A limit theorem for the Riemann zeta-function in the complex space

by

A. LAURINCÍKAS (Vilnius)

In memory of V. G. SPrindžuk

Let \( s = \sigma + it \) be a complex variable and let \( \zeta(s) \) denote, as usual, the Riemann zeta-function. It is known by [33], [7] that the function \( \zeta(s) \) has a limiting distribution in the half-plane \( \sigma > 1/2 \). In modern terminology it is formulated as follows. Let \( C \) be the complex space and let \( \mathcal{B}(C) \) denote the class of Borel sets of the space \( C \). For \( T > T_0 \) let

\[

v_T(\ldots) = \frac{1}{T} \text{mes}\{t \in [0, T], \ldots\}
\]

where instead of dots we will write the condition which is satisfied by \( t \) and \( \text{mes}\{A\} \) denotes the Lebesgue measure of the set \( A \). We define the probability measure

\[

P_T(A) = v_T(\zeta(\sigma + it) \in A), \quad A \in \mathcal{B}(C), \quad \sigma > 1/2.
\]

The function \( \zeta(s) \) has a limiting distribution if on the space \( (C, \mathcal{B}(C)) \) there exists a probability measure \( P \) such that \( P_T \) weakly converges to \( P \) as \( T \to \infty \).

More general results are obtained in [1] where it was proved that the function \( \zeta(s) \) has a limiting distribution in the space of functions meromorphic in the half-plane \( \sigma > 1/2 \).

The aim of this paper is to prove the limit theorem for the Riemann zeta-function in the complex space, when \( \sigma \) depends on \( T \) and tends to \( 1/2 \) as \( T \to \infty \).

In [13] the theorem of this kind has been obtained for the modulus of the function \( \zeta(s) \). It turns out that in this case some power norming is necessary. It has been proved there that the distribution function

\[

v_T(\zeta(\sigma + it) \leq x) \sim \log x (1 + \log \log T)^{-1/2}
\]

converges as \( T \to \infty \) to the lognormal distribution function, i.e. to the dis-
A limit theorem for the Riemann zeta-function

We put

\[ S_n(s) = \sum_{m \leq n} \frac{d_m(m)}{m^s}, \quad g(s) = \zeta(s) - S_\infty^\mu(s). \]

\( c_1, c_2, \ldots \) are suitably chosen positive constants, \( B \) denotes a number (not always the same) which is bounded by a constant. Further on it is assumed that \( T \to \infty \).

We will need the mean value theorem of Montgomery–Vaughan for Dirichlet polynomials:

Let \( a_1, \ldots, a_n \) be arbitrary complex numbers. Then

\[ \int_0^T \left| \sum_{\substack{m \leq T \atop (m, n) = 1}} a_m \right|^2 dt = T \sum_{\substack{m \leq T \atop (m, n) = 1}} |a_m|^2 + B \sum_{\substack{m \leq T \atop (m, n) = 1}} m|a_m|^2. \]

This theorem is the special case of the results of [13]. For the proof see [6], p. 130–134.

**Lemma 1.** We have

\[ \int_0^T \left| g(s) \right|^2 dt = BT \exp\left\{ -c_1 \ln T \right\}. \]

**Proof.** Following [5], [8] we will define the functions

\[ w(t) = \int_0^\infty \exp\left\{ -2x(t+\tau)^2 \right\} dt, \quad L(\sigma) = \int_{-\infty}^\infty |S_n(\sigma+it)|^2w(t)dt, \]

\[ J(\sigma) = \int_{-\infty}^\infty |\zeta(\sigma+it)|^2w(t)dt, \quad K(\sigma) = \int_{-\infty}^\infty |g(\sigma+it)|^2w(t)dt. \]

In [8] it was shown that

\[ J(\frac{1}{2}) = B \sqrt{x^{-1}} T \ln T \zeta^2. \]

It follows from the definition of \( w(t) \) that

\[ w(t) = \frac{1}{4 \sqrt{x} \zeta^2(t \ln x) - 1} e^{-t^2/2} dt. \]

Consequently,

\[ w(t) = \begin{cases} B \sqrt{x^{-1}} \exp\left\{ -c_9 x (2\ln x^2 T - t^2) \right\} & \text{for } t \leq 0, \\ B \sqrt{x^{-1}} \exp\left\{ -c_{10} x (2T - t^2) \right\} & \text{for } t \geq 2T + \ln^2 x T, \\ B \sqrt{x^{-1}} & \text{for } 0 \leq t \leq 2T + \ln^2 x T. \end{cases} \]

