \frac{i^{-mn/2}}{\sqrt{\Phi(L)}} N(\tau)^{-m/2}\big((-1)^{(m+1)/2}|\mathfrak{P}\big)(2\operatorname{disc} L_{\mathfrak{P}_0}|\mathfrak{P}_0)$$
$$\times\Bigg(\sum_{\alpha \in \mathfrak{P}^{-1}\mathfrak{P}_0\mathfrak{J}/\mathfrak{P}_0\mathfrak{J}} e\left(\frac{b_k}{d}2\alpha^2\right)\Bigg)\Bigg(\sum_{\substack{x \in \mathfrak{P}\mathfrak{P}_0\tilde{L}/\mathfrak{P}^2\mathfrak{P}_0\tilde{L} \\ y \in \mathfrak{P}^2\tilde{L}}} e\left(-\frac{c_k}{d}Q(x)\right)e\left(-Q(x+y)\frac{1}{\tau}\right)\Bigg).$$

If $x \in \mathfrak{P}_0\mathfrak{P}\tilde{L}$ such that $Q(x) \in \mathfrak{P}^3 N(\tilde{L})$, then we have $\frac{c_k}{d}Q(x) \in 2\partial^{-1}$; in this case the techniques of Proposition 3.3 yield

$$\sum_k e\left(-\frac{c_k}{d}Q(x)\right) \sum_{\alpha \in \mathfrak{P}^{-1}\mathfrak{P}_0\mathfrak{J}/\mathfrak{P}_0\mathfrak{J}} e\left(\frac{b_k}{d}2\alpha^2\right) = \left(\sum_k\left(\frac{b_k\eta^2}{\nu}\Big|\mathfrak{P}\right)\right)\left(\sum_{\alpha \in \mathcal{O}/\mathfrak{P}} e\left(\frac{2\nu\alpha^2}{d}\right)\right) = 0$$

where $\eta \in \mathfrak{P}^{-1}\mathfrak{P}_0\mathfrak{J} - \mathfrak{P}_0\mathfrak{J}$ and $\nu \in 2\mathfrak{P}^{-\operatorname{ord}_{\mathfrak{P}}\partial} - 2\mathfrak{P}^{1-\operatorname{ord}_{\mathfrak{P}}\partial}$. Let $x \in \mathfrak{P}_0\mathfrak{P}\tilde{L}$ such that $Q(x) \notin \mathfrak{P}^3 N(\tilde{L})$. Choose $\varrho \in \mathfrak{P}^{1-\operatorname{ord}_{\mathfrak{P}}\mathfrak{J}}$. Observe that

$$\left(\sum_k\left(\frac{b_k\eta^2}{\nu}\Big|\mathfrak{P}\right)\right) = (-c_k\varrho^2\nu|\mathfrak{P})$$

since $(b_k c_k|\mathfrak{P}) = 1$. Now,

$$(Q(x)\varrho^{-2}\nu^{-2}|\mathfrak{P})\sum_k e\left(-\frac{c_k}{d}Q(x)\right)\sum_{\alpha \in \mathfrak{P}^{-1}\mathfrak{P}_0\mathfrak{J}/\mathfrak{P}_0\mathfrak{J}} e\left(\frac{b_k}{d}2\alpha^2\right)$$
$$= \sum_k (Q(x)\varrho^{-2}\nu^{-2}|\mathfrak{P})\,e\left(-\frac{c_k}{d}Q(x)\right)\left(\frac{b_k\eta^2}{\nu}\Big|\mathfrak{P}\right)\sum_{\alpha \in \mathcal{O}/\mathfrak{P}} e\left(\frac{2\nu\alpha^2}{d}\right)$$
$$= \sum_k\left(-\frac{Q(x)c_k}{\nu}\Big|\mathfrak{P}\right)e\left(-\frac{c_k}{d}Q(x)\right)\sum_{\alpha \in \mathcal{O}/\mathfrak{P}} e\left(\frac{2\nu\alpha^2}{d}\right).$$

As k varies, $-Q(x)c_k/\nu$ runs over $(2\mathfrak{P}_0\partial^{-1}/\mathfrak{P}\mathfrak{P}_0\partial^{-1})^\times$, thus

$$\sum_k\left(-\frac{Q(x)c_k}{\nu}\Big|\mathfrak{P}\right)e\left(-\frac{c_k}{d}Q(x)\right) = \sum_{\beta \in \mathfrak{P}_0/\mathfrak{P}\mathfrak{P}_0} e\left(\frac{\beta\nu}{d}\right)(\beta|\mathfrak{P}) = \sum_{\beta \in \mathfrak{P}_0/\mathfrak{P}\mathfrak{P}_0} e\left(\frac{\beta^2\nu}{d}\right).$$

Proposition 3.3 gives us

$$\sum_{\beta \in \mathfrak{P}_0/\mathfrak{P}\mathfrak{P}_0} e\left(\frac{\beta^2\nu}{d}\right)\sum_{\alpha \in \mathcal{O}/\mathfrak{P}} e\left(\frac{2\nu\alpha^2}{d}\right) = (-2|\mathfrak{P})N(\mathfrak{P});$$

so we now have

$$\sum_k \Theta(L, \tau)|[\tilde{B}_k]_{m/2} = N(\mathfrak{P})^{(m+1)/2}\frac{i^{-mn/2}}{\sqrt{\Phi(L)}}N(\tau)^{-m/2}$$
$$\times\big((-1)^{(m-1)/2}|\mathfrak{P}\big)(2\operatorname{disc} L_{\mathfrak{P}_0}|\mathfrak{P}_0)$$
$$\times \sum_{\substack{x \in \mathfrak{P}\tilde{L} \\ Q(x) \notin \mathfrak{P}^3 N(\tilde{L})}} (2Q(x)\varrho^{-2}\nu^{-2}|\mathfrak{P})\,e\left(-Q(x)\frac{1}{\tau}\right).$$

