Note for subscribers: Volume LIII will have 5 fascicles; the last one will appear towards the end of 1990.

La revue est consacrée à la Théorie des Nombres
The journal publishes papers on the Theory of Numbers
Die Zeitschrift veröffentlicht Arbeiten aus der Zahlentheorie
Журнал посвящен теории чисел

L'adresse de la Rédaction et de l'échange	Address of the Editorial Board and of the exchange	Die Adresse der Schriftleitung und des Austausches	Адрес редакции и книгообмена

ACTA ARITHMETICA
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ISBN 83-01-09574-1 ISSN 0065-1036

PRINTED IN POLAND

W R O C Ł A W S K A D R U K A R N I A N A U K O W A

ACTA ARITHMETICA
LIII (1990)

On the frequency of Titchmarsh's phenomenon for $\zeta(s)$, VI

by

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To the memory of V. G. Sprindžuk

1. Introduction. This is the continuation of the paper V [2] with the same title. In paper V the following theorem was proved.

THEOREM 1. *Let α be a fixed constant satisfying $1/2 < \alpha < 1$ and $E > 1$ an arbitrary constant. Let $C \leqslant H \leqslant T/100$ and $K = \operatorname{Exp}\left(\frac{D\log H}{\log\log H}\right)$, where C is a large positive constant and D an arbitrary positive constant. Then there are $\geqslant TK^{-E}$ disjoint integrals I of length K each contained in $[T, 2T]$ such that*

$$\frac{(\log K)^{1-\alpha}}{(\log\log K)^{\alpha}} \ll \max_{t\text{ in }I} |\log\zeta(\alpha+it)| \ll \frac{(\log K)^{1-\alpha}}{(\log\log K)^{\alpha}}.$$

Remarks. Here $\log\zeta(s)$ $(s = \sigma+it)$ is the analytic continuation along lines parallel to the σ-axis (we choose only those lines which do not contain a zero or a pole of $\zeta(s)$) of $\log\zeta(s)$ in $\sigma \geqslant 2$. We had also remarked about extensions of this result.

In this paper we concentrate on $|\log\zeta(1+it)|$ and $|\zeta(1+it)|$ and prove some upper and lower bounds for the maximum of these functions as t varies over some t-intervals J described below. Our remarks made in paper V about extensions of Theorem 1 hold good with little or no modifications though we do not state them explicitly here.

2. Upper bounds. We begin with

THEOREM 2. *Let I be the interval for t referred to in Theorem 1. Let J be the t-interval obtained by removing intervals of length $(\log H)^2$ from both the ends of I. Then*

$$\max_{t\text{ in }J}|\log\zeta(1+it)| \leqslant \log\log\log K + \gamma + \frac{\log\log\log K}{\log\log K} + O\left(\frac{1}{\log\log K}\right)$$

where γ is the Euler's constant.

Proof. Let $s_0 = 1+it$ where t is in J. Define $\Lambda_1(n)$ by

$$\log\zeta(s) = \sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n^s} \quad \text{where } \operatorname{Re} s > 1.$$

The function $\log\zeta(s)$ can be continued analytically in $\sigma \geqslant 1/2+\delta$ for t in I (where $\delta > 0$ is an arbitrary constant) and in $\sigma \geqslant 1/2+2\delta$ it is $O(\log H)$ (this is proved in [2]). We start with

$$\sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n^{s_0}} e^{-n/x} = \frac{1}{2\pi i} \int_{\operatorname{Re} w = 1} \log\zeta(s_0+w)\,\Gamma(w)\, x^w\, dw$$

where $x = \log K \log\log K$. We deform the contour as follows:

$$L_1 = \{\operatorname{Re} w = 1,\ |\operatorname{Im} w| \geqslant (\log H)^2\},$$
$$L_2 = \{|\operatorname{Im} w| = (\log H)^2,\ 1 \geqslant \operatorname{Re} w \geqslant \alpha-1\},$$
$$L_3 = \{\operatorname{Re} w = \alpha-1,\ |\operatorname{Im} w| \leqslant (\log H)^2\}.$$

Here it is assumed that $\alpha \geqslant 1/2+2\delta$. We come across the residue at $w = 0$ from which the contribution is $\log\zeta(s_0)$. From the asymptotics of the gamma function and from the fact that on L_3, $\log\zeta(s_0+w) = O((\log K)^{1-\alpha}(\log\log K)^{-\alpha})$ we obtain

$$|\log\zeta(s_0)| \leqslant \sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n} e^{-n/x} + O\left(\frac{1}{\log\log K}\right).$$

We now observe

(1) $$\sum_{n \geqslant x} \frac{\Lambda_1(n)}{n} e^{-n/x} \leqslant x \sum_{n \geqslant x} \frac{\Lambda_1(n)}{n^2} = O\left(\frac{1}{\log\log K}\right),$$

(2) $$\sum_{n \leqslant x} \frac{\Lambda_1(n)}{n}(e^{-n/x}-1) = O\left(\frac{1}{x}\sum_{n \leqslant x} \Lambda_1(n)\right) = O\left(\frac{1}{\log\log K}\right)$$

since for $n \leqslant x$, $1-e^{-n/x} = O(n/x)$.