Since \( S_n(\frac{1}{2}+it) = B \sqrt{N} \), from these properties of \( w(t) \) we obtain

\[ L(\frac{1}{2}) = B \sqrt{x^{-1}} \int_0^{2T} |S(\frac{1}{2}+it)|^2 dt + BN x^{-1} \ln^4 T. \]
By use of the estimation [8]
\[ \sum_{m \in N} d^2(m)/m = B(\ln T)^{2\beta} \]
and applying the Montgomery–Vaughan theorem we have
\[ \int_0^{2T} \left| S_N(t+it) \right|^2 dt = 2T \sum_{m \in N} d^2(m)/m + B \sum_{m \in N} d^2(m) \]
\[ = BT \sum_{m \in N} d^2(m)/m = BT(\ln T)^{2\beta}. \]

Hence and from (4) we obtain
\[ L(\lambda) = B, \sqrt{x^{-1}T(\ln T)^{2\beta}}. \]
From the definition of the function \( g(s) \) the inequality
\[ K(\lambda) \leq L(\lambda) \]
follows. Then by (3) and (5) we conclude that
\[ K(\lambda) = B, \sqrt{x^{-1}T(\ln T)^{2\beta}}. \]

In [12] the following modification of Lemma 7 of [5] was obtained. Let
\[ 1/2 < \sigma < 3/4, \]
then
\[ K(\sigma) \leq (1 + B(\ln T)^{-1/2}) \left( K(\lambda)^{1/3 - 2^{-1/2}}(c_{11}\sqrt{x^{-1}T} - 9\ln(4\pi - 2)^{1/3}) + B K(\lambda)^{1/2 - 3^{-1/2}} \exp \left\{ -c_{12} x(2\sigma - 1)\ln T + c_{13} (2\sigma - 1) \ln T \right\} \right). \]

Now, by the use of (6), we find that
\[ K(\sigma) = B, \sqrt{x^{-1}T(\ln T)^{2\beta}} \exp \left\{ -c_2 \ln T \right\} = BT \exp \left\{ -c_1 \ln T \right\}, \]

And again, by the properties of \( w(t) \), we obtain
\[ K(\sigma) = B, \sqrt{x^{-1}T(\ln T)^{2\beta}} + \frac{2T}{c_1} \left| g(\sigma + it) \right|^{2\beta} dt + B N^{-1/2} \ln T. \]

Hence and from (7) the assertion of Lemma 1 follows easily.

Let
\[ \tilde{\mu}_T(A) = v_T(S_N(\sigma + it) \in A), \quad A \in \mathfrak{B}(C). \]

We will prove that the study of the measure \( \mu_T \) can be replaced by that of the measure \( \tilde{\mu}_T \).

**Lemma 2.** If for \( T \to \infty \) the measure \( \tilde{\mu}_T \) weakly converges to some measure, then the measure \( \mu_T \) also weakly converges to the same measure.

**Proof.** Let \( \varepsilon_T = (\ln T)^{-1/2}. \) Then by Lemma 1 and by the Chebyshev inequality
\[ v_T(|\sigma_T + it| \geq \varepsilon_T) \leq 2 \int_0^{2T} \left| g(\sigma_T + it) \right|^2 dt = o(1). \]

Then, since the distribution function (1) converges to \( G(x) \) as \( T \to \infty \), we have
\[ v_T(|\xi(\sigma_T + it)| < 2 \sqrt{\varepsilon_T}) = v_T(|\xi(\sigma_T + it)| < (2 \sqrt{\varepsilon_T})^2T) \]
\[ = G(2 \sqrt{\varepsilon_T}^2) + o(1) = o(1). \]

Now from (8) and (9) we deduce
\[ v_T(|S_N^{\lambda}(\sigma_T + it)| < \sqrt{\varepsilon_T}) \]
\[ \leq v_T(|\xi(\sigma_T + it)| - |\xi(\sigma_T + it) - S_N^{\lambda}(\sigma_T + it)| < \sqrt{\varepsilon_T}) \]
\[ \leq v_T(|\xi(\sigma_T + it) - |g(\sigma_T + it)| \sqrt{\varepsilon_T}) \]
\[ \leq v_T(|\sigma_T + it| < 2 \sqrt{\varepsilon_T} + o(1) = o(1). \]