Finally, we observe that d is the lower right entry of some matrix A_d in $\Gamma_1(\mathfrak{J}\mathfrak{J}^2\mathfrak{P}^{-2e}\partial, \mathfrak{J}^{-2}\mathfrak{P}^{2e}\partial^{-1})$ where $e = \operatorname{ord}_{\mathfrak{P}}\mathfrak{J}$. Thus by Theorem 3.7 we have

$$\Theta(\mathfrak{P}^{-e}L, \tau)|[\tilde{A}_d]_{m/2} = (2\operatorname{disc}(\mathfrak{P}^{-e}L)_{\mathfrak{P}_0}|\mathfrak{P}_0)(2\operatorname{disc}(\mathfrak{P}^{-e}L)_{\mathfrak{P}}|\mathfrak{P})\,\Theta(\mathfrak{P}^{-e}L, \tau)$$
$$= \Theta(\mathfrak{P}^{-e}L, \tau).$$

Hence

$$(2\operatorname{disc} L_{\mathfrak{P}_0}|\mathfrak{P}_0) = (2\operatorname{disc}(\mathfrak{P}^{-e}L)_{\mathfrak{P}_0}|\mathfrak{P}_0) = (2\operatorname{disc}(\mathfrak{P}^{-e}L)_{\mathfrak{P}}|\mathfrak{P}). \ \blacksquare$$

7. The action of $T(\mathfrak{P})$ and $T(\mathfrak{P}^2)$ on $\Theta(L, \tau)$ in terms of sublattices of L. To realize $\Theta(L, \tau)|T(\mathfrak{P})$ or $\Theta(L, \tau)|T(\mathfrak{P}^2)$ as a linear combination of theta series associated to sublattices of L, we examine particular sublattices of L which we define as follows.

DEFINITION. Let L be a lattice of rank m and let $\mathfrak{P}$ be a prime ideal not dividing 2. Consider $L/\mathfrak{P}L$ as a quadratic space over $\mathcal{O}/\mathfrak{P}$ by choosing $\alpha \in K$ such that $\alpha \gg 0$ and $\alpha N(L) \subseteq \mathcal{O}$ with $\operatorname{ord}_{\mathfrak{P}}\alpha N(L) = 0$; then with the quadratic form induced by Q^α, the space $L^\alpha/\mathfrak{P}L^\alpha$ becomes a quadratic space over $\mathcal{O}/\mathfrak{P}$. We say that a sublattice L' of L is a *$\mathfrak{P}$-sublattice* of L if $\mathfrak{P}L \subseteq L'$ and $(L')^\alpha/\mathfrak{P}L^\alpha$ is a maximal totally isotropic subspace of $L^\alpha/\mathfrak{P}L^\alpha$. (Notice that this definition is independent of the choice of α.)

If L' is a $\mathfrak{P}$-sublattice of L and L'' is a $\mathfrak{P}$-sublattice of L' with $\dim L''/(\mathfrak{P}L \cap L'') = \dim L'/\mathfrak{P}L$, then we say L'' is a $\mathfrak{P}^2$-sublattice of L.

We now describe which vectors of L these $\mathfrak{P}$-sublattices contain in the case that $\mathfrak{P} \nmid 2S(L)$. Proposition 2.1 implies that $L^\alpha/\mathfrak{P}L^\alpha$ is regular whenever $\mathfrak{P} \nmid S(L)$, so the maximal totally isotropic subspaces of $L^\alpha/\mathfrak{P}L^\alpha$ are of dimension equal to the Witt index of $L^\alpha/\mathfrak{P}L^\alpha$ (see Ch. III, §6 of [2]).

Proposition 7.1. *Let $\mathfrak{P}$ be a prime ideal such that $\mathfrak{P} \nmid S(L)$; let $m = \operatorname{rank} L$ and let α be as in the preceding definition. If $L^\alpha/\mathfrak{P}L^\alpha$ is anisotropic then $\mathfrak{P}L$ is the only $\mathfrak{P}$-sublattice of L. Suppose that $L^\alpha/\mathfrak{P}L^\alpha$ is isotropic; then*

$$N(L') = \mathfrak{P}N(L)$$

and

$$S(L') = \begin{cases} S(L) & \text{if } L^\alpha/\mathfrak{P}L^\alpha \text{ is hyperbolic,} \\ \mathfrak{P}S(L) & \text{otherwise.} \end{cases}$$

For L'' a $\mathfrak{P}$-sublattice of L', we have

$$N(L'') = \mathfrak{P}^2 N(L) \quad \text{and} \quad S(L'') = S(L).$$

(Notice that $L^\alpha/\mathfrak{P}L^\alpha$ can be anisotropic only if $m = 1$ or 2.)

