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A note on the Hasse principle

by

NGUYEN QUOC THANG (Hanoi)

I. Introduction. A celebrated theorem of Hasse–Minkowski states that over a global field k, any quadratic form, representing zero everywhere locally, represents zero globally (in this note we always understand zero to be non-trivial). This is the so-called Hasse principle for quadratic forms. It is natural to ask whether or not the Hasse principle still remains true for a system of quadratic forms over global fields. The first counter-example to the Hasse principle for systems of quadratic forms was constructed by Iskovskikh [14] over the rational numbers Q and later other counter-examples have been found (see [2], [4]–[12], [17], [18], [22] and references there).

In this note we are interested in the validity of the Hasse principle for systems of quadratic forms over any global field of characteristic $p \neq 2$; we give another approach to this, based on a simple connection between the Hasse principle for varieties and the (cohomological) Hasse principle for algebraic groups. We have

THEOREM. *Let k be any global field of characteristic $p \neq 2$ and m any natural number. Then there is a k-variety X, defined by a system of $m(2m+1)$ quadratic forms in $4m^2+1$ variables, for which the Hasse principle does not hold.*

It may be that using the arguments given here, it is possible to prove the analogous statement for the case $\operatorname{char} k = 2$ too. Finally, we will give various counter-examples, including a minimal possible one over a global field of any characteristic.

It is worth mentioning here that the systems constructed here are not obtained from the known counter-examples just by adding to the latter a number of quadratic forms, so they are non-trivial in a certain sense.

II. Definitions, notation. In this paper, k will be a field, k-algebraic groups are always linear algebraic k-groups, considered as matrix groups under some matrix representation in some linear group GL_n. If G is a k-group, G_0 denotes the connected component of G. $\bar{k}$ always denotes the algebraic closure of k, $\operatorname{Gal}(k'/k)$ is the Galois group of the extension $k' \supset k$, $M_n(k)$ is the algebra of all $n \times n$ matrices with coefficients in k, and μ_2 denotes the group (± 1). For D a division algebra of finite dimension over its centre k, denote by

Nrd: $D \to k$ the reduced norm, and if D is a quaternion algebra, J is its standard involution. If Φ is a non-degenerate (skew-)hermitian form with values in a division algebra D of finite dimension over its centre k, we denote by $GU(\Phi)$ the k-algebraic group defined by the group of similitudes $GU(\Phi, D)$ of the form Φ. Let m denote the homomorphism $GU(\Phi, D) \to k^{\times}$, mapping every unitary similitude to its multiplicator (or similitude factor); we denote by the same symbol the corresponding homomorphism $GU(\Phi \otimes k', D \otimes k') \to (k')^{\times}$, for any extension $k' \supset k$. k_v always denotes the completion of the field k at a valuation v of k.

For a k-variety X, we say that X satisfies the *Hasse principle* if the non-emptiness of $X(k_v)$ for all valuations v of k implies that $X(k)$ is also non-empty. For a k-algebraic group G, we denote by $H^1(k, G)$ (and if G is commutative, by $H^i(k, G)$) the 1-cohomology set of G (the ith (Galois) cohomology group of G). We say that the (*cohomological*) *Hasse principle* holds for G if the canonical map

$$H^1(k, G) \to \prod_v H^1(k_v, G)$$

is injective, where the product is taken over all valuations of the field k (cf. [20]).

III. Some remarks on the Hasse principle. Let

$$(*) \qquad 1 \to H \to G \to L \to 1$$

be an exact sequence of k-groups. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
G(k) \arrow[r, "\pi"] \arrow[d] & L(k) \arrow[r, "\delta"] \arrow[d] & H^1(k, H) \arrow[r, "\alpha"] \arrow[d, "\gamma"] & H^1(k, G) \arrow[d, "\beta"] \\
\prod_v G(k_v) \arrow[r] & \prod_v L(k_v) \arrow[r, "\bar\delta"] & \prod_v H^1(k_v, H) \arrow[r] & \prod_v H^1(k_v, G).
\end{tikzcd}$$

Consider the fibers $X_\lambda := \pi^{-1}(\lambda)$ for $\lambda \in L(k)$. They are k-subvarieties of G and one may ask whether the varieties X_λ satisfy the Hasse principle. First we have the following

PROPOSITION 1. *If, in the above diagram, the map β is injective, then H satisfies the Hasse principle if and only if*

$$\pi(G(k)) = \bigcap_v \big(\pi(G(k_v)) \cap L(k)\big).$$

Proof. Assume that H satisfies the Hasse principle. Let λ be in

$$\bigcap_v \big(\pi(G(k_v)) \cap L(k)\big).$$

We have $\gamma(\delta(\lambda)) = \bar\delta(\lambda)$, hence $\delta(\lambda) = 0$, i.e. $\lambda \in \pi(G(k))$. Conversely, if $x \in H^1(k, H)$ and $\gamma(x) = 0$ then $\theta(\gamma(x)) = \beta(\alpha(x)) = 0$, i.e. $\alpha(x) = 0$ since β is injective. Hence $x = \delta(y)$, $y \in L(k)$. Thus $\gamma(\delta(y)) = \bar\delta(y) = 0$, hence

$$y \in \bigcap_v \big(\pi(G(k_v)) \cap L(k)\big) = \pi(G(k)),$$

i.e. $y = \pi(g)$, $g \in G(k)$, hence $x = 0$. ∎

Remark. Note that

$$\lambda \in \pi(G(k)) \Leftrightarrow X_\lambda(k) \neq \varnothing,$$

$$\lambda \in \bigcap_v \big(\pi(G(k_v)) \cap L(k)\big) \Leftrightarrow X_\lambda(k_v) \neq \varnothing \quad \forall v.$$

Hence we have the following

COROLLARY. *The following three statements are equivalent:*

(a) $\lambda \in \bigcap_v \big(\pi(G(k_v)) \cap L(x)\big) \setminus \pi(G(k))$.

