V. P. Myakhishev, *Distribution of primitive integer points on certain cones*, Doklady ANSSSR 143 (1962), 785–786. Ich selbst füge hinzu: F. Fricker, Archiv d. Math. 28 (1977), 391–394.

Wie ich sehe ist in der Literatur nur der "eindimensionale" Fall behandelt. Die Sätze 2 und 3 kommen meines Wissens nach nicht vor, vor allem nicht die Anwendung auf die Gleichverteilung auf Sphären und auf numerische Berechnung von Integralen. Dies wird in einer weiteren Arbeit noch weiter ausgearbeitet werden. §3 ist an sich interessant.

Literatur

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Ramanujan's modular equations

by

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Dedicated to the memory of V. G. Sprindžuk

1. Let $\tau = x+iy$, $y > 0$ be a point in the upper half complex τ-plane H and let $\Gamma = \mathrm{SL}(2, Z)$ be the group of two-rowed integral matrices of determinant 1:

$$\Gamma = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \text{ integers}, \ ad-bc = 1 \right\}.$$

The group Γ acts on H discontinuously

$$\tau \to \tau_M = \frac{a\tau+b}{c\tau+d}.$$

Let G be a subgroup of finite index in Γ. A meromorphic function $\varphi(\tau)$ on H is called a *modular function belonging to G* if for every $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G,

$$\varphi(\tau_M) = \varphi\left(\frac{a\tau+b}{c\tau+d}\right) = \varphi(\tau)$$

and $\varphi(\tau)$ has nice behaviour at all parabolic vertices of G in H (see [3]). If $n > 0$ is a rational integer, $\varphi(\tau)$ and $\varphi(n\tau)$ are modular functions belonging to a subgroup of finite index in Γ and hence are algebraically related. This algebraic relation may be called a *modular equation* for $\varphi(\tau)$ of degree n.

Let K and iK' be the fundamental quarter periods of an elliptic function field so that

$$\tau = iK'/K$$

has positive imaginary part. Put $q = e^{\pi i\tau}$ and let k^2 and $k'^2 = 1-k^2$ be the squares of the moduli associated with K and K' respectively. Then

$$k^{1/4} = \sqrt{2}\cdot q^{1/8}\cdot\frac{1+q+q^3+q^6+\ldots}{1+2q+2q^4+2q^9+\ldots}. \tag{1}$$

Let us put, with Ramanujan, $k^2 = \alpha$ so that $k'^2 = 1-\alpha$. If $n > 0$ is a positive integer, put in the usual notation of elliptic functions

$$\frac{L'}{L} = n \cdot \frac{K'}{K}. \tag{2}$$

Let β and $1-\beta$ correspond to L and L' just as α and $1-\alpha$ correspond to K and K'. The relation (2) is said to be a *transformation of degree n* and *β of degree n relative to α*. Further $\alpha^{1/8}$ and $\beta^{1/8}$ are modular functions of τ belonging to the same subgroup G of finite index in Γ ([3], p. 491). Hence they satisfy an algebraic relation. The simplest such relation was obtained for n odd, in fact $n = 3$, by Legendre:

$$(\alpha\beta)^{1/4} + ((1-\alpha)(1-\beta))^{1/4} - 1 = 0. \tag{3}$$

Relations like (3) involving $\alpha\beta$ and $(1-\alpha)(1-\beta)$ where β is of degree n are called *irrational modular equations* of degree n. They are called $k\lambda - k'\lambda'$ modular equations by Russell ([7], p. 90). Such modular equations have been investigated by many mathematicians.

However, Jacobi and, following him, Sohnke ([3], p. 495 et seq.) obtained equations connecting $\alpha^{1/8}$ and $\beta^{1/8}$. Jacobi obtained equation for $n = 3$

$$v^4 - u^4 - 2vu(v^2u^2 - 1) = 0, \quad u = \alpha^{1/8}, \; v = \beta^{1/8} \tag{4}$$

and a similar equation for $n = 5$. Sohnke, inspired by Jacobi, ([3], p. 495 et seq.) obtained such equations for $n = 7, 11, 13, 17, 19$. These equations may be called the *Jacobi–Sohnke equations.*

In order to obtain simpler equations, Hermite and more importantly Schläfli ([9]) considered equations connecting the modular functions

$$f(\tau) = (2^4/\alpha(1-\alpha))^{1/24}, \quad f(n\tau) = (2^4/\beta(1-\beta))^{1/24}. \tag{5}$$

These equations are called the *Schläfli modular equations*. The simplest such equation for $n = 3$ is

$$(u/v)^6 + (v/u)^6 = u^3v^3 - 8/v^3u^3, \quad u = f(\tau), \; v = f(3\tau). \tag{6}$$

Schläfli went on to obtain such equations for $n = 3, 5, 7, 11, 13, 17$ and 19 and also for the composite degree $n = 9$ ([9], p. 368–369).