Proof. First, observe that it suffices to prove the assertions locally at $\mathfrak{P}$. Also, replacing $L_\mathfrak{P}$ with $L^\alpha_\mathfrak{P}$ if necessary, we can assume that $N(L_\mathfrak{P}) \subseteq \mathcal{O}_\mathfrak{P}$. Using the invariant factor theorem (see 81:11 of [12]) we can find $x_1, \ldots, x_m \in L_\mathfrak{P}$ such that

$$L_\mathfrak{P} = \mathcal{O}_\mathfrak{P} x_1 \oplus \ldots \oplus \mathcal{O}_\mathfrak{P} x_m$$

and

$$L'_\mathfrak{P} = \mathfrak{A}_1 x_1 \oplus \ldots \oplus \mathfrak{A}_m x_m$$

for some $\mathcal{O}_\mathfrak{P}$-ideals $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$ with $\mathfrak{A}_j | \mathfrak{A}_{j+1}$. Since $\mathfrak{P}L \subseteq L'$, we know $\mathfrak{P}\mathcal{O}_\mathfrak{P} \subseteq \mathfrak{A}_j \subseteq \mathcal{O}_\mathfrak{P}$; furthermore, $L_\mathfrak{P}/\mathfrak{P}L_\mathfrak{P} \simeq L/\mathfrak{P}L$ and $L'_\mathfrak{P}/\mathfrak{P}L'_\mathfrak{P} \simeq L'/\mathfrak{P}L'$ (via the canonical map $x + \mathfrak{P}L \mapsto x + \mathfrak{P}L_\mathfrak{P}$) so we must have $\mathfrak{A}_1 = \ldots = \mathfrak{A}_k = \mathcal{O}_\mathfrak{P}$ and $\mathfrak{A}_{k+1} = \ldots = \mathfrak{A}_m = \mathfrak{P}\mathcal{O}_\mathfrak{P}$ where $k = \dim L'/\mathfrak{P}L$.

Letting $\bar{x}_j$ denote $x_j + \mathfrak{P}L_\mathfrak{P}$, we have that $L'_\mathfrak{P}/\mathfrak{P}L'_\mathfrak{P}$ is spanned by $\{\bar{x}_1, \ldots, \bar{x}_k\}$. We can extend this set to a basis $\{\bar{x}_1, \ldots, \bar{x}_k, \bar{y}_{k+1}, \ldots, \bar{y}_m\}$ for $L_\mathfrak{P}/\mathfrak{P}L_\mathfrak{P}$ such that with respect to this basis

$$L_\mathfrak{P}/\mathfrak{P}L_\mathfrak{P} \simeq \begin{bmatrix} 0 & I_k & 0 \\ I_k & 0 & 0 \\ 0 & 0 & A \end{bmatrix} \in M_m(\mathcal{O}_\mathfrak{P}/\mathfrak{P}\mathcal{O}_\mathfrak{P})$$

where I_k is the $k \times k$ identity matrix and A is an $(m-2k) \times (m-2k)$ nonsingular matrix (see Ch. 1, § 4 of [10]). Let $B \in M_m(\mathcal{O}_\mathfrak{P}/\mathfrak{P}\mathcal{O}_\mathfrak{P})$ be the matrix which maps the basis $\{\bar{x}_1, \ldots, \bar{x}_m\}$ to $\{\bar{x}_1, \ldots, \bar{x}_k, \bar{y}_{k+1}, \ldots, \bar{y}_m\}$. Multiplying $\bar{y}_m$ by a suitable scalar from $\mathcal{O}_\mathfrak{P}/\mathfrak{P}\mathcal{O}_\mathfrak{P}$, we may assume that $B \in SL_m(\mathcal{O}_\mathfrak{P}/\mathfrak{P}\mathcal{O}_\mathfrak{P})$. Now we have

$$L_\mathfrak{P}/\mathfrak{P}L_\mathfrak{P} \simeq \begin{bmatrix} 0 & D & 0 \\ D & 0 & 0 \\ 0 & 0 & A \end{bmatrix} \in M_m(\mathcal{O}_\mathfrak{P}/\mathfrak{P}\mathcal{O}_\mathfrak{P})$$

with respect to the basis $\{\bar{x}_1, \ldots, \bar{x}_k, \bar{y}_{k+1}, \ldots, \bar{y}_m\}$, where D is a nonsingular diagonal $k \times k$ matrix and A is a nonsingular $(m-2k) \times (m-2k)$ matrix. The proof of Lemma 1.38 of [16] (p. 20) implies that we can find $B' \in SL_m(\mathcal{O}_\mathfrak{P})$ such that B' transforms the $\mathcal{O}_\mathfrak{P}$-basis $\{x_1, \ldots, x_m\}$ into an $\mathcal{O}_\mathfrak{P}$-basis $\{x'_1, \ldots, x'_m\}$ such that $x'_j - x_j \in \mathfrak{P}L_\mathfrak{P}$ for $1 \leqslant j \leqslant k$, and $x'_j - y_j \in \mathfrak{P}L_\mathfrak{P}$ for $j > k$; thus

$$L_\mathfrak{P} = \mathcal{O}_\mathfrak{P} x'_1 \oplus \ldots \oplus \mathcal{O}_\mathfrak{P} x'_m$$

and

$$L'_\mathfrak{P} = \mathcal{O}_\mathfrak{P} x'_1 \oplus \ldots \oplus \mathcal{O}_\mathfrak{P} x'_k \oplus \mathfrak{P}\mathcal{O}_\mathfrak{P} x'_{k+1} \oplus \ldots \oplus \mathfrak{P}\mathcal{O}_\mathfrak{P} x'_m.$$

Hence, taking $\varrho \in \mathfrak{P}^{-1}\mathcal{O}^\times_\mathfrak{P}$, we get

$$(L'_\mathfrak{P})^\varrho/\mathfrak{P}(L'_\mathfrak{P})^\varrho \simeq \begin{bmatrix} * & D & 0 \\ D & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_m(\mathcal{O}_\mathfrak{P}/\mathfrak{P}\mathcal{O}_\mathfrak{P})$$

with respect to the basis

$$\{x'_1 + \mathfrak{P}L'_\mathfrak{P}, \ldots, x'_k + \mathfrak{P}L_\mathfrak{P}, \varrho^{-1} x_{k+1} + \mathfrak{P}L'_\mathfrak{P}, \ldots, \varrho^{-1} x'_m + \mathfrak{P}L'_\mathfrak{P}\}.$$