(b) *X_λ does not satisfy the Hasse principle.*

(c) *H does not satisfy the cohomological Hasse principle.*

Now if $\pi(G(k))$ is a normal subgroup in $L(k)$, we may consider the factor group $B = \bigcap_v \big(\pi(G(k_v)) \cap L(k)\big)/\pi(G(k))$ and B can be considered as the obstruction for the family $\mathscr{C} = (X_\lambda\colon \lambda \in L(k))$ to satisfy the Hasse principle. In this case we have

PROPOSITION 2. *In the above notation, if $\pi(G(k))$ is a normal subgroup in $L(k)$ and G satisfies the Hasse principle, then there is a bijection $B \leftrightarrow Ш(H)$, where $Ш(H)$ denotes the Tate–Shafarevich group of H, and this bijection carries the group structure of B onto $Ш(H)$.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
L(k) \arrow[r, "\delta"] \arrow[d] & H^1(k, H) \arrow[r, "\alpha"] \arrow[d, "\gamma"] & H^1(k, G) \arrow[d, "\theta"] \\
\prod_v L(k_v) \arrow[r, "\bar\delta"] & \prod_v H^1(k_v, H) \arrow[r, "\beta"] & \prod_v H^1(k_v, G).
\end{tikzcd}$$

It is clear that the correspondence $x \cdot \pi(G(k)) \mapsto \delta(x)$ determines a well-defined map from $L(k)/\pi(G(k))$ to $H^1(k, H)$. For, $L(k)$ acts on the set $H^1(k, H)$ and $\delta(x)$ is the image of the trivial cohomology class under the action of x (see [20]), and $\delta(x) = \delta(y)$ for $x, y \in L(k)$ if and only if $x = y \cdot \pi(g)$, where $g \in G(k)$. Therefore, this map is injective. Now, if $z \in Ш(H)$ then $\gamma(z) = 0$ and $\beta(\gamma(z)) = \theta(\alpha(z)) = 0$, hence $\alpha(z) = 0$ and $z \in \delta(L(k))$, $z = \delta(p)$. Thus $\gamma(\delta(p)) = \bar\delta(p) = 0$, i.e. $p \in \bigcap_v \big(\pi(G(k_v)) \cap L(k)\big)$. ∎

We now discuss the validity of the Hasse principle for some algebraic groups over global fields.

LEMMA 3 (cf. [13], [19]). *Over any global field k, any almost simple k-group $G \neq E_8$ satisfies the Hasse principle.*

Proof. Denote by $\tilde G$ the k-universal covering of G and let F be the fundamental group of G. We have the commutative diagram

$$\begin{tikzcd}
H^1(k, \tilde G) \arrow[r] \arrow[d, "r"] & H^1(k, G) \arrow[r] \arrow[d, "q"] & H^2(k, F) \arrow[d, "p"] \\
\prod_v H^1(k_v, \tilde G) \arrow[r] & \prod_v H^1(k_v, G) \arrow[r] & \prod_v H^2(k_v, F)
\end{tikzcd}$$

with exact rows. It is well known that p and r are injective, hence by twisting, we get the injectivity of q.

LEMMA 4. *Let Φ be a non-degenerate hermitian (or skew-hermitian) T-form as in Section II, and let k be a global field. Then the group $\mathrm{GU}(\Phi)_0$ satisfies the Hasse principle over k.*

Proof. We use the exact sequence (*) above to deduce the following commutative diagram:

$$\begin{array}{ccccc} 0 \to & H^1(k, \mathrm{GU}(\Phi)_0) & \xrightarrow{j} & H^1(k, \mathrm{PGU}(\Phi)_0) & \\ & \downarrow p & & \downarrow q & \\ 0 \to & \prod_v H^1(k_v, \mathrm{GU}(\Phi)_0) & \to & \prod_v H^1(k_v, \mathrm{PGU}(\Phi)_0) & \end{array}$$

where $G = \mathrm{GU}(\Phi)_0$, $H = G_m$, $L = \mathrm{PGU}(\Phi)_0$. By Lemma 3, q is injective. If $x \in H^1(k, \mathrm{GU}(\Phi)_0)$ and $p(x) = 0$ then $q(j(x)) = 0$, hence $j(x) = 0$, i.e. $x = 0$. ∎

LEMMA 5. *If the group $\mathrm{PGU}(\Phi)$ satisfies the Hasse principle then so does the group $\mathrm{GU}(\Phi)$.*

Proof. It is the same as that of Lemma 4.

IV. Construction of some special forms. In this section we will construct a form Φ such that the group $\mathrm{GU}(\Phi)$ satisfies the Hasse principle over the global field considered. Let k be a field of characteristic $\neq 2$, let G be a k-group and F the group (± 1) lying in the centre of G and defined over k, where G is considered as a k-subgroup of GL_n for some n, and 1 denotes the identity matrix of GL_n. Let $G' = G/F$ and let π: $G \to G'$ be the projection. First we need the following lemma.

LEMMA 6. $\pi^{-1}(G'(k)) = G(k) \cup \bigcup_{k'} (g \in G(k')\colon g^{\sigma'} = -g)$, *where k' runs over all quadratic extensions of k contained in a fixed algebraic closure of k, and σ' denotes the unique non-trivial automorphism of* $\mathrm{Gal}(k'/k)$.