In two very interesting papers [7], [8], Russell obtained a large number of modular equations, some of them new at that time. An interesting feature of that work was his obtaining modular equations for composite degrees, that is irrational modular equations involving $\alpha\beta$ and $(1-\alpha)(1-\beta)$ where β is of composite degree n; in particular for $n = 9, 15, 35, 95$, and 119. At about the same time, Weber [12], independently, developed his function-theoretic methods to obtain irrational modular equations of degrees $n = 3, 5, 7, 11, 19, 23, 31, 47, 71$, and $n = 15$. By the same methods he obtained all the Schläfli equations of degrees $n = 3, 5, 7, 11, 13, 17$ and 19. However the most interesting aspect of Weber's important work [12] is that concerning, what are now

known as mixed modular equations for composite degrees $n = 15, 21, 33, 35, 39, 55$ and 105. He found that if $n = n' \cdot n''$ where n' and n'' are two distinct odd primes, then one has to consider simultaneously all the four modular functions

$$\begin{aligned} f(\tau) &= (2^4/\alpha(1-\alpha))^{1/24}, & f(n\tau) &= (2^4/\delta(1-\delta))^{1/24}, \\ f(n'\tau) &= (2^4/\beta(1-\beta))^{1/24}, & f(n''\tau) &= (2^4/\gamma(1-\gamma))^{1/24}. \end{aligned} \tag{7}$$

He found that, for example for $n = 15 = 3 \cdot 5$, one has the mixed modular equations

$$\begin{aligned} &2^{1/3}(\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/24} + (\alpha\beta\gamma\delta)^{1/8} \\ &\qquad + ((1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/8} = 1, \\ &\left\{\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}\right\} \\ &\times \left\{\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}\right\} + 1 = 0. \end{aligned} \tag{8}$$

These two equations are given by Weber on pages 358 and 359 respectively of his very important paper [12].

Indeed, Weber had stated many more mixed modular equations in his paper; more about them in the sequel below.

2. Ramanujan's work on modular equations is contained in two chapters (Chapter XIX and Chapter XX) of Notebook II and in several pages in Notebook I, especially pages 86–92 and pages 282 et seq. It spans all that has been said about irrational modular equations, the Jacobi–Sohnke equations, the Schläfli modular equations and above all the mixed modular equations.

Ramanujan begins his work on irrational modular equations on page 282 of Notebook I, giving equations of degrees 3, 5, ... by using identities between theta functions. On pages 302–304, however, he states a number of modular equations of degrees $n = 7, 23, 71$; $15, 31, 47, 95$ for $n+1 \equiv 0 \pmod 8$), $n = 11, 19, 35, 27$ for $n \equiv 3 \pmod 8$ and $n = 5, 9, 13, 17, 29$ for $n \equiv 1 \pmod 4$ in terms of functions P, Q, R of Russell ([7], p. 110), which in case $n+1 \equiv 0 \pmod 8$ are given by

$$\begin{aligned} P &= 1 + (-1)^{(n+1)/8}((\alpha\beta)^{1/8} + ((1-\alpha)(1-\beta))^{1/8}), \\ Q &= (\alpha\beta)^{1/8} + ((1-\alpha)(1-\beta))^{1/8} + (-1)^{(n+1)/8}(\alpha\beta(1-\alpha)(1-\beta))^{1/8}, \\ R &= (\alpha\beta(1-\alpha)(1-\beta))^{1/8}, \end{aligned} \tag{9}$$

where β is of degree n, in Ramanujan's terminology.

Greenhill's book [4] and Cayley's book on elliptic functions are two books which Ramanujan is supposed to have looked into and perhaps studied. Greenhill, in his book, has just two pages (327–328) on modular equations.

He simply quotes, not completely however, from Russell's longer paper [8] and gives modular equations of degrees $n = 15, 31, 47; 7, 23, 71; 3, 11, 19, 35; 9, 17; 5, 13, 29$. Russell, in his paper, has however equations for $n = 95, 119, 53$ which Greenhill does not give (see Russell [8], p. 388–390). Ramanujan, in the pages 302–304 gives all these results, not in the same order however (his equation for $n = 95$ lacks the last term). He does not give the equations for $n = 53$ and 119. He has, however, an equation for $n = 27$ which Russell does not have. In fact, this result for $n = 27$ according to Hanna ([5], p. 49–50) who states it in the same form as Ramanujan, seems to have been found by Fiedler in a paper, which is inaccessible. If one looks at these pages in Notebook I, it is obvious that Ramanujan tries, unsuccessfully in some cases, to give newer results. It does not appear that Ramanujan had looked at Russell's paper in the Proceedings of the London Mathematical Society, assuming that this journal was available in some library in Madras and that he had access to it. If he had seen Russell's paper, there seems no reason why he should not have recorded in his Notebook I Russell's equations for $n = 53$ and 119. Further sheer curiosity should have forced him to look into the later volumes of this periodical for more results on modular equations. He would have then stumbled across the paper of L. J. Rogers, in the volume for 1894, where Rogers proves, among others, the result of Rogers which is now known as the Rogers–Ramanujan identities. It is instructive to see that on page 160 of Notebook I Ramanujan states these identities, in his own notation, along with many q-identities of Euler and more importantly of Heine and others (rediscovered by him). He has in addition given many applications of these to continued fractions. Thus while he did gain by looking at the tabulated results in Greenhill's book, his whole outlook on modular equations has been completely original. This is obvious also from the way he derives the equations from theta function identities on pages 282 et seq. in Notebook I and in Chapters XIX and XX in Notebook II.