Let \( A \in \mathfrak{B}(C) \). Then in virtue of (8) and (10) we have uniformly in \( A \)
\[ v_T(|\xi(\sigma_T + it) \in A) \]
\[ = v_T(S_N^{\lambda}(\sigma_T + it) + g(\sigma_T + it) \in A) \]
\[ = v_T(S_N^{\lambda}(\sigma_T + it) + g(\sigma_T + it) \in A, |g(\sigma_T + it)| < \varepsilon_T) + o(1) \]
\[ = v_T(S_N(\sigma_T + it)(1 + g(\sigma_T + it)S_N^{\lambda}(\sigma_T + it)) \in A, \]
\[ |g(\sigma_T + it)| < \varepsilon_T, |S_N^{\lambda}(\sigma_T + it)| \geq \sqrt{\varepsilon_T} + o(1) \]
\[ = v_T(S_N(\sigma_T + it) + B \ln |g(\sigma_T + it)|S_N(\sigma_T + it)^{-1/2} \in A, \]
\[ |g(\sigma_T + it)| < \varepsilon_T, |S_N^{\lambda}(\sigma_T + it)| \geq \sqrt{\varepsilon_T} + o(1) \]
\[ = v_T(S_N(\sigma_T + it) + o(1) \in A, |g(\sigma_T + it)| < \varepsilon_T, |S_N^{\lambda}(\sigma_T + it)| \geq \sqrt{\varepsilon_T} + o(1) \]
\[ = v_T(S_N(\sigma_T + it) + o(1) \in A) + o(1) \]

Hence we see that from weak convergence of the measure \( \tilde{\mu}_T \) the weak convergence of the measure
\[ \tilde{\mu}_T(A) = v_T(|\xi(\sigma_T + it) \in A), \quad A \in \mathfrak{B}(C), \]
follows. It remains to pass from the measure \( \tilde{\mu}_T \) to the measure \( \mu_T \). For that we will make use of the mappings at weak convergence of measures. Let \( \tilde{\mu}_T \) converge weakly to the measure \( \mu \) and let \( h_T : C \to C \) be defined by
\[ h_T(s) = s(4\pi - 2)^{1/2}, \quad s \in C, \quad s \neq 0, \quad h_T(0) = 0. \]
Then by Theorem 5.5 from [2] we find...
that the measure \( \tilde{\nu}_T h_T^{-1} \) weakly converges to the measure \( \mu h^{-1} \) where \( h(s) = s \), i.e. \( \mu_T \) weakly converges to \( \mu \).

The sum \( S_N(\sigma_T + it) \) is rather long and it is difficult to approximate it by a product. Later on this sum will be replaced by a shorter one. For this we will need the asymptotics of some Dirichlet polynomial with multiplicative coefficients. Let \( g(m) = g_x(m; T) \) be a multiplicative function, \( |g(m)| \leq 1 \) and \( g(p) = c(\tau, k)x^{\frac{\sigma - 1}{2}} \). Here \( c(\tau, k) \) is some function of the parameters \( \tau \in \mathbb{R} \) and \( k \in \mathbb{Z} \).

**Lemma 3.** Let \( T \geq n \geq \frac{1}{2} \ln T \). Then uniformly for \( T \) and \( \tau, k \) in the domain \( |c(\tau, k)| \leq c_3 \)

\[
\sum_{m \leq x} g(m) m^{-2\tau} = \frac{1}{2} \left( H(1) + BR_T \right)
\]

where

\[
H(s) = \prod_p \left( 1 - \frac{1}{p^s} \right) \sum_{n=0}^{\infty} g(p^n), \quad \sigma > 1/2,
\]

\[ R_T = \max \left( \ln \varphi_T, \frac{1}{\sqrt{\ln l_T}} \right). \]

Proof. Consider the Dirichlet series

\[ Z(s) = \sum_{m \leq x} g(m)/m^s, \quad \sigma > 1. \]

Since \( g(p) = g \), we have by a simple calculation

\[ Z(s) = \zeta(s)H(s). \]

The function \( Z(s) \) satisfies all the conditions of the theorem from [4]. Let

\[ M(x) = \sum_{m \leq x} g(m)/m. \]