Proof. Let $g \in \pi^{-1}(G'(k))$. Then we have $\pi(g)^\sigma = \pi(g^\sigma)$, or equivalently $g^\sigma = \pm g$, for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Let $\Gamma = (\sigma \in \mathrm{Gal}(\bar{k}/k)\colon g^\sigma = g)$. Clearly Γ is a normal subgroup of $\mathrm{Gal}(\bar{k}/k)$ of index not greater than 2. Let k' be the extension of k corresponding to Γ. Hence $[k'\colon k] \leqslant 2$ and $g \in G(k')$. If $k' = k$, then $g \in G(k)$. Otherwise, let σ' be the unique non-trivial automorphism of $\mathrm{Gal}(k'/k)$. Then we have $g^{\sigma'} = -g$ and the lemma follows. ∎

We shall now prove that over any field k of characteristic $\neq 2$, there is a hermitian (or skew-hermitian) form Φ such that $\mathrm{PGU}(\Phi)(k) \neq \mathrm{PGU}(\Phi)_0(k)$. From now on we consider only forms over a quaternion algebra D of type D_n, denoted by Φ. Let $\mathrm{U}(\Phi)$ (resp. $\mathrm{SU}(\Phi)$) be the unitary (resp. special unitary) k-group of the form Φ. Then $\mathrm{PGU}(\Phi)$ (resp. $\mathrm{PGU}(\Phi)_0$) is the adjoint group of $\mathrm{U}(\Phi)$ (resp. of $\mathrm{SU}(\Phi)$) and they are obtained by factoring the group $\mathrm{U}(\Phi)$ (resp. $\mathrm{SU}(\Phi)$) by the subgroup (± 1). From Lemma 6 we see that it is sufficient to construct a form Φ such that

$$\pi^{-1}(\mathrm{PGU}(\Phi)(k)) \neq \pi^{-1}(\mathrm{PGU}(\Phi)_0(k)),$$

or equivalently, such that

$$\exists k'\colon [k'\colon k] = 2, \quad \exists u \in \mathrm{U}(\Phi)(k') \setminus \mathrm{SU}(\Phi)(k')\colon\ u^\sigma = -u,\ \sigma \in \mathrm{Gal}(k'/k),\ \sigma \neq 1.$$

First we choose k' such that k' splits D. Then $\mathrm{U}(\Phi)(k') = O(f, k')$, $\mathrm{SU}(\Phi)(k') = \mathrm{SO}(f, k')$ for some quadratic form f defined over k'. Moreover, it is clear that the following 2 statements are equivalent:

(1) $\quad (u \in O(f, k') \setminus \mathrm{SO}(f, k'),\ u^\sigma = -u)$.

(2) $\quad \left[\begin{array}{l} \det(u) = -1 \text{ and } u = a\sqrt{\theta}, \text{ where } k' = k(\sqrt{\theta}), \sqrt{\theta} \text{ satisfies} \\ (\sqrt{\theta})^\sigma = -\sqrt{\theta}, \text{ and } a \text{ belongs to the group } \mathrm{GO}(f, k) \text{ of} \\ \text{similitudes of } f, \text{ rational over } k. \end{array}\right]$

For $g \in \mathrm{GO}(f, k')$, let $\boldsymbol{m}(g)$ be the multiplicator of g and let $M(f)$ be the group of all multiplicators of similitudes in $\mathrm{GO}(f, k')$. For u in (2) we see that since $u \in O(f, k')$, $\theta \in M(f)$ and $\boldsymbol{m}(a) = \theta^{-1}$. We know that $(\det(a))^2 = \theta^{-2m}$, where m is the dimension of Φ, and clearly $\det(u) = -1$ iff $\det(a) = -\theta^{-m}$. To show the existence of u in (2), we need the following lemmas.

LEMMA 7 (cf. [21]). *$\mathrm{GO}(f, k')$ is generated by $O(f, k')$ and its intersection with the groups of the form $\mathrm{GO}(f_1, k') \times \ldots \times \mathrm{GO}(f_m, k')$, where $f = f_1 \perp f_2 \perp \ldots \perp f_m$ is any orthogonal decomposition of f into subforms of dimension 2.*

LEMMA 8. *Let φ be a quadratic form over k', $\alpha \in M(\varphi)$, $\dim(\varphi) = 2$. Then there are $a_1, a_2 \in \mathrm{GO}(\varphi, k')$ such that*

$$\det(a_1) = -\det(a_2) = \boldsymbol{m}(a_1) = \boldsymbol{m}(a_2) = \alpha.$$

Proof. Let $a \in \mathrm{GO}(\varphi, k')$, $\alpha = \boldsymbol{m}(a)$. Let $\{e_1, e_2\}$ be an orthogonal basis of the vector space associated with φ, $\varphi(e_1) = \gamma$, $\varphi(e_2) = \beta$. Then for

$$a = \begin{bmatrix} x & z \\ y & t \end{bmatrix}, \quad x, y, z, t \in k',$$

we have

$$\begin{cases} a \in \mathrm{GO}(\varphi, k') \\ \boldsymbol{m}(a) = \alpha \end{cases} \Leftrightarrow \begin{cases} \gamma x^2 + \beta y^2 = \gamma\alpha, \\ \gamma x \cdot z + \beta y \cdot t = 0, \\ \gamma z^2 + \beta t^2 = \beta\alpha, \end{cases}$$

where we can assume that $\beta\gamma \neq 0$ for simplicity. Clearly $x \cdot z = 0 \Leftrightarrow y \cdot t = 0$. If $x = t = 0$, then $\gamma z = \pm\beta y$, and $\det(a) = \mp\alpha$. The same holds if $y = z = 0$. If $x \cdot y \cdot z \cdot t \neq 0$, then easy calculation shows that $t = \pm x$ and $\gamma z = \pm\beta y$, and the lemma follows from this. ∎