(Notice that $(L'_\mathfrak{P})^\varrho/\mathfrak{P}(L'_\mathfrak{P})^\varrho$ has Witt index k and a radical of dimension $m-2k$.) Since $L_\mathfrak{P}$ is unimodular,

$$\operatorname{vol}(L'_\mathfrak{P}) = \mathfrak{P}^{2m-2k} \operatorname{vol}(L_\mathfrak{P}) = \mathfrak{P}^{2m-2k}\mathcal{O}_\mathfrak{P}$$

(see 82:11 of [12]). We also have that

$$L'_\mathfrak{P} \simeq \langle \beta_1, \ldots, \beta_m \rangle$$

for some $\beta_1, \ldots, \beta_m \in \mathfrak{P}\mathcal{O}_\mathfrak{P}$. Thus $\operatorname{vol}(L'_\mathfrak{P}) = \beta_1 \ldots \beta_m$. Also,

$$(L'_\mathfrak{P})^\varrho/\mathfrak{P}(L'_\mathfrak{P})^\varrho \simeq \langle \varrho\beta_1, \ldots, \varrho\beta_m \rangle.$$

Since this space has a radical of dimension $m-2k$, it follows that exactly $m-2k$ of the β_j are in $\mathfrak{P}^2\mathcal{O}^\times_\mathfrak{P}$, so $N(L'_\mathfrak{P}) = \mathfrak{P}\mathcal{O}_\mathfrak{P}$. We know that $(L'_\mathfrak{P})^\# \simeq \langle \beta_1^{-1}, \ldots \ldots, \beta_m^{-1} \rangle$, which means that

$$N((L'_\mathfrak{P})^\#) = \begin{cases} \mathfrak{P}^{-1}\mathcal{O}_\mathfrak{P} & \text{if } m = 2k, \\ \mathfrak{P}^{-2}\mathcal{O}_\mathfrak{P} & \text{if } m > 2k. \end{cases}$$

A similar analysis shows that for L'' a $\mathfrak{P}$-sublattice of L',

$$L''_\mathfrak{P} \simeq \langle \alpha_1, \ldots, \alpha_m \rangle$$

with $\alpha_j \in \mathfrak{P}^2\mathcal{O}_\mathfrak{P}$ for each j and $\operatorname{vol}(L''_\mathfrak{P}) = \mathfrak{P}^{2m}\mathcal{O}_\mathfrak{P}$. Thus $L''_\mathfrak{P}$ is $\mathfrak{P}^2$-modular, so by Proposition 2.1, $N(L''_\mathfrak{P}) = \mathfrak{P}^2\mathcal{O}_\mathfrak{P}$ and $N((L''_\mathfrak{P})^\#) = \mathfrak{P}^{-2}\mathcal{O}_\mathfrak{P}$. ∎

Proposition 7.2. *Let L be a lattice of rank m and let $\mathfrak{P}$ be a prime, $\mathfrak{P} \nmid 2S(L)$. Let α be as in the preceding proposition, and let k denote the Witt index of $L^\alpha/\mathfrak{P}L^\alpha$.*

1. *If $k = 0$ then the only $\mathfrak{P}$-sublattice of L is $\mathfrak{P}L$.*
2. *Suppose $k > 0$. For $x \in L$, x is in a $\mathfrak{P}$-sublattice of L if and only if*

$Q(x)\in\mathfrak{P}N(L)$. *If* $x\in\mathfrak{P}L$ *then* x *is in every* $\mathfrak{P}$*-sublattice of* L, *of which there are*

$$\begin{cases}(N(\mathfrak{P})^{k-1}+1)\dots(N(\mathfrak{P})^{0}+1) & \text{if } m=2k,\\ (N(\mathfrak{P})^{k}+1)\dots(N(\mathfrak{P})^{1}+1) & \text{if } m=2k+1,\\ (N(\mathfrak{P})^{k+1}+1)\dots(N(\mathfrak{P})^{2}+1) & \text{if } m=2k+2.\end{cases}$$

If $x\in L-\mathfrak{P}L$ *and* $Q(x)\in\mathfrak{P}N(L)$ *then the number of* $\mathfrak{P}$*-sublattices of* L *containing* x *is*

$$\begin{cases}(N(\mathfrak{P})^{k-2}+1)\dots(N(\mathfrak{P})^{0}+1) & \text{if } m=2k,\\ (N(\mathfrak{P})^{k-1}+1)\dots(N(\mathfrak{P})^{1}+1) & \text{if } m=2k+1,\\ (N(\mathfrak{P})^{k}+1)\dots(N(\mathfrak{P})^{2}+1) & \text{if } m=2k+2.\end{cases}$$

Proof. We construct all the $\mathfrak{P}$-sublattices of L by constructing all the maximal totally isotropic (nonzero) subspaces of $L^{\alpha}/\mathfrak{P}L^{\alpha}$. To ease the notation we assume L has already been appropriately scaled so that $N(L)\subseteq\mathcal{O}$ and $\mathfrak{P}\nmid N(L)$.