We make a remark on equation of degree $n = 31$. According to Watson [10] a form of this equation was known to Schröter. However identical forms of the equation for $n = 31$, different from that of Schröter's were given by Russell ([8], p. 388) and Weber ([12], p. 352). Russell and Weber give it in the form

$$(P^2 - 4Q)^2 - 4PR = 0 \tag{10}$$

where P, Q, R are given by (9) with $n+1 = 32 = 4\cdot 8$. On page 302 of Notebook I, Ramanujan gives for $n = 31$ the equation

$$P^2 - Q - \sqrt{PR} = 0 \tag{10'}$$

where he has Q instead of $4Q$ and R instead of $4R$ of Russell. In Chapter XX of Notebook II Ramanujan gives three forms of the equation of degree $n = 31$ of which the middle one is precisely (10′) with P, Q, R substituted from (9); namely

$$\begin{aligned} &1 + (\alpha\beta)^{1/4} + ((1-\alpha)(1-\beta))^{1/4} - 2\left((\alpha\beta)^{1/8} + ((1-\alpha)(1-\beta))^{1/8}\right. \\ &\qquad\qquad \left. + (\alpha\beta(1-\alpha)(1-\beta))^{1/8}\right) \\ &= 2(\alpha\beta(1-\alpha)(1-\beta))^{1/16}\sqrt{1 + (\alpha\beta)^{1/8} + ((1-\alpha)(1-\beta))^{1/8}}. \end{aligned} \tag{11}$$

This is the form in which Russell gives the equation for $n = 31$ in his first paper ([7], p. 98).

The other two forms of the equation given by Ramanujan seem to be new (see remark of Bruce Berndt, Chapter XX, p. 177 top line).

3. Ramanujan has given the Jacobi–Sohnke equations only for $n = 3, 5, 7$. They are stated on page 92 of Notebook I and also in the comprehensive accounts in Chapter XIX of Notebook II, specifically pages 231, 5(xiii), 237, 13(xv) and 240, 19(x) in the forms given in (12) below. As is well known, the Jacobi–Sohnke equations are equations involving α and β but not $1-\alpha$, $1-\beta$ explicitly. They are

$$\begin{aligned} &n = 3, \quad Q - Q^{-1} = 2(P - P^{-1}), \quad P = (\alpha\beta)^{1/8}, \quad Q = (\beta/\alpha)^{1/8}, \\ &n = 5, \quad (Q - Q^{-1})^3 + 8(Q - Q^{-1}) = 4(P - P^{-1}), \quad P = (\alpha\beta)^{1/4}, \quad Q = (\beta/\alpha)^{1/8}, \\ &n = 7, \quad P + P^{-1} = Q + Q^{-1} + (P^{1/8} - P^{-1/8})^8, \quad P = (\alpha\beta)^{1/2}, \quad Q = (\beta/\alpha)^{1/2}. \end{aligned} \tag{12}$$

These are stated in slightly different forms on page 92 of Notebook I. We shall prove these in the sequel using the statements of Ramanujan in Chapter XIX.

Ramanujan knew the Schläfli modular equations. Indeed on page 90 of Notebook I he states the equations for $n = 3, 5, 7, 11, 13, 17$ and 19. In trying to obtain the Schläfli equation for $n = 13$, using Ramanujan's equations for $n = 13$, Watson ([11], p. 191, footnote) remarks: "Although Ramanujan constructed numerous equations of Schläfli's type, he left no indication that he was acquainted with Schläfli's equation of order 13". This is strange in view of the results on page 90 of Notebook I.

The most important work of Ramanujan's on modular equations is undoubtedly the one concerning mixed modular equations. As mentioned in Section 1, Weber was the pioneer in this regard. Ramanujan has most of the results of Weber but more importantly goes much much further. He has, in addition, several forms of these mixed equations for every degree he has considered. Bruce Berndt who proved many of Ramanujan's results left unproved by Watson makes the rather unfortunate statement [Chapter XX, p. 196 of preprint]: "Entry 24 completes Ramanujan's work on mixed modular equations. We *emphasize* that not only are all of Ramanujan's results original but that *the literature contains no results of this kind whatsoever*. The only mixed modular equations mentioned in the literature are two results from Entry 11 of this chapter quoted by Hardy in his description of Ramanujan's work in the field of elliptic functions" (italics ours). Ironically both Watson and, following him, Bruce Berndt quote this important paper of Weber's which contains these mixed modular equations. See also a similar remark by Bruce Berndt ([2], p. 317 last line) in his recent paper.

Just as there are Schläfli modular equations of prime degree, Ramanujan has listed Schläfli-like modular equations for mixed modular equations, i.e. for composite degrees. On pages 86 and 88 of Notebook I Ramanujan gives such equations. For example for $n = 35 = 5\cdot 7$ he gives on page 86 of Notebook I the equation

$$Q^4+Q^{-4}-(Q^2+Q^{-2})-2(P^2+P^{-2}) = 0 \tag{13}$$

where

$$P = (256\,\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/48}, \quad Q = \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/48}.$$

We prove below two equations of this type which can be deduced from results in Chapter XX of Notebook II.

In this paper we shall prove many statements of Ramanujan within the framework of Ramanujan's results in Notebook II. For example we use the method of Watson to prove, in particular, the Schläfli modular equation of degree 7 which can be proved also by using other results of Ramanujan. We also show the connection between the results of Weber and that of Ramanujan.

A more detailed account of the problems raised here will be given in a continuation of this paper.

Ramanujan's many-sided genius and the astonishing results he obtains shine forth from his work on modular equations. How maddening it is to learn that at the time Ramanujan was working there was, in India — mathematically barren, nobody even to suggest to him literature to be looked into.

