On the greatest prime factor of an arithmetical progression (II)

by

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To the memory of Professor V. G. Sprindžuk

1. For an integer $v > 1$, we denote by $P(v)$ the greatest prime factor of $v$ and we write $\omega(v)$ for the number of distinct prime factors of $v$. We put $P(1) = 1$ and $\omega(1) = 0$. Let $a$, $d$ and $k$ be positive integers satisfying $\gcd(a, d) = 1$ and $k \geq 3$. We put

$$\chi = a + (k-1)d, \quad x_1 = \chi/k, \quad x_2 = \max(x_1, 3),$$

$$\Delta(a; d) = a(a+d)\ldots(a+(k-1)d)$$

and

$$P = P(\Delta(a; d)), \quad \omega = \omega(\Delta(a; d)).$$

A classical theorem of Sylvester [5] states that

(1) \hspace{1cm} P > k \quad \text{if} \quad a \geq d + k.

Langevin [2] improved (1) to

(2) \hspace{1cm} P > k \quad \text{if} \quad a > k.

Further, Shorey and Tijdeman [4] showed that

(3) \hspace{1cm} P > k \quad \text{if} \quad d \geq 2 \quad \text{and} \quad (a, d, k) \neq (2, 3, 7).

Also, Langevin [3] obtained results which imply that, under suitable conditions, there exists some number $r > 1$ such that

$$P > C_1 k \log \log \log a \quad \text{if} \quad a > k^r$$

where $C_1 > 0$ is an effectively computable number depending only on $r$. This is an immediate consequence of the following result.

**Theorem 1.** Let $\varepsilon > 0$ and

$$\chi > k^{1+\varepsilon}. \quad \square$$
There exists an effectively computable number \( C_2 > 0 \) depending only on \( \varepsilon \) such that
\[
P > C_2 k \log \log \chi.
\]

By taking \( a = 1 \) and \( d = (\log \log \chi)^{1/2} \), we observe that the restriction (3) cannot be relaxed to \( \chi > k \). On the other hand, the assumption (3) is no more necessary if we are contented with \( \log \log \chi_2 \) instead of \( \log \log \chi \) in estimate (4).

See Corollary 1.

More precisely, we prove:

**Theorem 2.** Let \( \varepsilon > 0 \). There exist effectively computable numbers \( C_3 \) and \( C_4 > 0 \) depending only on \( \varepsilon \) such that for \( \chi_1 \geq C_3 \), either
\[
\omega \geq (1 - \varepsilon) k \log \log \chi_1 \log k
\]
or
\[
P \geq C_4 k \log \log \chi.
\]

The following result follows immediately from Theorem 2 and (2).

**Corollary 1.** There exists an effectively computable absolute constant \( C_6 > 0 \) such that
\[
P \geq C_6 k \log \log \chi_2.
\]

For deriving Corollary 1, we observe that (7) is a consequence of (2) if \( \chi_1 \) is bounded. In fact, in this case, it follows immediately from the Prime Number Theorem for arithmetical progressions that for \( \varepsilon > 0 \) and \( \psi > 0 \) with \( \chi_1 \leq \psi \), there exists an effectively computable number \( C_6 \) depending only on \( \varepsilon \) and \( \psi \) such that for \( k \geq C_6 \), we have
\[
\omega \geq (1 - \varepsilon) k / \log k \quad \text{and} \quad P \geq (1 - \varepsilon) \chi.
\]

2. In this section, we state results that we shall use from other sources for the proof of our theorems. Let \( F(X, Y) \in \mathbb{Z}[X, Y] \) be a binary form with at least three distinct linear factors in its factorisation over \( \mathbb{C} \). Denote by \( L \) the splitting field of \( F \) and we write \( I, R \) and \( h \), respectively, for the degree, regulator and class number of \( L \). Let \( H(F) \) be the maximum of the absolute values of the coefficients of \( F \). Let \( p_1, \ldots, p_n \) be distinct prime numbers and \( A \) some non-zero rational integer. We start with the following theorem of Győry [1] on an estimate for integer solutions of Thue-Mahler equations.