We know by Proposition 2.1 that $L/\mathfrak{P}L$ is regular. If $L/\mathfrak{P}L=(A_j\oplus B_j)\perp U$ where A_j is totally isotropic (nonzero) of dimension j and $A_j\oplus B_j\simeq jH$ (where H denotes a hyperbolic plane – see [2]), then the Witt index of U is $k-j$. Using the formulae from § 6 of [2], we find that there are $\varphi(m,k,j)$ isotropic vectors in U where

$$\varphi(m,k,j)=\begin{cases}(N(\mathfrak{P})^{k-j}-1)(N(\mathfrak{P})^{k-j-1}+1) & \text{if } m=2k,\\ N(\mathfrak{P})^{2(k-j)}-1 & \text{if } m=2k+1,\\ (N(\mathfrak{P})^{k-j+1}+1)(N(\mathfrak{P})^{k-j}-1) & \text{if } m=2k+2.\end{cases}$$

Thus there are $N(\mathfrak{P})^{j}\varphi(m,k,j)$ isotropic vectors in $A_j^{\perp}-A_j$. Using induction on j, we find that there are

$$N(\mathfrak{P})\dots N(\mathfrak{P})^{k-1}\varphi(m,k,0)\varphi(m,k,1)\dots\varphi(m,k,k-1)$$

ways to construct a basis for a totally isotropic k-dimensional subspace A_k. Hence there are

$$\frac{N(\mathfrak{P})\dots N(\mathfrak{P})^{k-1}\varphi(m,k,0)\varphi(m,k,1)\dots\varphi(m,k,k-1)}{(N(\mathfrak{P})^{k}-1)\dots(N(\mathfrak{P})^{k}-N(\mathfrak{P})^{k-1})}$$

such subspaces. ∎

We also have

Proposition 7.3. *Let* L *be a lattice of rank* m *and let* $\mathfrak{P}$, α *and* k *be as in the preceding proposition.*

1. *If* $k=0$, *then* $\mathfrak{P}L$ *is the only* $\mathfrak{P}^2$*-sublattice of* L.

2. *Suppose* $k>0$. *Then each* $\mathfrak{P}^2$*-sublattice of* L *is contained in exactly one* $\mathfrak{P}$*-sublattice of* L. *For* $x\in L$, x *is in a* $\mathfrak{P}^2$*-sublattice of* L *if and only if* $Q(x)\in\mathfrak{P}^2N(L)$. *If* $x\in\mathfrak{P}^2L$, *then* x *is in every* $\mathfrak{P}^2$*-sublattice of* L, *of which there are*

$$\begin{cases}N(\mathfrak{P})^{k-1}\dots N(\mathfrak{P})^{0}(N(\mathfrak{P})^{k-1}+1)\dots(N(\mathfrak{P})^{0}+1) & \text{if } m=2k,\\ N(\mathfrak{P})^{k-1}\dots N(\mathfrak{P})^{0}(N(\mathfrak{P})^{k}+1)\dots(N(\mathfrak{P})+1) & \text{if } m=2k+1,\\ N(\mathfrak{P})^{k-1}\dots N(\mathfrak{P})^{0}(N(\mathfrak{P})^{k+1}+1)\dots(N(\mathfrak{P})^{2}+1) & \text{if } m=2k+2.\end{cases}$$

If $x\in L-\mathfrak{P}L$ *and* $Q(x)\in\mathfrak{P}^2N(L)$, *then the number of* $\mathfrak{P}^2$*-sublattices of* L *containing* x *is*

$$\begin{cases}N(\mathfrak{P})^{k-2}\dots N(\mathfrak{P})^{0}(N(\mathfrak{P})^{k-2}+1)\dots(N(\mathfrak{P})^{0}+1) & \text{if } m=2k \text{ and } k>1,\\ N(\mathfrak{P})^{k-2}\dots N(\mathfrak{P})^{0}(N(\mathfrak{P})^{k-1}+1)\dots(N(\mathfrak{P})+1) & \text{if } m=2k+1 \text{ and } k>1,\\ N(\mathfrak{P})^{k-2}\dots N(\mathfrak{P})^{0}(N(\mathfrak{P})^{k}+1)\dots(N(\mathfrak{P})^{2}+1) & \text{if } m=2k+2 \text{ and } k>1,\\ 1 & \text{otherwise.}\end{cases}$$

If $x\in\mathfrak{P}L-\mathfrak{P}^2L$ *and* $Q(x)\in\mathfrak{P}^3N(L)$, *then the number of* $\mathfrak{P}^2$*-sublattices of* L *containing* x *is*

$$\begin{cases}N(\mathfrak{P})^{k-1}\dots N(\mathfrak{P})^{0}(N(\mathfrak{P})^{k-2}+1)\dots(N(\mathfrak{P})^{0}+1) & \text{if } m=2k,\\ N(\mathfrak{P})^{k-1}\dots N(\mathfrak{P})^{0}(N(\mathfrak{P})^{k-1}+1)\dots(N(\mathfrak{P})+1) & \text{if } m=2k+1 \text{ and } k>1,\\ N(\mathfrak{P})^{k-1}\dots N(\mathfrak{P})^{0}(N(\mathfrak{P})^{k}+1)\dots(N(\mathfrak{P})^{2}+1) & \text{if } m=2k+2 \text{ and } k>1,\\ 1 & \text{otherwise.}\end{cases}$$

If $x\in\mathfrak{P}L-\mathfrak{P}^2L$ *and* $Q(x)\notin\mathfrak{P}^3N(L)$, *then the number of* $\mathfrak{P}^2$*-sublattices of* L *containing* x *is*

$$\begin{cases}0 & \text{if } m=2k,\\ N(\mathfrak{P})^{k-1}\dots N(\mathfrak{P})^{0}(N(\mathfrak{P})^{k-1}+1)\dots(N(\mathfrak{P})^{0}+1) & \text{if } m=2k+1,\\ N(\mathfrak{P})^{k-1}\dots N(\mathfrak{P})^{0}(N(\mathfrak{P})^{k}+1)\dots(N(\mathfrak{P})+1) & \text{if } m=2k+2.\end{cases}$$

Proof. As in the proof of Proposition 7.2, we assume $N(L)\subseteq\mathcal{O}$ and $\mathrm{ord}_{\mathfrak{P}}N(L)=0$.