LEMMA 9. *Let f be a quadratic form of even dimension $2m$ over k'. For every $\alpha \in M(f)$ there are $a_1, a_2 \in \mathrm{GO}(f, k')$ such that*

$$\det(a_1) = -\det(a_2) = \alpha^m, \quad \boldsymbol{m}(a_1) = \boldsymbol{m}(a_2) = \alpha.$$

Proof. We proceed by induction on m. Lemma 8 settles the case of $m = 1$. Let $m > 1$. From Lemma 7 we deduce that there is

$$a \in \mathrm{GO}(f, k') \cap (\mathrm{GO}(f_1, k') \times \mathrm{GO}(f_2, k') \times \ldots \times \mathrm{GO}(f_m, k'))$$

such that $\alpha = \boldsymbol{m}(a)$, i.e. $a = \mathrm{diag}(b_1, b_2, \ldots, b_m)$, where for all $1 \leqslant i \leqslant m$, $b_i \in \mathrm{GO}(f_i, k')$ and $\boldsymbol{m}(b_i) = \alpha$. Since $\det(a) = \prod_{1 \leqslant i \leqslant m} \det(b_i)$, and by Lemma 8, $\det(b_i) = \varepsilon_i \alpha$, where $\varepsilon_i = \pm 1$, we can choose ε_i such that $\det(a)$ equals α^m or $-\alpha^m$ as required. ∎

Now we are able to find a form Φ such that the group $\mathrm{GU}(\Phi)$ satisfies the Hasse principle.

LEMMA 10. *Let k be any field of characteristic $\neq 2$ and let D be a non-trivial quaternion division algebra over k. There is a hermitian (or skew-hermitian) form Φ with values in D such that*

$$\mathrm{PGU}(\Phi)(k) \neq \mathrm{PGU}(\Phi)_0(k).$$

Proof. As we remarked above, it is sufficient to choose Φ such that if $f = \Phi \otimes k'$, where $D = \left(\frac{\theta, \eta}{k}\right)$, $k' = k(\sqrt{\theta})$, then there is an element $a \in \mathrm{GO}(f, k)$ such that $m(a) = \theta^{-1}$, $\det(a) = -\theta^{-m}$, where $m = \dim \Phi$. We identify $D \otimes k'$ with the matrix algebra $M_2(k')$ in such a way that if $\{1, i, j, ij\}$ is the canonical basis of D, $i^2 = \theta$, $j^2 = \eta$, and if $x_0, x_1, x_2, x_3 \in k$, then $X = x_0 + x_1 i + x_2 j + x_3 (ij)$ is mapped to the matrix

$$\begin{bmatrix} x_0 + x_1\sqrt{\theta} & \eta(x_2 + x_3\sqrt{\theta}) \\ x_2 - x_3\sqrt{\theta} & x_0 - x_1\sqrt{\theta} \end{bmatrix}.$$

Clearly, we can choose the form Φ so that the matrix of the form f under the above identification has the diagonal form $\mathrm{diag}(a_1, a_2, \ldots, a_{2m})$ such that $a_{2t-1} = -a_{2t}$ for $1 \leqslant t \leqslant m$. For example, we can take $\Phi = \mathrm{diag}(1, -1, \ldots)$ if Φ is a hermitian form and $\Phi = \mathrm{diag}(i, i, \ldots, i)$ if Φ is a skew-hermitian form. The form f is then hyperbolic, hence $M(f) = (k')^\times$. From the proofs of Lemmas 8 and 9 it is easy to choose a as required. ∎

COROLLARY 11. *With the notation as above, for the form Φ in Lemma 10 and for any extension $K \supset k$, we have $\mathrm{PGU}(\Phi)(K) \neq \mathrm{PGU}(\Phi)_0(K)$.*

Proof. This follows from the fact that $[\mathrm{PGU}(\Phi) : \mathrm{PGU}(\Phi)_0] = 2$, since $[\mathrm{U}(\Phi) : \mathrm{SU}(\Phi)] = 2$. ∎

Now we assume that k is a global field of characteristic $\neq 2$.

LEMMA 12. *For the form Φ as in Lemma 10, the group $\mathrm{PGU}(\Phi)$ satisfies the Hasse principle over k.*

Proof. Consider the exact sequence of k-groups

$$1 \to \mathrm{PGU}(\Phi)_0 \to \mathrm{PGU}(\Phi) \to \mu_2 \to 1.$$

From Lemma 10 it follows that we also have the following exact sequence:

$$1 \to \mathrm{PGU}(\Phi)_0(k) \to \mathrm{PGU}(\Phi)(k) \to \mu_2 \to 1,$$

hence also the following commutative diagram with exact lines:

$$\begin{array}{ccccccc} 1 \to & H^1(k, \mathrm{PGU}(\Phi)_0) & \to & H^1(k, \mathrm{PGU}(\Phi)) & \to & H^1(k, \mu_2) \\ & \downarrow^{\alpha} & & \downarrow^{\beta} & & \downarrow^{\gamma} \\ 1 \to & \prod_v H^1(k_v, \mathrm{PGU}(\Phi)_0) & \to & \prod_v H^1(k_v, \mathrm{PGU}(\Phi)) & \to & \prod_v H^1(k_v, \mu_2). \end{array}$$

We know that α is injective, and γ is too because of the global square theorem, hence β is injective as required. ∎

COROLLARY 13. *With the notation as in Lemma 12, the group $\mathrm{GU}(\Phi)$ satisfies the Hasse principle over k.*

Proof. This follows from Lemmas 5 and 12. ∎

The following lemma will give us an explicit description of systems of quadratic forms mentioned in the introduction.