4. Let, in the usual notation of elliptic functions,

$$L'/L = nK'/K \tag{14}$$

represent a transformation of degree n and let α, $1-\alpha$ correspond to K, K' and β, $1-\beta$ to L and L' respectively so that β is of degree n relative to α. Let

$$m = \frac{K}{L} \tag{15}$$

where m is the *multiplier* of the transformation (14). If $\varphi(\alpha, \beta, m) = 0$ is an equation connecting α, β and m then

$$\varphi(1-\beta, 1-\alpha, n/m) = 0 \tag{16}$$

is also satisfied. This is known as the *allied* equation to $\varphi(\alpha, \beta, m) = 0$.

Let us now consider a transformation of degree 3. Ramanujan has given theta function identities, proved by Watson [10], from which he obtained the results

$$\left(\frac{\alpha^3}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} = 1 \quad \text{(XIX, 5(i)),}^{(1)} \tag{17}$$

$$1+2\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} = \frac{3}{m} \quad \text{(XIX, 5(iii)),} \tag{18}$$

β being of degree 3 in relation to α. These identities are already stated in Notebook I. One has, of course, the allied equations

$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^3}{\alpha}\right)^{1/8} = 1, \tag{17'}$$

$$1+2\left(\frac{\beta^3}{\alpha}\right)^{1/8} = m. \tag{18'}$$

From (17′) and (18′) we get

$$\begin{aligned} m &= 1+\left(\frac{\beta^3}{\alpha}\right)^{1/8}+\left(\frac{\beta^3}{\alpha}\right)^{1/8} = 1+\left(\frac{\beta^3}{\alpha}\right)^{1/8}+\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8}-1 \\ &= \left(\frac{\beta^3}{\alpha}\right)^{1/8}+\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} \end{aligned} \tag{19}$$

and the allied result

$$\frac{3}{m} = \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8}+\left(\frac{\alpha^3}{\beta}\right)^{1/8}. \tag{19'}$$

Multiplying (17) with (17′) and (19) with (19′) we have the two relations

$$\left(\left(\frac{\beta^3}{\alpha}\right)^{1/8}-\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8}\right)\left(\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8}-\left(\frac{\alpha^3}{\beta}\right)^{1/8}\right) = 1,$$

$$\left(\left(\frac{\beta^3}{\alpha}\right)^{1/8}+\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8}\right)\left(\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8}+\left(\frac{\alpha^3}{\beta}\right)^{1/8}\right) = 3.$$

Multiplying out and adding and subtracting, we get

$$(\alpha\beta)^{1/4}+((1-\alpha)(1-\beta))^{1/4}-1 = 0 \quad \text{(XIX, 5(ii))} \tag{20}$$

which is Legendre's equation, and

$$\left(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\right)^{1/8}+\left(\frac{\alpha^3(1-\beta)^3}{\beta(1-\alpha)}\right)^{1/8} = 2.$$

After removing denominators we get

$$(\beta(1-\alpha))^{1/2}+(\alpha(1-\beta))^{1/2} = 2(\alpha\beta(1-\alpha)(1-\beta))^{1/8} \quad \text{(XIX, 5(ix)).}$$

(1) The entries within the brackets denote Chapter number, Entry in that Chapter and the subentry.

Squaring (19′) and subtracting from it the square of (17), gives

$$\frac{9}{m^2} = 1+4\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8} \qquad \text{(XIX, 5(v))} \tag{21}$$

and the allied result

$$m^2 = 1+4\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8} \qquad \text{(XIV, 5(v))}.$$

These two equations give, by multiplication, the Schläfli modular equation of degree 3:

$$9 = \left(1+4\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}\right)\left(1+4\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}\right). \tag{22}$$

Multiplying, clearing denominators and dividing by $(\alpha\beta(1-\alpha)(1-\beta))^{1/4}$ gives

$$Q+Q^{-1}+2\sqrt{2}(P-P^{-1}) = 0 \qquad \text{(XIX, 5(xii))} \tag{23}$$

where

$$P = (16\alpha\beta(1-\alpha)(1-\beta))^{1/8}, \qquad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}.$$

Equation (23) is precisely the Schläfli modular equation of degree 3 given by Ramanujan. If we put $u = f(\tau) = (2^4/\alpha(1-\alpha))^{1/24}$ and $v = f(3\tau) = (2^4/\beta(1-\beta))^{1/24}$ we get the Schläfli equation given by Weber ([12], p. 349).

The equation (23) is also given on page 90 of Notebook I.

Similar results are true in the cases 5 and 7. For example in the case $n = 5$, if we multiply the two equations in (XIX, 13 (iv)) we get

$$5 = \left(1+2\cdot 2^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}\right)\left(1+2\cdot 2^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}\right). \tag{24}$$

Multiplying out, clearing the denominators and dividing by $(\alpha\beta(1-\alpha)(1-\beta))^{1/8}$ throughout, we get

$$Q+Q^{-1}+2(P-P^{-1}) = 0 \qquad \text{(XIX, 13(xiv))} \tag{24′}$$

where

$$P = (16\alpha\beta(1-\alpha)(1-\beta))^{1/12}, \qquad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}.$$

This is also given on page 90 of Notebook I.

It is interesting to notice that on page 287 top line, Notebook I, Ramanujan states precisely (24). It is possible that this is how Ramanujan obtained the Schläfli equations of degrees 3, 5, 7 and 11.