Fix a $\mathfrak{P}$-sublattice L' of L. Then for $\varrho\in\mathfrak{P}^{-1}-\mathcal{O}$ and D some nonsingular, diagonal $k\times k$ matrix, we have

$$(L'_{\mathfrak{P}})^{\varrho}/\mathfrak{P}(L'_{\mathfrak{P}})^{\varrho}\simeq\begin{bmatrix}* & D & 0\\ D & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

with respect to some basis $\{y_1+\mathfrak{P}L'_{\mathfrak{P}},\dots,y_m+\mathfrak{P}L'_{\mathfrak{P}}\}$ of $(L'_{\mathfrak{P}})^{\varrho}/\mathfrak{P}(L'_{\mathfrak{P}})^{\varrho}$ where $\{y_1+\mathfrak{P}L_{\mathfrak{P}},\dots,y_k+\mathfrak{P}L_{\mathfrak{P}},\varrho y_{k+1}+\mathfrak{P}L_{\mathfrak{P}},\dots,\varrho y_m+\mathfrak{P}L_{\mathfrak{P}}\}$ is a basis for $L_{\mathfrak{P}}/\mathfrak{P}L_{\mathfrak{P}}$. So the $\mathfrak{P}^2$-sublattices of L which are contained in L' are in one-to-one correspondence with the maximal totally isotropic subspaces of $(L'_{\mathfrak{P}})^e/\mathfrak{P}(L'_{\mathfrak{P}})^e$ which have trivial intersection with the subspace $\langle y_{k+1}+\mathfrak{P}L'_{\mathfrak{P}}, \dots, y_{2k}+\mathfrak{P}L'_{\mathfrak{P}}\rangle$. Notice that each of these maximal totally isotropic subspaces must contain the radical of the space $(L'_{\mathfrak{P}})^e/\mathfrak{P}(L'_{\mathfrak{P}})^e$.

Suppose A_j is a totally isotropic j-dimensional subspace of $\langle y_1+\mathfrak{P}L'_{\mathfrak{P}}, \dots, y_{2k}+\mathfrak{P}L'_{\mathfrak{P}}\rangle$ such that $A_j \cap \langle y_{k+1}+\mathfrak{P}L'_{\mathfrak{P}}, \dots, y_{2k}+\mathfrak{P}L'_{\mathfrak{P}}\rangle = \{0\}$ and

$$\langle y_1+\mathfrak{P}L'_{\mathfrak{P}}, \dots, y_{2k}+\mathfrak{P}L'_{\mathfrak{P}}\rangle = (A_j \oplus B_j) \perp U$$

where $A_j \oplus B_j \simeq jH$. It follows that $U \simeq (k-j)H$, and hence there are

$$N(\mathfrak{P})^{k-1}(N(\mathfrak{P})^{k-j}-1)$$

isotropic (nonzero) vectors which are in $A_j^{\perp}$ but not in A_j or in $\langle y_{k+1}+\mathfrak{P}L'_{\mathfrak{P}}, \dots, y_m+\mathfrak{P}L'_{\mathfrak{P}}\rangle$. Thus an argument similar to that used to prove Proposition 7.2 shows that there are

$$N(\mathfrak{P})^{k-1}N(\mathfrak{P})^{k-2}\dots N(\mathfrak{P})^0$$

$\mathfrak{P}^2$-sublattices of L contained in each $\mathfrak{P}$-sublattice L' of L.

Suppose $x \in \mathfrak{P}L-\mathfrak{P}^2L$ and $Q(x) \in \mathfrak{P}^3N(L)$. Then x is in a $\mathfrak{P}^2$-sublattice L'' of L if and only if ϱx is in the $\mathfrak{P}$-sublattice L' of L which contains L''; in fact, if $\varrho x \in L'$ then x is in every $\mathfrak{P}$-sublattice of L'.

Now suppose $x \in \mathfrak{P}L-\mathfrak{P}^2L$ and $Q(x) \notin \mathfrak{P}^3N(L)$. Then $\varrho x+\mathfrak{P}L$ is anisotropic in $L/\mathfrak{P}L$, so x is in a $\mathfrak{P}^2$-sublattice L'' of L if and only if $\varrho x+\mathfrak{P}L$ is orthogonal to $L'+\mathfrak{P}L$ in $L/\mathfrak{P}L$ where L' is the $\mathfrak{P}$-sublattice of L which contains L''. Furthermore, if $\varrho x+\mathfrak{P}L$ is orthogonal to $L'+\mathfrak{P}L$ then x is in every $\mathfrak{P}$-sublattice of L'. The $\mathfrak{P}$-sublattices L' of L with $\varrho x+\mathfrak{P}L$ orthogonal to $L'+\mathfrak{P}L$ are in one-to-one correspondence with the k-dimensional totally isotropic subspaces of $\langle \varrho x+\mathfrak{P}L_{\mathfrak{P}}\rangle^{\perp} \subseteq L_{\mathfrak{P}}/\mathfrak{P}L_{\mathfrak{P}}$; Proposition 7.2 tells us the number of such subspaces. ∎

Now we combine the results of this section with those of the preceding section to get the main result of this paper. For $K = \mathbf{Q}$ and $m = \operatorname{rank} L$ even, this result is the same as Theorem 21.3 of [5]; see also [13]. It is interesting to note that in Lemma 2 of [14], the following theorem is implicitly assumed in the case that $m = 3$.