LEMMA 14. *Let Φ be an m-dimensional non-degenerate skew-hermitian form, defined over a D-vector space V, where D is a quaternion division algebra with standard involution J. Let $\boldsymbol{m}$: $\mathrm{GU}(\Phi) \to G_m$ be the k-epimorphism associating to every similitude g its multiplicator $\boldsymbol{m}(g)$. Then for every $\lambda \in k^\times$, the fiber $X_\lambda = \boldsymbol{m}^{-1}(\lambda)$ is defined by a system of $m(2m+1)$ quadratic forms in $4m^2+1$ variables over k.*

Proof. Let $\{e_1, e_2, \ldots, e_m\}$ be an orthogonal basis of V with respect to Φ and let $\Phi(e_i) = \alpha_i$. For $g \in \mathrm{GU}(\Phi)(k)$, $\boldsymbol{m}(g) = \lambda$ if and only if the following system of equations holds:

$$\begin{cases} \Phi(ge_i, ge_i) = \lambda\alpha_i, & \text{for } 1 \leqslant i \leqslant m, \\ \Phi(ge_i, ge_j) = 0, & \text{for } 1 \leqslant i < j \leqslant m. \end{cases}$$

Since every α_i is a skew-quaternion, each equation $\Phi(ge_i, ge_i) = \lambda\alpha_i$ gives us a system of 3 quadratic forms and each equation $\Phi(ge_i, ge_j) = 0$ gives a system of 4 quadratic forms, all of which are defined over k. Taking the homogenization of the system obtained we will have a system of $m(2m+1)$ quadratic forms in $4m^2+1$ variables. ∎

V. Counter-examples to the Hasse principle over a global field. In this section we will prove the following theorem, which gives various systems of quadratic forms not satisfying the Hasse principle over any global field k of characteristic $\neq 2$. We end the section by considering some examples.

THEOREM. *Let k be any global field of characteristic $\neq 2$ and let m be a natural number. Then there is a k-variety X, defined by a system of $m(2m+1)$ quadratic forms in $4m^2+1$ variables, for which the Hasse principle does not hold.*

First proof. First we state some results concerning the classification of skew-hermitian forms of type D_n over global fields of characteristic $\neq 2$. Let Φ be a skew-hermitian form of type D_n with values in a quaternion division algebra with centre a global field k of characteristic $\neq 2$. Let $\mathrm{SU}(\Phi)$ and $\mathrm{U}(\Phi)$ be the special unitary and unitary k-groups respectively of Φ. We have the following exact sequence of k-groups:

$$1 \to \mathrm{SU}(\Phi) \to \mathrm{U}(\Phi) \to \mu_2 \to 1.$$

We know that if D is not trivial over k then $\mathrm{SU}(\Phi)(k) = \mathrm{U}(\Phi)(k)$ and the following exact sequence holds:

$$(1) \qquad 1 \to \mu_2 \xrightarrow{v} H^1(k, \mathrm{SU}(\Phi)) \xrightarrow{\alpha} H^1(k, \mathrm{U}(\Phi)) \xrightarrow{d} H^1(k, \mu_2) = k^\times/k^{\times 2}.$$

This remains true if we replace k by any extension K of k provided D is not trivial over K. The following lemma is known, but for completeness we also give a proof here.

LEMMA 15. *With the assumptions as above, if D is non-trivial over a v-adic completion k_v of k and if we replace k by k_v in (1), then the map d has trivial kernel. Thus forms of type D_n over a quaternion division algebra over a local field are classified by their dimension and discriminant.*

Proof. In our case, we have the following exact sequence:

$$(2) \qquad 0 \to H^1(k_v, \mathrm{SU}(\Phi)) \xrightarrow{\beta} H^2(k_v, \mu_2),$$

deduced from the exact sequence $1 \to \mu_2 \to \mathrm{Spin}(\Phi) \to \mathrm{SU}(\Phi) \to 1$, and from the triviality of $H^1(k_v, \mathrm{Spin}(\Phi))$ by Kneser–Bruhat–Tits' theorem (cf. [2], [16]). We have to show that $\mathrm{Im}(\alpha) = 0$. First we claim that $\mathrm{Card}(H^1(k_v, \mathrm{SU}(\Phi))) \leqslant 2$. Note that since $H^2(k_v, \mu_2) \simeq \mu_2$, it is sufficient to show that β is injective. Indeed, if $\beta(a) = \beta(b)$, then by twisting with the cocycle b, we see that $a = b$ since $\mathrm{Ker}\,\beta = 0$. Further, since v is injective and $\mathrm{Card}(H^1(k_v, \mathrm{SU}(\Phi))) \leqslant 2$, we conclude that v is onto, i.e. $\mathrm{Ker}\,\alpha = H^1(k_v, \mathrm{SU}(\Phi))$. ∎

Now consider the exact sequence

$$1 \to \mathrm{U}(\Phi) \to \mathrm{GU}(\Phi) \xrightarrow{m} G_m \to 1,$$

where m is the map as in Lemma 14, and the commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{GU}(\Phi)(k) \arrow[r, "m"] \arrow[d] & G_m(k) \arrow[r] \arrow[d] & H^1(k, \mathrm{U}(\Phi)) \arrow[r] \arrow[d, "\omega"] & H^1(k, \mathrm{GU}(\Phi)) \arrow[d] \\
\prod_v \mathrm{GU}(\Phi)(k_v) \arrow[r] & \prod_v G_m(k_v) \arrow[r] & \prod_v H^1(k_v, \mathrm{U}(\Phi)) \arrow[r] & \prod_v H^1(k_v, \mathrm{GU}(\Phi)).
\end{tikzcd}$$