Then in [4] it has been shown that

\[ A(x) = \int_1^x M(u) du + \frac{xH(1)\ln x^p}{\Gamma(g+1)} + r(x) \]

where \( r(x) = Bx(\ln x)^{\sigma - 1} \). By this formula it is easy to find the asymptotics of the function \( M(x) \). Let \( A = x(\ln x)^{-1/2} \). In view of the identity

\[ M(x) = \frac{1}{A} \left( A(x + A) - A(x) - \int_x^{x+A} (M(u) - M(x)) du \right) \]

and of the estimate

\[ M(u) - M(x) = BAx^{-1} \quad \text{for } x < u \leq x + A \]

we deduce

\[ M(x) = (A(x + A) - A(x)) \frac{1}{A} + \frac{BA}{x}. \]

Thus, applying (11) we obtain

\[ M(x) = x(\ln x)^p \left( 1 + \frac{A}{x} \right) \left( \left( \frac{\ln(x + A)}{\ln x} \right)^{\sigma} - 1 \right) \frac{H(1)}{\Gamma(g+1)} + \frac{Bx(\ln x)^{\sigma - 1}}{\Gamma(g+1)} + \frac{BA}{x} \]

\[ = H(1) \ln x^p + Bx(\ln x)^{\sigma - 1} + B(\ln x)^{1/2} \]

Hence, summing by parts and taking into account the equalities

\[ \int_0^1 \frac{u^{\sigma - 1/2}}{u^{2\sigma}} du = Bx \sqrt{\ln x} + B(\sigma - 1)^{-\sigma - 1} \Gamma(\sigma + 1) \]

\[ \int_0^1 \frac{(\ln u)^p du}{u^{2\sigma}} = Bx \sqrt{\ln x} + B(\sigma - 1)^{-\sigma - 1} \Gamma(\sigma + 1) \]

we find

\[ \sum_{m \leq x} g(m)/m^{2\sigma} = H(1) \ln x^p + Bx \sqrt{\ln x} + B(\sigma - 1)^{-\sigma - 1} \Gamma(\sigma + 1) \]

\[ + Bx^{-2\sigma}(\ln x)^{\sigma - 1} + B(\ln x)^{1/2}. \]

Putting \( x = n \) in view of the obvious estimate

\[ \Gamma(g + 1, (2\sigma - 1)\ln n) = 1 + B \exp \{- c_4 \varphi_T \sqrt{\ln l_T} \}
\]

we obtain the assertion of the lemma.

Let \( M = \frac{1}{\ln T} \) and

\[ g_T(A) = v_T(S_M(\sigma + it) \in A), \quad A \in \mathcal{B}(\mathbb{C}). \]

**Lemma 4.** If the measure \( g_T \) weakly converges to some measure \( T \rightarrow \infty \), then \( \mu_T \) also weakly converges to the same measure.

Proof. By Lemma 2, it is sufficient to prove that the weak convergence of \( g_T \) implies the weak convergence of \( \mu_T \). By the Montgomery–Vaughan theorem and Lemma 3 we deduce that

\[ \frac{1}{T} \int_0^1 \left| S_N(\sigma_T + it) - S_M(\sigma_T + it) \right|^2 dt = \sum_{M < m < N} \frac{d^2(m)}{m^{2\sigma} T} + o \left( \sum_{M < m < N} \frac{d^2(m)}{m^{2\sigma}} \right) = BR_T. \]

Thus, by the Chebyshev inequality we obtain

\[ v_T(\left| S_N(\sigma_T + it) - S_M(\sigma_T + it) \right| \geq \sqrt{R_T}) \geq B \sqrt{R_T}. \]
Let \( A \in \mathcal{B}(C) \). From the estimate (12) it follows that uniformly in \( A \)
\[
v_T(S_M(\sigma_T+i)) = v_T(S_M(\sigma_T+i) + o(1)) \in A, |S_M(\sigma_T+i) - S_M(\sigma_T+i)| < 4\sqrt{T} + o(1)
\]
\[
= v_T(S_M(\sigma_T+i) + o(1)) \in A + o(1).
\]
The latter equality shows that the weak convergence of \( \tilde{\nu}_T \) to some measure implies the weak convergence of \( \tilde{\nu}_T \) to the same measure, which proves the lemma.