In the case of the Schläfli equation of degree 11, one has to add and subtract the two equations XX, 7 (vi) and XX, 7 (vii) and multiply the two resulting equations. We then get the Schläfli equation of degree 11 given by Ramanujan on page 90 of Notebook I.

It is not known how Ramanujan obtained the other equations on page 90 of Notebook I. For $n = 13$ Watson used a method which is interesting ([11], p. 192). For $n = 13$ Ramanujan gave only the two equations

$$\begin{aligned} m &= \left(\frac{\beta}{\alpha}\right)^{1/4}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/4}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}-4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}, \\ \frac{13}{m} &= \left(\frac{\alpha}{\beta}\right)^{1/4}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/4}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}-4\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6}. \end{aligned} \tag{25}$$

Equations like (25) are stated by Ramanujan for $n = 3, 5, 7, 9, 13, 25$ on page 291 of Notebook I. Perhaps Ramanujan had a uniform method of deriving all of them and to use them to obtain the Schläfli equations of respective degrees.

We shall now show how equations like (25) lead to the Schläfli modular equation. We shall take, for example, $n = 7$ and use the method of Watson ([11], pp. 192–194).

Let us take the equations (XIX, 19 (v))

$$\begin{aligned} m^2 &= \left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/2}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}-8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3}, \\ \frac{49}{m^2} &= \left(\frac{\alpha}{\beta}\right)^{1/2}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/2}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}-8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3} \end{aligned} \tag{26}$$

given by Ramanujan, β being of degree 7. We write the second equation as

$$\frac{49}{m^2}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} = \left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{1-\beta}{1-\alpha}\right)^{-1}-1-8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}. \tag{27}$$

Multiplying together (27) and the first equation in (26), we get

$$\left(\left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/2}\right)^2-(1+8Q^4+8Q^8+Q^{12})\left(\left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/2}\right) +8Q^8(1+Q^4)^2 = 0, \tag{28}$$

where

$$Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/24}. \tag{29}$$

Let us put

$$Q^2+Q^{-2} = v > 0,$$

and

$$\theta = \tfrac{1}{2}\left(v^3+5v+\sqrt{(v^3-3v)^2+16v(v^3-3v)+32v^2}\right).$$

Then the positive root $\left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/2}$ of the quadratic in (28) is given by

$$\left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/2}=\theta\cdot Q^6. \tag{30}$$

If we put

$$\bar\theta = \tfrac{1}{2}\left(v^3+5v-\sqrt{(v^3-3v)^2+16v(v^3-3v)+32v^2}\right)$$

then

$$\begin{aligned}\theta+\bar\theta &= v^3+5v,\\ \theta\bar\theta &= 8v^2,\\ \theta^2 &= (v^3+5v)\,\theta-8v^2.\end{aligned} \tag{31}$$

Let us now put

$$P=\left(16\alpha\beta(1-\alpha)(1-\beta)\right)^{1/24}.$$

Then

$$4\alpha(1-\alpha)=(PQ^{-1})^{12},\qquad 4\beta(1-\beta)=(PQ)^{12}$$

and so

$$(1-2\alpha)^2=1-(PQ^{-1})^{12},\qquad (1-2\beta)^2=1-(PQ)^{12}.$$

Taking square roots we can write

$$\frac{\beta}{\alpha}+\frac{1-\beta}{1-\alpha}=\frac{1-\sqrt{1-(PQ)^{12}}}{1-\sqrt{1-(PQ^{-1})^{12}}}+\frac{1+\sqrt{1-(PQ)^{12}}}{1+\sqrt{1-(PQ^{-1})^{12}}},$$

provided $0<\beta<1/2$, $0<\alpha<1/2$. If, however $\alpha>1/2$ or $\beta>1/2$ or both, then the sign of the square root has to be changed. In any case

$$P^{12}=4\frac{\left((v^3-3v)^2-\theta^2\right)}{\theta^2(4-\theta^2)}. \tag{32}$$

Using (31) we obtain

$$\begin{aligned}(4-\theta^2)(4-\bar\theta^2) &= -4(v^2-1)^2(v^2-4),\\ \left((v^3-3v)^2-\theta^2\right)(4-\bar\theta^2) &= -\bar\theta^2(v^2-1)^2(v^2-4)\end{aligned}$$

and hence

$$P^{12}=\bar\theta^2/\theta^2=\bar\theta^4/\theta^2\bar\theta^2.$$

Since P, $\bar\theta$ and v are all positive

$$P^3=\bar\theta/2\sqrt{2}\,v \tag{33}$$

and so

$$P^{-3}=\frac{2\sqrt{2}v}{\bar\theta}=\frac{2\sqrt{2}\,v\,\theta}{\theta\bar\theta}=\frac{\theta}{2\sqrt{2}\,v},$$

$$P^3+P^{-3}=\frac{\theta+\bar\theta}{2\sqrt{2}v}=\frac{v^2+5}{2\sqrt{2}}.$$

Restoring the value of v, we have

$$Q^4+Q^{-4}+7-2\sqrt{2}(P^3+P^{-3})=0,$$

which is precisely the Schläfli modular equation stated by Ramanujan on page 90 of the Notebook I.

In (XIX, 19 (ix)) one has Q instead of Q^4 and P instead of P^3.

This method of Watson can be applied to all n given on page 291 of Notebook I.