It is well known that the map ω need not be injective over any field (Kneser–Springer, cf. [1], [15]). In fact, it is shown that if k is a number field and D is a non-trivial quaternion division algebra over its centre k, Φ is any non-degenerate skew-hermitian form of type D_n with values in D, S is the finite set of all valuations of k at which D is not ramified and $\mathrm{Card}(S) \geqslant 4$, then $\mathrm{Ker}(\omega)$ consists of exactly 2^{s-2} elements, where $s = \mathrm{Card}(S)$ (see [1], [15]). Notice that if $\mathrm{char}\, k \neq 2$, the counter-example of Kneser–Springer still holds, since all the facts that are needed in the functional case have been proved above (Lemma 15); for details we refer to [1], [15]. Now, if D is non-ramified at $s \geqslant 4$ valuations of k and if we choose the form Φ as in Lemma 10, then the results of Sections III and IV show that there is $\lambda \in G_m(k)$ such that the k-variety $X_\lambda = m^{-1}(\lambda)$, which is defined by a system of $m(2m+1)$ quadratic forms in $4m^2+1$ variables over k, does not satisfy the Hasse principle over k. ∎

Second proof. The idea is the same as before, but the proof differs slightly from the above one.

Let $\mathrm{U}(\Phi)$ be the unitary k-group associated with a non-degenerate skew-hermitian form Φ of type D_n over a global field k of characteristic $\neq 2$ with values in a non-trivial quaternion division algebra D as above. Now take any k-group G such that there is a k-isomorphism from $\mathrm{U}(\Phi)$ into G and the group G satisfies the Hasse principle over k. We have the exact sequence of sets

$$1 \to \mathrm{U}(\Phi) \to G \xrightarrow{\pi} G/\mathrm{U}(\Phi) \to 1,$$

and the associated commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{U}(\Phi)(k) \arrow[r] \arrow[d] & G(k) \arrow[r, "\pi"] \arrow[d] & (G/\mathrm{U}(\Phi))(k) \arrow[r] \arrow[d] & H^1(k, \mathrm{U}(\Phi)) \arrow[d] \\
1 \arrow[r] & \prod_v \mathrm{U}(\Phi)(k_v) \arrow[r] & \prod_v G(k_v) \arrow[r] & \prod_v (G/\mathrm{U}(\Phi))(k_v) \arrow[r] & \prod_v H^1(k_v, \mathrm{U}(\Phi))
\end{tikzcd}$$

with exact rows. The same argument as in the proof of Proposition 1 shows that there is an element $\lambda \in (G/\mathrm{U}(\Phi))(k)$ such that the Hasse principle does not hold for $X_\lambda = \pi^{-1}(\lambda)$. If we take G to be the general linear group containing U, i.e. $\mathrm{U}(\Phi)(k) = \mathrm{U}_n(\Phi, D)$, $G(k) = \mathrm{GL}_n(D)$, we see that $X_\lambda = g \cdot \mathrm{U}(\Phi)$, $g \in G$. Since $x \in X_\lambda$ if and only if $g^{-1} \cdot x \in \mathrm{U}(\Phi)$ and the defining equations of $\mathrm{U}(\Phi)$ can be given by quadratic forms, X is then defined by a system of quadratic forms, which is obtained by "twisting" the defining system of $\mathrm{U}(\Phi)$ by means of g. The number of quadratic forms and the number of variables are the same as before. ∎

Remark. The counter-examples constructed differ from the known counter-examples by that they give various systems with many quadratic forms and variables and they are non-trivial in the sense that they cannot be obtained from the known counter-examples to the Hasse principle just by adding a number of quadratic forms. It is worth noticing that since all the k-varieties constructed have the form $g \cdot \mathrm{U}(\Phi)$, hence they become the union of two k'-rational subvarieties for any quadratic extension $k' \supset k$ splitting the quaternion division algebra D associated with Φ. For, over k', the group $\mathrm{U}(\Phi)$ becomes $O(f)$, where f is a k'-quadratic form, and since the cohomological Hasse principle holds for $O(f)$ over k', the k'-variety $g \cdot O(f)$ has a k'-point. Since $O(f) = \mathrm{SO}(f) \cup \mathrm{SO}(f) \cdot g_0$, where g_0 is a k'-rational point of $O(f)$ (a matrix with determinant -1), we have $g \cdot O(f) = g \cdot \mathrm{SO}(f) \cup g \cdot \mathrm{SO}(f) \cdot g_0$.

Thus $g \cdot \mathrm{SO}(f)$ also has a k'-point, hence is k'-rational, since so is $\mathrm{SO}(f)$ by virtue of the Cayley transformation. By the way, these varieties also serve as counter-examples to the smooth Hasse principle.

Now we will consider some examples basing on the first proof of the theorem. As we have seen, that proof is more constructive than the second one.

EXAMPLE 1. A minimal example. It is obtained from the Kneser–Springer counter-example to the Hasse principle for a skew-hermitian form of dimension 1 (see [1], [15]). Let $D = \left(\frac{\theta, \eta}{k}\right)$ be a quaternion division algebra over a global field k of characteristic $\neq 2$ and let S be the finite set of all valuations of k at which D is non-ramified. Assume that $s = \mathrm{Card}(S) \geqslant 4$. For any skew quaternion $\alpha \in D$ such that $d = -\mathrm{Nrd}(\alpha) \notin k_v^{\times 2}$ for $v \in S$ (e.g. we can take $\alpha = i$, $i^2 = \theta$) we define 2^{s-2}-tuples $\varepsilon_j = (\varepsilon_{j,v})$, $1 \leqslant j \leqslant 2^{s-2}$, where $\varepsilon_{j,v} = \pm 1$ for all j, v, $\varepsilon_{j,v} = 1$ for $v \notin S$ and $\prod_v \varepsilon_{j,v} = 1$ such that for any j, there is at least one $v \in S$ with $\varepsilon_{j,v} = 1$. Choose for every j an element $\lambda_j \in k^\times$ such that

$$(\lambda_j, d)_v = \varepsilon_{j,v} \quad \text{for any } v,$$

where $(\cdot, \cdot)_v$ denotes the quadratic Hilbert symbol in k_v; this is possible (cf. A. Weil, *Basic number theory*, Springer-Verlag, 1967, ch. XIII, §6, Theorem 4, or J.-P. Serre, *Cours d'arithmétique*, Paris, 1970, ch. III, §2, Théorème 4, where the results are also valid in the global case).