**Lemma 1.** All solutions of the Thue-Mahler equation
\[
F(x, y) = A p_1^{s_1} \cdots p_n^{s_n}
\]
in \( x, y, z_1, \ldots, z_n \), with \( \gcd(x, y) = 1, z_1 \geq 0, \ldots, z_n \geq 0 \) satisfy
\[
\log(\max(\{|x|, |y|\})) \leq C_7(s + 1)^{C_8 s + 1} k^{2(1 + \log(|A| H(F)))}
\]

where \( C_7 \) and \( C_8 \) are effectively computable numbers such that \( C_7 \), depends only on \( I, R, \), \( h \), and \( C_8 \) only on \( I \).

The next lemma contains an elementary fundamental argument of Erdős.

**Lemma 2.** Let
\[
S \subseteq \{a, a + d, \ldots, a + (k - 1)d\}, \quad s_0 = \min S
\]

Denote by \( T \) the set of all elements \( s \in S \) such that \( P(s) \leq k \). Then
\[
\|T\| \leq \frac{k \log k}{\log s_0 + \pi(k)}.
\]

**Proof.** For every \( p \leq k \), we choose an \( f(p) \in T \) such that \( p \) does not appear to a higher power in the factorisation of any other element of \( T \). Denote by \( T_1 \) the set obtained from \( T \) by deleting all \( f(p) \) with \( p \leq k \). Then
\[
|T_1| \geq |T| - \pi(k)
\]
and
\[
s_0^{\pi(k)} \leq \prod_{p \leq k} \left( 1 + \left( \frac{p}{k} \right)^{\frac{1}{p}} \right) \cdots = k!
\]

which implies (8).

3. We apply Lemma 2 to obtain the following result.

**Lemma 3.** Let \( \varepsilon > 0, V \geq 2 \) and \( \chi > e^\varepsilon \). If
\[
\log k \leq (\log \log \chi)^V, \quad P \leq (\log \chi)^{1/2 - \varepsilon}
\]

then
\[
\omega \geq C_9 k \log \log \chi
\]
and
\[
P \geq C_{10} k \frac{\log \log \chi}{\log \log \chi}
\]

where \( C_9 > 0 \) and \( C_{10} > 0 \) are effectively computable numbers depending only on \( \varepsilon \) and \( V \).

**Proof.** We denote by \( C_{11}, \ldots, C_{16} \) effectively computable positive numbers depending only on \( \varepsilon \) and \( V \). If \( k \leq C_{11} \), we apply Lemma 1 with
\[
F(X, Y) = X(X + Y)(X + 2Y)
\]
to conclude (10). Thus, we may assume that \( k \geq C_{12} \) with \( C_{12} \) sufficiently large.
Now, we apply Lemma 1 to the binary form (12) to derive from (9) that
\[ \omega((a+\mu d)(a+\mu d+d)(a+\mu d+2d)) \geq C_{13} \frac{\log \log \chi}{\log \log \log \chi} \]
for \( k/2 \leq \mu < k \). Consequently, we obtain
\[ \omega \geq C_{14} k \frac{\log \log \chi}{\log \log \log \chi} - \sum_{p \leq k} \left( \frac{k}{p} + 1 \right) \]
\[ \geq C_{15} k \left( \frac{\log \log \chi}{\log \log \log \chi} - \log \log k \right) \geq C_{16} k \frac{\log \log \chi}{\log \log \log \chi} \]
the last inequality follows from (9). Then, the assertion (11) follows immediately from Prime Number Theory.

Lemma 3 admits the following consequence.

**Corollary 2.** Let \( \varepsilon > 0 \) and \( \chi > \varepsilon^2 \). If
\[ k < (\log \chi)^{1/2 - \varepsilon} \]
then
\[ P \geq C_{17} k \frac{\log \log \chi}{\log \log \log \chi} \]
\[ \text{where } C_{17} \text{ is an effectively computable number depending only on } \varepsilon. \]

**Proof.** In view of Lemma 3 with \( V = 2 \) and \( \varepsilon \) replaced by \( \varepsilon/2 \), we may assume that
\[ P \geq (\log \chi)^{1 - \varepsilon/2} \]
which, together with (13), proves Corollary 2.