5. The Jacobi–Sohnke modular equations are equations that involve α and β but not $1-\alpha$, $1-\beta$ explicitly. Ramanujan gives these in Notebook II only for $n=3, 5, 7$. We have stated them in (12). They can be proved very simply using the results of Ramanujan already stated in Chapter XIX.

Let us take, for example, $n=3$. The equations (17) and (18) give

$$\frac{3}{m}=1+2\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8}=1+2\left(\left(\frac{\alpha^3}{\beta}\right)^{1/8}-1\right)=2\left(\frac{\alpha^3}{\beta}\right)^{1/8}-1.$$

Furthermore by (18′)

$$m=1+2\left(\frac{\beta^3}{\alpha}\right)^{1/8}.$$

Multiplying these two equations together, we have

$$3=\left(2\left(\frac{\alpha^3}{\beta}\right)^{1/8}-1\right)\left(1+2\left(\frac{\beta^3}{\alpha}\right)^{1/8}\right).$$

Multiplying out and dividing throughout by $(\alpha\beta)^{1/8}$, we get

$$\left(\frac{\alpha}{\beta}\right)^{1/4}-\left(\frac{\beta}{\alpha}\right)^{1/4}=2\left(\frac{1}{(\alpha\beta)^{1/8}}-(\alpha\beta)^{1/8}\right).$$

With the definition of P and Q in (12), this reduces to

$$Q-Q^{-1}=2(P-P^{-1})$$

which is Ramanujan's equation. If we put

$$u = \alpha^{1/8}, \quad v = \beta^{1/8}$$

we obtain at once

$$\frac{u^2}{v^2} - \frac{v^2}{u^2} = 2\left(\frac{1}{uv} - uv\right)$$

(see [3], p. 501, equation (23)).

The case $n = 5$ is similar. Let us take $n = 7$. We have from (XIX, 19(i))

$$((1-\alpha)(1-\beta))^{1/8} = 1 - (\alpha\beta)^{1/8}$$

and

$$\left(\frac{1 + (\alpha\beta)^{1/2} + ((1-\alpha)(1-\beta))^{1/2}}{2}\right)^{1/2} = 1 - (\alpha\beta(1-\alpha)(1-\beta))^{1/8}$$
$$= 1 - (\alpha\beta)^{1/8}(1 - (\alpha\beta)^{1/8}) = \lambda \text{ say.}$$

From (XIX, 19(iii)) and above

$$\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^7}{\alpha}\right)^{1/8} = m\lambda.$$

Using (XIX, 19(iv)) we have

$$(\alpha\beta)^{1/8}\left(\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^7}{\alpha}\right)^{1/8}\right) = \left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} - 1$$
$$= \left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^7}{\alpha}\right)^{1/8} + \left(\frac{\beta^7}{\alpha}\right)^{1/8} - 1.$$

This gives

$$1 - \left(\frac{\beta^7}{\alpha}\right)^{1/8} = m\lambda(1 - (\alpha\beta)^{1/8}).$$

In a similar way

$$\left(\frac{\alpha^7}{\beta}\right)^{1/8} - 1 = \frac{7\lambda}{m}(1 - (\alpha\beta)^{1/8}).$$

Multiplying the above two equations, we have

$$P^{3/2}(P^{-3/4} - Q)(Q^{-1} - P^{-3/4}) = 7(1 - P^{1/4})^2(1 - P^{1/4} + P^{1/2})^2$$

with P and Q as in (12). It is not hard to see that this is the same as (12).

6. Regarding modular equations of composite degrees, there are two types, one concerns equations with β of degree n relative to α. We obtain equations involving $\alpha\beta$ and $(1-\alpha)(1-\beta)$ as in the case of irrational modular equations of prime degree. Russell seems to be the person with most equations of this type. He has equations for $n = 9$, 15, 35, 95 and 119. Weber ([12], p. 352) gives an equation for $n = 15$ which is substantially the same as Russell's ([8], p. 388). Ramanujan also gives the equation for $n = 15$ (XX, 21 (i)) which is equivalent to Russell's.

We now consider the mixed modular equations. Let n be an odd positive integer

(34) $$n = n' \cdot n''$$

where n' and n'' are two distinct primes. Weber and Ramanujan seem to have independently realised that one should consider simultaneously the four modular functions given in (7). Weber considers two cases for such n; the first of which is

(35) $$(n'+1)(n''+1) = 8\mu \equiv 0 \pmod 8.$$

In this case he defines two functions A and B in terms of P, Q, R given as follows

(36)
$$P = 1 + (-1)^\mu((\alpha\beta\gamma\delta)^{1/8} + ((1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/8}),$$
$$Q = 4((\alpha\beta\gamma\delta)^{1/8} + ((1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/8}$$
$$+ (-1)^\mu(\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/8}),$$
$$R = (256\,\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/24}.$$

Then

(37) $$A = P/R, \quad B = Q/R^2$$

so that A and B are functions of α, β, γ, δ. Note that the P, Q, R defined in (36) resemble the P, Q, R defined in (9) for modular equations of prime degree by Russell.

Among the numbers n satisfying (35) Weber takes 5 examples ([12], p. 358). We give them below.

(38) $$n = 15 = 3\cdot 5, \quad A = 1.$$

Since $\mu = 3$, we have from the definition of A in (37)

(39) $$(\alpha\beta\gamma\delta)^{1/8} + ((1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/8}$$
$$+ 2^{1/3}(\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/24} = 1.$$

It is interesting to see that this equation of Weber is given by Ramanujan on page 307 of Notebook I. It is also given in XX, 11 (xiv) on page 247 of Notebook II.