Then, for every j, $1 \leqslant j \leqslant 2^{s-2}$, the solution of the equation

$$x^J \cdot \alpha \cdot x = \lambda_j \cdot \alpha$$

is in k_v for all v, but not in k. We can rewrite this equation in the form of a system of quadratic forms; for example if $\alpha = i$, then the system looks as follows:

$$\begin{cases} x_0^2 - \theta x_1^2 + \eta x_2^2 - \theta\eta x_3^2 = \lambda_j, \\ x_0 x_3 - x_1 x_2 = 0, \\ x_0 x_2 - \theta x_1 x_3 = 0. \end{cases}$$

This is of course a minimal example among the ones obtained above. It corresponds to the case $\dim \Phi = 1$. Let us consider this system more closely. Assume that $x_0 x_3 \neq 0$. Then $x_1 x_2 \neq 0$ and we see that the right-hand side of the first equation is equal to zero, which is impossible. Hence we must have $x_0 x_3 = 0$. Then if $x_0 = x_1 = 0$, the system becomes

$$\begin{cases} x_0 = x_1 = 0, \\ \eta x_2^2 - \theta\eta x_3^2 = \lambda_j. \end{cases}$$

If $x_0 = 0$, $x_1 \neq 0$, then $x_2 = x_3 = 0$ and the system becomes

$$\begin{cases} x_0 = x_2 = x_3 = 0, \\ -\theta x_1^2 = \lambda_j. \end{cases}$$

If $x_0 \neq 0$, $x_3 = 0$, then $x_2 = 0$ and the system becomes

$$\begin{cases} x_2 = x_3 = 0, \\ x_0^2 - \theta x_1^2 = \lambda_j. \end{cases}$$

Now it is clear that the k-variety X defined by our system is the union of two plane conics X_1, X_2, where X_1 is defined by the system

$$X_1\colon \begin{cases} x_0 = x_1 = 0, \\ \eta x_2^2 - \theta\eta x_3^2 = \lambda_j, \end{cases}$$

and X_2 is defined by the system

$$X_2\colon \begin{cases} x_2 = x_3 = 0, \\ x_0^2 - \theta x_1^2 = \lambda_j. \end{cases}$$

It is not hard to see that, using Serre's theorem cited above, one can choose λ such that any subsystem of the system constructed satisfies the Hasse principle over k.

EXAMPLE 2. Let $\dim(\Phi) = 2$. Take $\Phi = \mathrm{diag}(i, i)$. Then for λ_j, $1 \leqslant j \leqslant 2^{s-2}$, as above, we also obtain 2^{s-2} system of quadratic forms for which the Hasse principle does not hold:

$$\begin{cases} x^J ix + y^J iy = \lambda_j i, \\ x^J iz + y^J it = 0, \\ z^J iz + t^J it = \lambda_j i, \end{cases}$$

or equivalently, one can write these systems explicitly as follows:

$$\begin{cases} x_0^2 - \theta x_1^2 + \eta x_2^2 - \theta\eta x_3^2 + y_0^2 - \theta y_1^2 + \eta y_2^2 - \theta\eta y_3^2 = \lambda_j, \\ x_0 x_3 - x_1 x_2 + y_0 y_3 - y_1 y_2 = 0, \\ x_0 x_2 - \theta x_1 x_3 + y_0 y_2 - \theta y_1 y_3 = 0, \\ x_0 z_1 - x_1 z_0 - \eta x_2 z_3 + \eta x_3 z_2 + y_0 t_1 - y_1 t_0 - \eta y_2 t_3 + \eta y_3 t_2 = 0, \\ x_0 z_0 - \theta x_1 z_1 + \eta x_2 z_2 - \theta\eta x_3 z_3 + y_0 t_0 - \theta y_1 t_1 + \eta y_2 t_2 - \theta\eta y_3 t_3 = 0, \\ x_0 z_3 - x_1 z_2 - x_2 z_1 + x_3 z_0 + y_0 t_3 - y_1 t_2 - y_2 t_1 + y_3 t_0 = 0, \\ x_0 z_2 - \theta x_1 z_3 + x_2 z_0 - \theta x_3 z_1 + y_0 t_2 - \theta y_1 t_3 + y_2 t_0 - \theta y_3 t_1 = 0, \\ z_0^2 - \theta z_1^2 + \eta z_2^2 - \theta\eta z_3^2 + t_0^2 - \theta t_1^2 + \eta t_2^2 - \theta\eta t_3^2 = \lambda_j, \\ z_0 z_3 - z_1 z_2 + t_0 t_3 - t_1 t_2 = 0, \\ z_0 z_2 - \theta z_1 z_3 + t_0 t_2 - \theta t_1 t_3 = 0. \end{cases}$$

EXAMPLE 3. Now we consider the system

$$\begin{cases} x_0^2-\theta x_1^2+\eta x_2^2-\theta\eta x_3^2=\lambda, \\ x_0x_3-x_1x_2=0, \\ x_0x_2-x_1x_3=0 \end{cases}$$

which can be regarded as a modification of the system in Example 1. We show that with a suitable choice of θ, η, λ, this system does not satisfy the Hasse principle. If $x\in k$, we set $S(x):=$ (valuations v of k such that $(\theta, x)_v=-1$), where k is any global field and $(\cdot,\cdot)_v$ denotes the quadratic Hilbert symbol. First we choose θ, η such that $\operatorname{Card}(S(\eta))\geq 4$. Now we consider the following cases:

(a) $S(1+\eta)=\emptyset$. Then we choose λ such that $S(\eta)\supsetneqq S(\lambda)\neq\emptyset$.