(3) $$\sum_{n \leqslant x} \frac{\Lambda_1(n)}{n} = \log\log x + \gamma + O\left(\frac{1}{\log x}\right)$$

by the result on page 58 of [6] (it is not hard to improve the error term $o(1)$ given there). See also Theorem 5.3, Chapter III of the book *Primzahlverteilung* by K. Prachar.

Now

$$\log\log x = \log\log(\log K \log\log K)$$
$$= \log(\log\log K + \log\log\log K)$$
$$= \log\log\log K + \frac{\log\log\log K}{\log\log K} + O\left(\frac{1}{\log\log K}\right)$$

and this completes the proof of Theorem 2.

The following theorem is a simple corollary to Theorem 2.

THEOREM 3. *Let J be as in Theorem 2. Then*

$$\max_{t \text{ in } J} |\zeta(1+it)| < e^{\gamma}(\log\log K + \log\log\log K + O(1)).$$

Proof. Follows by $\log|\zeta(1+it)| \leqslant |\log\zeta(1+it)|$, and taking exponentials. This completes the proof of Theorem 3.

In Section 2 of paper V the following result was proved. Let β be an arbitrary constant satisfying $1/2 < \beta < 1$. Let $\theta = \delta/2$, where δ is a positive constant depending only on β. Put $H_1 = H^{\theta}$. Then there are $\gg T/H_1$ intervals each of length $H_1 + 20(\log H)^2$ which are disjoint and all contained in $[T, 2T]$ such that if we denote a typical interval I_1 by $[T_0 - 10(\log H)^2, T_0 + H_1 + 10(\log H)^2]$, then in $\{\sigma \geqslant \beta,\ t \text{ in } I_1\}$ $\log\zeta(s)$ is analytic and further $\log\zeta(s) = O(\log H)$. Hence for t in $[T_0, T_0+H_1]$, we have, arguing as in the proof of Theorem 2 and choosing β close to $1/2$, the following theorem.

THEOREM 4. *For t belonging to $[T_0, T_0+H_1]$, we have*

$$|\log\zeta(1+it)| \leqslant \log\log\log H + \gamma + \log 2 + o(1).$$

COROLLARY. *We have, for t as in the Theorem,*

$$|\zeta(1+it)| \leqslant 2e^{\gamma}\log\log H + o(\log\log H).$$

3. Lower bounds for $|\zeta(1+it)|$ for some values of t. It follows from our method given above, that for t in J, we have

$$\max_{\sigma \geqslant 1,\, t \text{ in } J} |\zeta(\sigma+it)| = O(\log\log K).$$

The length of the interval J is $K - 2(\log H)^2 = M$, say. The following theorem is a simple corollary to a theorem essentially due to K. Ramachandra [4] (see also [3]).

THEOREM 5. *Let C_0 be a large constant and $C_0 \leqslant M \leqslant T$. Let k be a positive integer not exceeding $\log M$ and let J be an interval (contained in $[T, 2T]$) of length M, and*

$$\max_{\sigma \geqslant 1,\, t \text{ in } J} |\zeta(\sigma+it)|^{2k} \leqslant \operatorname{Exp}\operatorname{Exp}\{M/80A\},$$

where A is a large constant. Then

$$\frac{1}{M}\int_{t \text{ in } J} |\zeta(1+it)|^{2k}\, dt \geqslant \frac{C_1}{\log\log M} \sum_{n \leqslant M/200} \frac{(d_k(n))^2}{n^2},$$

where C_1 is a positive constant depending only on A.

Remark. We wish to apply this theorem to the special case. The upper bound on $\max_{\sigma \geqslant 1,\, t \text{ in } J} (\ldots)$ is certainly satisfied.

From this theorem we try to get a lower bound for $\max_{t \text{ in } J} |\zeta(1+it)|$. For this purpose we observe the following facts.

(1) $d_k(n)/n$ is multiplicative and

$$\frac{d_k(p^m)}{p^m} = \frac{k(k+1)\dots(k+m-1)}{m!\,p^m},$$

$$\frac{d_k(p^{m+1})}{p^{m+1}} = \left(\frac{d_k(p^m)}{p^m}\right)\left(\frac{k+m}{(m+1)p}\right) < \left(\frac{3}{4}\right)\left(\frac{d_k(p^m)}{p^m}\right),$$

provided $4k+4m < 3mp+3p$, i.e. $m > (4k-3p)/(3p-4) = m_p$ say.