**Lemma 4.** Let \( \varepsilon > 0 \). There exist effectively computable numbers \( C_{18} \) and \( C_{19} > 0 \) depending only on \( \varepsilon \) such that for \( k \geq C_{18} \) and \( \chi = C_{18} \) we have
\[ \omega \geq k \min \left( (1 - \varepsilon) \frac{\log \log \chi}{\log \log k}, C_{19} \right). \]

**Proof.** We may assume that \( 0 < \varepsilon < 1 \) and that \( C_{18} \) is sufficiently large. We put \( \varepsilon = \varepsilon/4 \). Suppose \( a < \varepsilon \chi \). Then, we write \( \mu \) for the least positive integer such that \( A := a + \mu d \geq \varepsilon \chi \). Thus \( A - d < \varepsilon \chi \) and
\[ (k - 1 - \mu)d = \chi - A > (1 - \varepsilon)\chi - d \]
which implies that
\[ k - \mu > (1 - \varepsilon_1)(k - 1) \geq (1 - 2\varepsilon_1)k. \]
Therefore, we conclude that there exists some \( \mu \) with \( 0 \leq \mu < 2\varepsilon_1 k \) such that \( a + \mu d \geq \varepsilon_1 \chi \).

We put \( K = k - \mu \) and we denote by \( S \) the set of all integers \( A, A + d, \ldots, A + (K - 1) d \). Let \( T \) be the set of all elements \( s \in S \) with \( P(s) \leq K \).
Then
\[ \omega \geq |S| - |T| = K - |T|. \]

Further, we apply Lemma 2 with \( s_0 \geq \varepsilon_1 \chi \) to derive that
\[ |T| \leq \frac{K \log K}{\log s_0} + \pi(K) = \left( \frac{\log K}{\log k} \right)^{1 + \varepsilon} \pi(K) \ll \frac{K}{1 + \varepsilon} \pi(K) \]
where
\[ \nu = \frac{\log(\varepsilon_1 \chi)}{\log k} \leq (1 - \varepsilon) \frac{\log \chi}{\log k} \]
if \( C_{18} \) is sufficiently large.

First, we suppose that
\[ \chi < k^{1/(1 - \varepsilon)}. \]
Then \( \nu < \varepsilon/(1 - \varepsilon) \) and \( |T| \leq K(1 - (1 - \varepsilon)\nu) + \pi(K) \). Therefore
\[ \omega \geq K(1 - \varepsilon_1)\nu - \pi(K) \geq (1 - 2\varepsilon_1)K \frac{\log \chi}{\log k} \geq (1 - \varepsilon)k \frac{\log \chi}{\log k}, \]
since \( K \geq (1 - 2\varepsilon_1)k \). Thus, we may assume that (15) is not valid. Then \( \nu \geq \varepsilon_1 \)
and
\[ |T| \leq C_{20} K \]
where \( 0 < C_{20} < 1 \) is an effectively computable number depending only on \( \varepsilon \).
Thus
\[ \omega \geq (1 - C_{20})(1 - 2\varepsilon_1)k. \]

This completes the proof of Lemma 4.

**Proof of Theorem 1.** We may assume that \( 0 < \varepsilon < 1 \). By Corollary 2 and (3), we may suppose that \( k \geq C_{18}^{1/2} > C_{18} \) and \( \chi \geq k^1 \geq C_{18} \). Now, we apply Lemma 4 and Prime Number Theory to derive that
\[ P > C_{21} \log k \]
where \( C_{21} > 0 \) is an effectively computable number depending only on \( \varepsilon \).
Furthermore, in view of Corollary 2, we may suppose that
\[ k \geq (\log \chi)^{1/2 - \varepsilon}. \]
Finally, we combine (16) and (17) to obtain (4).

**Proof of Theorem 2.** We refer to Corollary 2 to assume that \( k \geq C_{18} \) and (17). Let \( C_3 > C_{18} \). Then, if (5) is not valid, we refer to Lemma 4 to derive that
\[ \omega \geq C_{19} k \]
which implies (16). Hence, the assertion (6) follows from (16) and (17).
References


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Received on 22.11.1988 (1984)
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ISBN 83-01-09787-6 ISSN 0065-1036