(b) $S(\eta)\subseteq S(1+\eta)$ (resp. $S(\eta)\supsetneqq S(1+\eta)\neq\emptyset$). Then we choose λ such that $\emptyset\neq S(\lambda)\subsetneqq S(\eta)$ (resp. $\emptyset\neq S(\lambda)=S(\eta)\setminus S(1+\eta)$).

(c) In any other case, we choose λ such that

$$S(\lambda):=(S(\eta)\setminus S(1+\eta))\cup(S(1+\eta)\setminus S(\eta)).$$

Then, for λ chosen as above, the system considered does not satisfy the Hasse principle over k. Indeed a standard calculation shows that the above system has solutions in k_v if and only if one of the following relations holds:

(A) $(\theta, \lambda\eta)_v=1$,

(B) $(\theta, \lambda)_v=1$,

(C) $(\theta, \lambda(1+\eta))_v=1$,

and it has a solution in k if and only if one of these relations holds for all v. But by the choice of λ each relation (A), (B), (C) fails for some v, and for every v, one of them holds.

Geometrically, as a standard calculation shows, the k-variety X defined by this system is the union of the following plane conics $X_1, X_2, \dots, X_6$:

$$X_1\colon \begin{cases} x_0=x_1=0, \\ \eta x_2^2-\theta\eta x_3^2=\lambda, \end{cases} \qquad X_2\colon \begin{cases} x_2=x_3=0, \\ x_0^2-\theta x_1^2=\lambda, \end{cases}$$

$$X_3\colon \begin{cases} x_0-x_1=0, \\ x_2-x_3=0, \\ x_1^2+\eta x_3^2=\lambda/(1-\theta), \end{cases} \qquad X_4\colon \begin{cases} x_0-x_1=0, \\ x_2+x_3=0, \\ x_1^2+\eta x_3^2=\lambda/(1-\theta), \end{cases}$$

$$X_5\colon \begin{cases} x_0+x_1=0, \\ x_2-x_3=0, \\ x_1^2+\eta x_3^2=\lambda/(1-\theta), \end{cases} \qquad X_6\colon \begin{cases} x_0+x_1=0, \\ x_2+x_3=0, \\ x_1^2+\eta x_3^2=\lambda/(1-\theta). \end{cases}$$

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ACTA ARITHMETICA
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Sur les extensions totalement décomposées de certains corps de fonctions

par

JEAN CLAUDE DOUAI (Paris)

Soient X une courbe projective, lisse, géométriquement irréductible, définie sur un corps k complet pour une valuation discrète à corps résiduel fini k_0, K son corps des fonctions.

S. Saito et K. Kato ont défini dans [9] et [4] à partir du K_2 un groupe C_K (resp. $C_{m,K}$) (noté C_K^2 (resp. $C_{m,K}^2$) dans [4]) jouant le rôle de groupe de classes d'idèles (resp. de classes d'idèles pour la relation d'équivalence associée à un module m). Ils ont établi les suites exactes

$$(1) \qquad 0 \to (Q/Z)^r \to H^1(K, Q/Z) \to (C_K)^* \to 0,$$

$$(2) \qquad 0 \to (Q/Z)^r \to H^1(U_{\text{ét}}, Q/Z) \to (\varprojlim C_{m,K})^* \to 0$$

où r désigne un entier lié à la réduction de X modulo l'idéal maximal de k, nul dans le cas de bonne réduction, U un ouvert non vide de X, m dans la limite projective parcourant les modules de X tels que $U \cap m =$ ensemble vide. Que deviennent ces résultats dans le cas où $k = k_0((t))$ avec k_0 algébriquement clos et non plus fini? L'objectif de ce papier est de montrer qu'il existe alors des suites exactes

$$(1)' \qquad 0 \to (Q/Z)^{2g-\varepsilon} \to H^1(K, Q/Z) \to (C_K)^* \to 0,$$

$$(2)' \qquad 0 \to (Q/Z)^{2g-\varepsilon} \to H^1(U_{\text{ét}}, Q/Z) \to (\varprojlim C_{m,K})^* \to 0$$

du type précédent où cette fois C_K (resp. $C_{m,K}$) désigne le vrai groupe d'idèles de K (resp. des classes d'idèles pour la relation d'équivalence associé à un module m de X) mais qu'elles sont anti-analogues à (1) et (2) en ce sens que le rôle de l'entier r est joué par $2g-\varepsilon$ où g est le genre de X, ε l'invariant de Ogg associé à la réduction de X modulo t et est maximum précisément dans le cas de bonne réduction. Si $U = X$, l'exactitude de (2)' (resp. (2)) traduit exactement le fait que les revêtements abéliens complétement décomposés de X correspondent biunivoquement aux revêtements de même nature de la courbe réduite $\mathfrak{X}_s$ où $\mathfrak{X}$ désigne un modèle régulier de X sur $\operatorname{Spec} k_0[[t]]$ (resp. $\operatorname{Spec} O_k$). Les résultats obtenus ici précisent et complétent ceux de [3].