Now we turn the sum \( S_M(\sigma_T+i) \) into a product. It follows from the multiplicativity of \( d_{\sigma_M}(m) \) that
\[
S_M(\sigma_T+i) = \prod_{p \leq M} \left( 1 + \sum_{a \leq M/M(\ln p)^{1/2}} \frac{d_{\sigma_M}(m)}{m^{\sigma_T+i}} \right) - \sum_{1 \leq m \leq M} \frac{d_{\sigma_M}(m)}{m^{\sigma_T+i}}
\]
where the prime indicates that the sum is extended over those \( m \) whose all prime divisors are smaller than \( M \). It is easily seen that
\[
d \leq \prod_{p \leq M} p^s = B T^{\sigma_M}
\]
where \( c_s < 1 \). Therefore, by Lemma 3 we have
\[
v_T\left( \sum_{1 \leq m \leq M} \frac{d_{\sigma_M}(m)}{m^{\sigma_T+i}} \right) = o(1).
\]
Thus, in a manner similar to that used in the proof of Lemma 4 we can see that if the measure
\[
\tilde{\nu}_T : A \in \mathcal{B}(C),
\]
weakly converges to some measure as \( T \to \infty \), then \( \mu_T \) also weakly converges to the same measure.

Proof of the Theorem. We will write the product defining \( \tilde{\nu}_T \) in a simpler form. We have
\[
\Pi_M(t) \overset{\text{def}}{=} \prod_{p \leq M} \left( 1 + \sum_{a \leq M/\ln p^{1/2}} \frac{d_{\sigma_M}(p^a)}{p^{\sigma_T+i}} \right) \in A \in \mathcal{B}(C),
\]
and for an odd \( l \) we have
\[
\Pi_{M(\sigma_T+i)^2}^{(l+1)/2} = \prod_{p \leq M} \left( 1 + \sum_{a \leq M/\ln p^{1/2}} \frac{d_{\sigma_M}(p^a)}{p^{\sigma_T+i}} \right) \in A \in \mathcal{B}(C),
\]
and for an odd \( l \) we have
\[
\Pi_{M(\sigma_T+i)^2}^{(l+1)/2} = \prod_{p \leq M} \left( 1 + \sum_{a \leq M/\ln p^{1/2}} \frac{d_{\sigma_M}(p^a)}{p^{\sigma_T+i}} \right) \in A \in \mathcal{B}(C),
\]
Here \( \mathcal{C}_1 \) are binomial coefficients. If \( T \) is sufficiently large, and \( \tau_T \) and \( k_T \) tend to infinity slowly as \( T \to \infty \), then for all \( |t| \leq \tau_T, |k| \leq k_T \) the estimates \( h_{\tau,T}(p^a) = O(1), d_T(p^a) = O(1) \) for \( a \geq 1 \) are valid. From (13), (15) and (16) we find that uniformly in \( t \) and \( |t| \leq c_s, |k| \leq c_s \),
\[
\Pi_{M(\sigma_T+i)^2}^{(l+1)/2} = (1 + o(1)) \prod_{p \leq M} \left( 1 + \sum_{a \leq M/\ln p^{1/2}} \frac{h_{\tau,T}(p^a) + o(1)}{p^{\sigma_T+i}} \right) \cdot \prod_{p \leq M} \left( 1 + \sum_{a \leq M/\ln p^{1/2}} \frac{h_{\tau,T}(p^a) + o(1)}{p^{\sigma_T+i}} \right)
\]
Now we will find the characteristic transform \( w_T(\kappa, k) \) of \( \tilde{\nu}_T \). We have [11]
\[
w_T(\kappa, k) = \int \frac{1}{c(i\eta)} e^{\text{Re}s} d\tilde{\nu}_T
\]
\[
= \frac{1}{T} \int_0^T |\Pi_M(t)|^a \exp \{ ik \arg \Pi_M(t) \} dt.
\]
which can be easily seen that
\[
|\Pi_M(t)|^a = |\Pi_M^{(l+1)/2}(t)|^a \cdot |\Pi_M^{-1/2}(t)|^a
\]
For each prime \( p \) we have
\[
\left( 1 + \sum_{a \leq M/\ln p^{1/2}} \frac{d_{\sigma_M}(p^a)}{p^{\sigma_T+i}} \right) = \sum_{a \leq M/\ln p^{1/2}} \frac{d_{\sigma_M}(p^a)}{p^{\sigma_T+i}}
\]
where
\[
c_{\tau,T}(l) = \frac{\tau + k}{2} \left( \frac{\tau + k}{2} - 1 \right) \cdots \left( \frac{\tau + k}{2} - l + 1 \right) \frac{1}{l!}
\]
Therefore
\[
\Pi_{M(\sigma_T+i)^2}^{(l+1)/2} = \prod_{p \leq M} \left( 1 + \sum_{a \leq M/\ln p^{1/2}} \frac{d_{\sigma_M}(p^a)}{p^{\sigma_T+i}} \right) \in A \in \mathcal{B}(C),
\]
and for an odd \( l \) we have
\[
\Pi_{M(\sigma_T+i)^2}^{(l+1)/2} = \prod_{p \leq M} \left( 1 + \sum_{a \leq M/\ln p^{1/2}} \frac{d_{\sigma_M}(p^a)}{p^{\sigma_T+i}} \right) \in A \in \mathcal{B}(C),
\]
Here \( \mathcal{C}_1 \) are binomial coefficients. If \( T \) is sufficiently large, and \( \tau_T \) and \( k_T \) tend to infinity slowly as \( T \to \infty \), then for all \( |t| \leq \tau_T, |k| \leq k_T \) the estimates \( h_{\tau,T}(p^a) = O(1), d_T(p^a) = O(1) \) for \( a \geq 1 \) are valid. From (13), (15) and (16) we find that uniformly in \( t \) and \( |t| \leq c_s, |k| \leq c_s \),
\[
= \left(1 + o(1)\right) \prod_{p \leq \sqrt{M}} \left(1 + \sum_{1 \leq m \leq M} \frac{h_{r,k}(p')}{p^{\nu r + 1}}\right) \\
\times \prod_{\sqrt{M} < p \leq M} \left(1 + \frac{h_{r,k}(p)}{p^{\nu r + 1}} + \frac{h_{r,k}(p^2)}{p^{2(\nu r + 1)}}\right) \\
= \left(1 + o(1)\right) \left(\sum_{m \leq M} \frac{g_{r,k}(m)}{m^{\nu r + 1}} + \sum_{M < m \leq \sqrt{M}} g_{r,k}(m)\right).
\]