THEOREM 7.4. *Let L be a lattice of rank m with norm $N(L)$ and stufe $S(L)$; take $\mathfrak{J}$ as in Theorem 3.7. Let $\mathfrak{P}$ be a prime such that $\mathfrak{P} \nmid 2S(L)$ if m is even, and $\mathfrak{P} \nmid S(L)N(L)\mathfrak{J}^{-2}$ if m is odd. Take $\alpha \in \mathbf{K}$ such that $\alpha \gg 0$, $\alpha N(L) \subseteq \mathcal{O}$ and $\operatorname{ord}_{\mathfrak{P}} \alpha N(L) = 0$; let k be the Witt index of $L^{\alpha}/\mathfrak{P}L^{\alpha}$. For $x \in L-\mathfrak{P}L$ with $Q(x) \in \mathfrak{P}N(L)$, let λ denote the number of $\mathfrak{P}$-sublattices of L which contain x. For $x \in L-\mathfrak{P}L$ with $Q(x) \in \mathfrak{P}^2N(L)$, let $\varkappa$ denote the number of $\mathfrak{P}^2$-sublattices of L which contain x. (Notice that by Propositions 7.2 and 7.3, λ and $\varkappa$ are independent of the choice of x.)*

1. *If $m = 2k$ (i.e. $L^{\alpha}/\mathfrak{P}L^{\alpha}$ is hyperbolic) then*

$$\Theta(L,\tau)|T(\mathfrak{P}) = \lambda^{-1}\sum_{L'}\Theta(\mathfrak{P}^{-1}L',\tau)$$

where the sum is taken over all $\mathfrak{P}$-sublattices L' of L. Furthermore, $\Theta(L,\tau)|T(\mathfrak{P})$ and $\Theta(\mathfrak{P}^{-1}L',\tau)$ are modular forms for the group $\Gamma_0(\mathfrak{P}^{-1}S(L)N(L)\partial, \mathfrak{P}N(L)^{-1}\partial^{-1})$ with character χ as described in Theorem 3.7.

2. *If $m = 2k+1$ then*

$$\Theta(L,\tau)|T(\mathfrak{P}^2) = \begin{cases} \left(N(\mathfrak{P})^{-1/2}+N(\mathfrak{P})^{-3/2}\right)\Theta(\mathfrak{P}^{-1}L,\tau) & \text{if } m=1,\\ N(\mathfrak{P})^{-m/2}\varkappa^{-1}\sum_{L''}\Theta(\mathfrak{P}^{-2}L'',\tau)+\left(N(\mathfrak{P})^{-m}-N(\mathfrak{P})^{-(m+3)/2}\right)\Theta(\mathfrak{P}^{-1}L,\tau) & \text{if } m \geqslant 3 \end{cases}$$

where the sum is taken over all $\mathfrak{P}^2$-sublattices L'' of L. Furthermore, $\Theta(L,\tau)|T(\mathfrak{P}^2)$ and $\Theta(\mathfrak{P}^{-2}L'',\tau)$ are modular forms for the group $\tilde{\Gamma}_0(\mathfrak{P}^{-2}S(L)\times N(L)\partial, \mathfrak{P}^2\mathfrak{J}^{-2}\partial^{-1})$ with character χ as described in Theorem 3.7.

3. *If $m = 2k+2$ (i.e. $L^{\alpha}/\mathfrak{P}L^{\alpha}$ is not hyperbolic) then*

$$\Theta(L,\tau)|T(\mathfrak{P}^2) = \begin{cases} 0 & \text{if } m=2,\\ \varkappa^{-1}\sum_{L''}\Theta(\mathfrak{P}^{-2}L'',\tau)-\left(2N(\mathfrak{P})^k+N(\mathfrak{P})^{k-1}-1\right)\Theta(\mathfrak{P}^{-1}L,\tau) & \text{if } m \geqslant 4 \end{cases}$$

where the sum is taken over all $\mathfrak{P}^2$-sublattices L'' of L. Furthermore, $\Theta(L,\tau)|T(\mathfrak{P}^2)$ and $\Theta(\mathfrak{P}^{-2}L'',\tau)$ are modular forms for the group $\Gamma_0(\mathfrak{P}^{-2}S(L)\times N(L)\partial, \mathfrak{P}^2N(L)^{-1}\partial^{-1})$ with character χ as described in Theorem 3.7.

Proof. First consider the case where $m = 2k$. Due to our restrictions on $\mathfrak{P}$, Proposition 2.1 implies that

$$L/\mathfrak{P}L \simeq L_{\mathfrak{P}}/\mathfrak{P}L_{\mathfrak{P}} \simeq \tilde{L}_{\mathfrak{P}}/\mathfrak{P}\tilde{L}_{\mathfrak{P}} \simeq \tilde{L}/\mathfrak{P}\tilde{L}.$$

Then with Propositions 6.2 and 7.2 we find that

$$\Theta(L,\tau)|T(\mathfrak{P}) = \frac{i^{-mn/2}}{\sqrt{\Phi(L)}}N(\tau)^{-m/2}N(\mathfrak{P})^{m/2}\lambda^{-1}\sum_{L_1}\Theta(L_1, 1/\tau)$$

where the sum is over all $\mathfrak{P}$-sublattices L_1 of $\tilde{L}$. Now using (1) we get

$$\Theta(L,\tau)|T(\mathfrak{P}) = \lambda^{-1}N(\mathfrak{P})^{m/2}\sum_{L_1}\frac{\sqrt{\Phi(\tilde{L}_1)}}{\sqrt{\Phi(L)}}\Theta(\tilde{L}_1,\tau).$$

Whenever L_1 is a $\mathfrak{P}$-sublattice of $\tilde{L}$, it follows from the invariant factor theorem that $\mathfrak{P}\tilde{L}_1$ is a $\mathfrak{P}$-sublattice of L. Hence

$$\Theta(L,\tau)|T(\mathfrak{P}) = \lambda^{-1}\sum_{L'}\Theta(\mathfrak{P}^{-1}L',\tau)$$

where the sum is over all $\mathfrak{P}$-sublattices L' of L.