Compare (39) with the irrational modular equation of degree $n = 23$ given by Ramanujan Chapter XX, p. 248.

Next we have from Weber ([12], p. 358)

$$n = 21 = 3\cdot 7, \quad (A^2-B)^2 = A$$

which, since $\mu = 4$, is

(41) $$(P^2-Q)^2-PR^3 = 0$$

where P, Q, R are given in (36). If we substitute for P, Q, R, we have

(42) $$\{1+(\alpha\beta\gamma\delta)^{1/4}+((1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/4}-2((\alpha\beta\gamma\delta)^{1/8}$$
$$+((1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/8}+(\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/8})\}^2$$
$$=(1+(\alpha\beta\gamma\delta)^{1/8}+((1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/8})$$
$$\times(256\,\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/8}.$$

Compare (42) with Russell's and Weber's and also with Ramanujan's equation for $n = 31$ given in (11).

This equation of Weber's is not given by Ramanujan.

Next Weber gives two equations, the first of which is

(43) $$n = 33 = 3\cdot 11, \quad A^2-B-A = 4,$$

which, since $\mu = 6$, is in terms of P, Q, R

(43′) $$P^2-Q-PR-4R^2 = 0.$$

The second equation of Weber's is

(44) $$n = 35 = 5\cdot 7, \quad A^2-B-A = 2$$

and with $\mu = 6$ is, in terms of P, Q, R, the same as (43′) except for the last term:

(44′) $$P^2-Q-PR-2R^2 = 0.$$

These two equations can be written in terms of $\alpha, \beta, \gamma, \delta$ since P, Q, R as given in (36) are functions of $\alpha, \beta, \gamma, \delta$. They are mixed modular equations.

It looks as though Ramanujan knew the two equations (43′) and (44′). In Notebook I, on the middle of page 309, Ramanujan writes "N.B. For 1, 5, 7, 35 same and for 1, 3, 11, 33, $4R^{2/3}$ instead of $2R^{2/3}$". The allusion is to the difference in the form of the last term in the equations (43′) and (44′). Also as in the case of equations of prime degree he takes $R^{1/3}$ instead of R. He further seems to have recognized the similarity in the forms of the modular equations of prime degree and mixed modular equations.

Lastly Weber gives a result for $n = 55$ ([12], p. 358).

$$n = 55 = 5\cdot 11, \quad A^3-B-4A^2-A+4 = 0$$

which, in terms of P, Q, R, is

(45) $$P^3-QR-4P^2R-PR^2+4R^3 = 0.$$

It is interesting to observe that on page 309 of Notebook I, Ramanujan writes

$$\text{“1, 5, 11, 55.} \quad P^3-R^{1/3}(4P^2+Q)-PR^{2/3}+4R = 0\text{”}$$

which is the same as (45) except that, as mentioned earlier, Ramanujan writes $R^{1/3}$ where Weber writes R. Ramanujan says nothing about P, Q, R but it is obvious that they are the same as (36) with $R^{1/3}$ instead of R.

It is thus obvious that Ramanujan had independently discovered the Weber results given above. How did he do this?

Ramanujan has one more mixed modular equation of type (35) given on page 309 of Notebook I. It is not given in Notebook II nor is it given by Weber. It is, with the previous notation,

(46) $$1, 3, 17, 51. \quad P^3-R^{1/3}(7P^2+Q)+13R^{2/3}P-12R = 0.$$

In the notation of Weber, it would be

(47) $$A^3-7A^2-B-13A = 12.$$

7. In the second case, Weber takes ([12], p. 358–359)

(48) $$n = n'\cdot n'', \quad (n'-1)(n''-1) = 8\mu \equiv 0 \pmod 8.$$

Here again Weber gives two functions A and B with

(49)
$$A = \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24}$$
$$+(-1)^\mu\left\{\left(\frac{\beta\gamma(1-\alpha)^2(1-\delta)^2}{\alpha\delta(1-\beta)^2(1-\gamma)^2}\right)^{1/24}+\left(\frac{\alpha^2\delta^2(1-\beta)(1-\gamma)}{\beta^2\gamma^2(1-\alpha)(1-\delta)}\right)^{1/24}\right\},$$
$$B = \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/24}$$
$$+(-1)^\mu\left\{\left(\frac{\alpha\delta(1-\beta)^2(1-\gamma)^2}{\beta\gamma(1-\alpha)^2(1-\delta)^2}\right)^{1/24}+\left(\frac{\beta^2\gamma^2(1-\alpha)(1-\delta)}{\alpha^2\delta^2(1-\beta)(1-\gamma)}\right)^{1/24}\right\}.$$

We can rewrite the two equations in (49) in the following compact and more suggestive forms:

(50)
$$A = \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/12}$$
$$\times\left\{(-1)^\mu\left(\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8}+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8}\right)+\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}\right\},$$
$$B = \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12}$$
$$\times\left\{(-1)^\mu\left(\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8}\right)+\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}\right\}.$$

Weber gives three modular equations of mixed type satisfying (48). His first mixed modular equation is ([12], p. 359)

(51) $$n = 15 = 3\cdot 5, \quad AB+1 = 0.$$

Here $\mu = 1$ and so from (50) and (51)

(52) $$\left\{\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}\right\}$$
$$\times\left\{\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}\right\}+1 = 0$$

which is an equation due to Weber.