Here

\[
h_{r,k}(p') = c_{r,k}(l) d_{r}(p), \quad g_{r,k}(m) = \prod_{p||m} g_{r,k}(p'), \quad g_{r,k}(p') = \begin{cases} h_{r,k}(p'), & p \leq \sqrt{M}, \\ h_{r,k}(p'), & M < p \leq M. \end{cases}
\]

We note that in (17) \( d \leq T^n \) where \( c_8 < 1 \). Similarly we find that uniformly for all \( t \) and \( |t| \leq c_6, \ |k| \leq c_7 \)

\[
(18) \quad I^I_{d}(t) = \left(1 + o(1)\right) \left(\sum_{m \leq M} \frac{g_{r,k}(m)}{m^{\nu r + 1}} + \sum_{M < m \leq \sqrt{M}} g_{r,k}(m)\right).
\]

Consequently, from (14), (15) and (18) we deduce that uniformly in \( |t| \leq c_6, \ |k| \leq c_7 \)

\[
(19) \quad w_T(t, k) = \left(1 + o(1)\right) \int_0^T \int_{m \leq M} \frac{g_{r,k}(m)}{m^{\nu r + 1}} \, dt \\
+ B \frac{\sqrt{T}}{2} \sum_{m \leq M} \left| \sum_{M < m \leq \sqrt{M}} \frac{g_{r,k}(m)}{m^{\nu r + 1}} \right| \, dt \\
+ B \frac{\sqrt{T}}{2} \sum_{m \leq M} \left| \sum_{M < m \leq \sqrt{M}} \frac{g_{r,k}(m)}{m^{\nu r + 1}} \right| \, dt \\
+ B \frac{\sqrt{T}}{2} \sum_{m \leq M} \left| \sum_{M < m \leq \sqrt{M}} \frac{g_{r,k}(m)}{m^{\nu r + 1}} \right| \, dt \\
= I_1 + I_2 + I_3 + I_4.
\]

By the Cauchy–Schwarz inequality

\[
I_2 \leq \left(\frac{1}{T} \int_0^T \sum_{m \leq M} \left| \frac{g_{r,k}(m)}{m^{\nu r + 1}} \right|^2 \, dt\right)^{1/2} \left(\frac{1}{T} \int_0^T \sum_{M < m \leq \sqrt{M}} \left| \frac{g_{r,k}(m)}{m^{\nu r + 1}} \right|^2 \, dt\right)^{1/2}.
\]