An analogous argument using Propositions 4.2, 6.1, 6.3 and 7.3 proves the theorem in the case that $m = 2k+2$.

Now we consider the case where $m = 2k+1$. Take e and ϱ as in Proposition 6.4. Then for $x \in \mathfrak{P}\tilde{L}$ with $Q(x) \notin \mathfrak{P}^3 N(\tilde{L})$,

$$\varrho x \in \varrho \mathfrak{P} \tilde{L}_{\mathfrak{P}} = \mathfrak{P}^e \partial \tilde{L}_{\mathfrak{P}} = \mathfrak{P}^e L^{\sharp}_{\mathfrak{P}} = L_{\mathfrak{P}}.$$

(Recall that by Proposition 2.1, $L_{\mathfrak{P}}$ is $N(L)\mathcal{O}_{\mathfrak{P}}$-modular.) Hence ϱx is anisotropic in the space $\mathfrak{P}^{-e} L_{\mathfrak{P}}/\mathfrak{P}^{1-e} L_{\mathfrak{P}}$, and so

$$(\operatorname{disc}(\mathfrak{P}^{-e} L)_{\mathfrak{P}} | \mathfrak{P}) = ((-1)^{(m-1)/2} Q(\varrho x) | \mathfrak{P}).$$

Now Propositions 6.4 and 7.3 yield the result of the theorem. ∎

Remark. In the case that $m = 2k+2$, we could use the methods used above to write $\Theta(L, \tau)|T(\mathfrak{P})$ as a linear combination of theta series $\{\Theta(\mathfrak{P}^{-1} L', \tau) |\ L' \text{ is a } \mathfrak{P}\text{-sublattice of } L\}$ and $\Theta(\mathfrak{P}^{-1} L, \tau)$. However, these theta series do not lie in the same space of modular forms as $\Theta(L, \tau)|T(\mathfrak{P})$.

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DEPARTMENT OF MATHEMATICS
BATES COLLEGE
Lewiston, Maine 04240
U.S.A.

Received on 31.3.1988 (1806)

ACTA ARITHMETICA
LIV (1990)

On Dirichlet's theorem concerning diophantine approximation

by

PETER THURNHEER (Orsay)

1. Introduction.

(i) Let $\alpha_1, \ldots, \alpha_n$, $n \geq 2$, be given real numbers. According to Dirichlet there exist infinitely many integer points $(\xi_1, \ldots, \xi_{n+1}) \in \mathbf{Z}^{n+1}$ such that

$$|\alpha_1 \xi_1 + \ldots + \alpha_n \xi_n + \xi_{n+1}| \leq (\max_{1 \leq \nu \leq n} |\xi_\nu|)^{-n}.$$

We will show that essentially this still holds, if for the approximation of $\alpha_1, \ldots, \alpha_n$ one allows only integer points $(\xi_1, \ldots, \xi_{n+1})$ in certain subsets of $\mathbf{R}^{n+1}$. In other words, we shall prove that the effectivity in Dirichlet's theorem can be replaced by a condition concerning the position of the approximating integer points.

(ii) In what follows, an integer point is always an element of $\mathbf{R}^{n+1}$ with integer coordinates $\xi_1, \ldots, \xi_{n+1}$ and ε and δ are any positive real numbers. For $\mathscr{X} = (\xi_1, \ldots, \xi_{n+1}) \in \mathbf{R}^{n+1}$ put

$$L(\mathscr{X}) = \sum_{\nu=1}^{n} \alpha_\nu \xi_\nu + \xi_{n+1}, \quad \{\mathscr{X}\} = \max_{1 \leq \nu \leq n} |\xi_\nu|, \quad \varrho(\mathscr{X}) = \Big(\sum_{\nu=1}^{n-1} \xi_\nu^2\Big)^{1/2}.$$

For real w let

$$\Phi(w) = \{\mathscr{X} \in \mathbf{R}^{n+1} |\ |\xi_n| \leq (1+\varepsilon)\varrho(\mathscr{X})^w\} \cup \{\mathscr{X} \in \mathbf{R}^{n+1} |\ \varrho(\mathscr{X}) \leq 1\};$$
$$\Psi = \{\mathscr{X} \in \mathbf{R}^{n+1} |\ |\xi_n| \leq \varepsilon \varrho(\mathscr{X})\}.$$

THEOREM 1. (a) *If*

(0) $$w = w(n) = 1 + 1/n + 1/n^2,$$

then there exist infinitely many integer points $\mathscr{G}$ *such that*

$$\mathscr{G} \in \Phi(w) \quad \text{and} \quad |L(\mathscr{G})| \leq (1+\delta)\{\mathscr{G}\}^{-n}.$$

(b) *If*

(1) $$v = v(n) = \tfrac{1}{2}(n - 1 + \sqrt{n^2 + 2n - 3})$$