In Chapter XX, Entry 11 (viii), Entry 11 (ix), Ramanujan gives two equations whose product is precisely (52). It is very interesting to observe Ramanujan giving an expression for each term of the product in terms of multipliers.

The integer $n = 15$ satisfies both conditions (35) and (48) and thus there are 2 mixed modular equations associated with it. They are given by Ramanujan in Notebook I on top of page 307 which shows, perhaps, that Ramanujan had understood what lies behind these mixed modular equations. It looks like he had, independently, ideas similar to Weber's.

The next equation of Weber's is

(53) $$n = 35 = 5\cdot 7, \quad 2(A+B)-AB = 5.$$

This equation can be written in the form

$$(A-2)(B-2)+1 = 0.$$

Substituting for A and B from (50) and noticing that $\mu = 3$, we have

(54) $$\left\{\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} + 2\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/12}\right\}$$
$$\times\left\{\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} + 2\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12}\right\}+1 = 0.$$

If we take the statements XX, 18 (vi) and XX, 18 (vii) and multiply them, we get precisely (54).

If we take $n = 39$, we have Weber's result

(55) $$n = 39 = 3\cdot 13, \quad 2(A+B)-AB = 3$$

which is the same as

$$(A-2)(B-2)-1 = 0.$$

We then have the same product as (54) except that outside the product we have the additive term -1 instead of $+1$.

This is mentioned by Ramanujan both in Notebook I, page 307 and in Notebook II, Chapter XX, Entry 19 (iv).

Ramanujan has many more beautiful formulae for $n = 21, 33, \ldots$ We will defer a detailed consideration of these, in the light of Weber's method, to a succeeding paper.

Weber has a mixed modular equation for $n = 105 = 1\cdot 3\cdot 5\cdot 7$ which is complicated. This will be put in a general set up in our next paper.

8. Ramanujan had also discovered Schläfli-like modular equations associated with mixed modular equations. Indeed Ramanujan gives such equations on pages 86 and 88 of Notebook I and at one or two places in Chapter XX of Notebook II. We shall content ourselves with proving two such Schläfli-like equations of degrees 15 and 35.

Let

$$P = (256\,\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/48}, \qquad Q = \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/48};$$

then, the two equations are

(56) $$Q^4+Q^{-4}-(Q^2+Q^{-2})-2(P^2+P^{-2}) = 0. \quad (1, 5, 7, 35),$$

(57) $$Q^3+Q^{-3}-\sqrt{2}(P+P^{-1}) = 0. \quad (1, 3, 5, 15).$$

These are given on page 86 of Notebook I.

Let us take (57) first. In order to prove this observe that by XX, 11 (iv) and XX, 11 (v)

(58) $$\left(2^{2/3}\left(\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24}-1\right)\left(1-2^{2/3}\left(\frac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/24}\right)$$
$$= ((\beta\gamma)^{1/8}+((1-\beta)(1-\gamma))^{1/8})((\alpha\delta)^{1/8}+((1-\alpha)(1-\delta))^{1/8}).$$

By XX, 11 (i) and XX, 11 (ii) the right side of (58) equals 1. If we use the values of P and Q above, we have

$$PQ = 2^{1/6}(\beta\gamma(1-\beta)(1-\gamma))^{1/24}, \quad PQ^{-1} = 2^{1/6}(\alpha\delta(1-\alpha)(1-\delta))^{1/24}.$$

Inserting these values in (58), we have

(59) $$(\sqrt{2}\,PQ^3-1)(1-\sqrt{2}\,PQ^{-3}) = 1$$

which is precisely (57).

In order to prove (56) consider the equation XX, 18 (v) in Ramanujan's Notebook II, namely

$$(60)\quad \frac{(16\beta\gamma(1-\beta)(1-\gamma))^{1/24}-(16\alpha\delta(1-\alpha)(1-\delta))^{1/8}}{(16\beta\gamma(1-\beta)(1-\gamma))^{1/24}+(16\beta\gamma(1-\beta)(1-\gamma))^{1/8}} = \frac{(16\alpha\delta(1-\alpha)(1-\delta))^{1/8}+(16\alpha\delta(1-\alpha)(1-\delta))^{1/24}}{(16\beta\gamma(1-\beta)(1-\gamma))^{1/8}-(16\alpha\delta(1-\alpha)(1-\delta))^{1/24}}.$$

This formula of Ramanujan's was not proved by Watson [10] but Bruce Berndt gives a proof (XX, p. 149–152). Using the above values of P and Q and inserting them in (60), we get

$$\frac{PQ-P^3Q^{-3}}{PQ+P^3Q^3} = \frac{P^3Q^{-3}+PQ^{-1}}{P^3Q^3-PQ^{-1}}.$$

Clearing denominators we find that this is precisely (56).

On page 90 of Notebook I, Ramanujan says "These are true even for even functions though the signs are changed in many cases". This concerns the Schläfli modular equations which have analogues for $f_1(\tau)$ instead of $f(\tau)$. Ramanujan calls $f_1(\tau)$ an 'even function' since in problems of singular moduli $f_1(\sqrt{-r})$ is generally evaluated for $r > 0$ an even integer.

Similar results can be obtained for $f_1(\tau), f_1(r\tau), \ldots$ for (56) and (57).

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