Therefore, by the Montgomery–Vaughan theorem and Lemma 3, \( I_2 = o(1) \) uniformly in \( |t| \leq c_6, \ |k| \leq c_7 \). Similarly, \( I_3 = o(1) \) and \( I_4 = o(1) \). Taking into account the equality

\[
\sum_{m \leq M} \frac{g_{r,k}(m)}{m^{\nu r + 1}} + \sum_{M < m \leq \sqrt{M}} g_{r,k}(m) \left(\frac{n}{m}\right)^{k/2} = \sum_{m \leq M} \frac{g_{r,k}(m)}{m^{\nu r + 1}} + \sum_{m \leq M} \sum_{m < n \leq \sqrt{M}} g_{r,k}(m) \left(\frac{n}{m}\right)^{k/2}
\]

and Lemma 3 we obtain

\[
I_1 = \sum_{m \leq M} \frac{g_{r,k}(m)}{m^{\nu r + 1}} + \frac{B}{2} \sum_{m \leq M} \sum_{m < n \leq \sqrt{M}} \left| g_{r,k}(m) \frac{n}{m}\right| \ln \frac{n}{m} \sim \exp\left\{-\left(\frac{T^2}{2} + \frac{k^2}{2}\right)^2\right\} + o(1)
\]

uniformly in \( |t| \leq c_6, \ |k| \leq c_7 \). Hence and from (19) it follows that

\[
w_T(t, k) = \exp\left\{-\left(\frac{T^2}{2} + \frac{k^2}{2}\right)^2\right\} + o(1)
\]

uniformly in \( |t| \leq c_6, \ |k| \leq c_7 \). By the properties of characteristic transforms, we find that the measure \( \mu_T \) weakly converges as \( T \to \infty \) to the measure defined by the characteristic transform \( \exp\left\{-\left(\frac{T^2}{2} + \frac{k^2}{2}\right)^2\right\} \). It is obvious that the limit measure is non-degenerate. It follows from the lemmas proved above that the measure \( \mu_T \) also weakly converges to the same measure. The theorem is proved.

Here we have considered only the value \( \sigma = \sigma_T \). It is easily seen that a similar result with appropriate changes is also valid for \( \sigma > \sigma_T \). The case \( \sigma < \sigma_T \) is more complicated.

The author expresses his gratitude to Professor J. Kubilius for constant attention to the present work.

References

Обобщенные тэта-функции с характеристиками и представление чисел квадратичными формами

Т. В. Вепхвадзе (Тбилиси)

Светлой памяти Владимира Геннадьевича Стрипундже
посвящается

Пусть

\[ f = f(x) = f(x_1, x_2, \ldots, x_\nu) = \frac{1}{2} x^T A x = \frac{1}{2} \sum_{j,k=1}^{\nu} A_{jk} x_j x_k \]

- целочисленная положительная квадратичная форма, где \( x \) — вектор-столбец с компонентами \( x_1, x_2, \ldots, x_\nu \), а \( x^T \) — вектор-столбец; \( A \) — определитель симметричной матрицы \( A = (A_{jk}) \) с четными диагональными элементами; \( N \) — степень формы \( f \), т.е. наименьшее натуральное число, для которого \( NA^{-1} \) — симметричная целочисленная матрица с четными диагональными элементами. Далее, пусть \( r(n; f) \) обозначает число представлений натурального числа \( n \) формой \( f \), т.е. число решений в целых числах уравнения

\[ n = f(x_1, x_2, \ldots, x_\nu). \]

Количество работ, посвященных т.н. точным формулам для функции \( r(n; f) \), весьма велико. Эта тема привлекала внимание математиков еще в прошлом веке (Гаусс, Эйнштейн, Линевиль и др.).

Задача получения формулы для \( r(n; f) \), годной для всех \( n \), сводится к задаче получения формулы для тэта-ряда

\[ g(\tau; f) = 1 + \sum_{n=1}^{\infty} r(n; f)e^{2\pi i n \tau} \]

являющегося целой модулярной формой некоторого типа, здесь и всюду в дальнейшем \( \tau \in \mathbb{H} \) (\( \mathbb{H} \) — верхняя полуплоскость). Схема метода получения такой формулы заключается в следующем. Тэта-ряд представляется в виде суммы двух слагаемых:

\[ g(\tau; f) = E(\tau; f) + X(\tau), \]