(2) m_p may not be an integer, but

$$m_p + 1 = \frac{4k-4}{3p-4} \leqslant \frac{4k}{p}.$$

(3) $$\left(1-\frac{1}{p}\right)^{-k} = \sum_{m=0}^{\infty} \frac{d_k(p^m)}{p^m} < (m_p+1) \max_{m \leqslant m_p} \frac{d_k(p^m)}{p^m} + 3 \max_{m \leqslant m_p} \frac{d_k(p^m)}{p^m},$$

since $\frac{3}{4}+(\frac{3}{4})^2+(\frac{3}{4})^3+\dots = 3$.

(4) Thus if m denotes the integer (to avoid complicated notations) not exceeding m_p for which the maximum of $d_k(p^m)/p^m$ is attained, we have

$$\left(\frac{d_k(p^m)}{p^m}\right)^2 \geqslant \left(\frac{1}{m_p+4}\right)^2 \left(1-\frac{1}{p}\right)^{-2k}.$$

(5) We choose k (as large as possible) to satisfy

$$\prod_{p \leqslant k} p^{m_p} \leqslant \frac{M}{200}, \quad \text{i.e. } \log\frac{M}{200} \geqslant \sum_{p \leqslant k} \frac{4k\log p}{p} \sim 4k\log k, \quad \text{i.e. } k \sim \frac{1}{4}\frac{\log M}{\log\log M}.$$

(6) Hence

$$\max_{t \text{ in } J} |\zeta(1+it)| \geqslant \left(\frac{C_1}{\log\log M}\right)^{1/(2k)} \left(\prod_{p \leqslant k} \frac{1}{m_p+4}\right)^{1/k} \prod_{p \leqslant k}\left(1-\frac{1}{p}\right)^{-1}$$
$$= \left(1+O\left(\frac{\log\log M \log\log\log M}{\log M}\right)\right)\left(1+O\left(\frac{1}{\log\log M}\right)\right) Y$$

where $Y = e^{\gamma}(\log\log M - \log\log\log M + O(1))$. Since $M \sim K$ we obtain the following theorem.

THEOREM 6. *We have*

$$\max_{t \text{ in } J} |\zeta(1+it)| \geqslant e^{\gamma}(\log\log K - \log\log\log K + O(1)).$$

We record a corollary to Theorems 3 and 6:

THEOREM 7. *We have*

$$e^{\gamma}(\log\log K - \log\log\log K + O(1)) \leqslant \max_{t \text{ in } J} |\zeta(1+it)|$$
$$\leqslant e^{\gamma}(\log\log K + \log\log\log K + O(1)).$$

Remark. In proving Theorem 6, we have followed the method of proof of Theorem 5 of [5]. It should be mentioned that the method of [1] also works (see the last result in Remark 3 on p. 342 and its proof sketched in § 3).

4. Two theorems. We state two theorems and indicate their proof. Our notation in this section will be independent of the previous sections.

THEOREM 8. (a) *Let I be any interval for t contained in $[T, 2T]$. Then*

$$\max_{\sigma \geqslant 1, t \text{ in } I} |\zeta(\sigma+it)| \geqslant e^{\gamma}(\log\log K - \log\log\log K + O(1))$$

where $K = |I| + 10000$.

(b) *Let I be any interval for t contained in $[T, 2T]$ and K (as defined just now) exceed $C\log\log\log T$ where C is a large constant. Then provided $T \geqslant C$,*

$$\max_{t \text{ in } I} |\zeta(1+it)| \geqslant e^{\gamma}(\log\log K - \log\log\log K + O(1)).$$

Proof. To prove (a) we observe that we may assume that

$$\max_{\sigma \geqslant 1, t \text{ in } I} |\zeta(\sigma+it)| \leqslant 10\log\log K$$

(for otherwise there is nothing to prove). K is essentially (M in the notation of Theorem 5) the length of I and we choose (as in the proof of Theorem 6) k to be $\sim \frac{\log K}{4\log\log K}$. The condition for the applicability of Theorem 5 is satisfied and we obtain

$$\max_{t \text{ in } I} |\zeta(1+it)| \geqslant \left(\frac{C_1}{\log\log K} \sum_{n \leqslant K/200} \frac{(d_k(n))^2}{n^2}\right)^{1/(2k)}.$$

Thus we obtain part (a) as in the proof of Theorem 6.

To prove (b) we recall the well-known result that

$$\max_{\sigma \geqslant 1, t \text{ in } [T, 2T]} |\zeta(\sigma+it)| = O(\log T).$$

So the conditions of Theorem 5 are satisfied provided the length of I (M of Theorem 5) namely K exceeds $C\log\log\log T$. The part (b) follows as before.

THEOREM 9. *Let C be any large positive constant and $C \leqslant H \leqslant C\log\log\log T$, $T \geqslant C$. Let the interval $[T, 2T]$ be divided into disjoint intervals I each of length H (ignoring an interval of length $< H$ at one end). Put $X = \operatorname{Exp}\operatorname{Exp}\operatorname{Exp}(\alpha H)$, where α is a small positive absolute constant. Then* *with the possible exception of at most* $O(TX^{-1/2})$ *intervals* I, *we have*

$$\max_{t \text{ in } I} |\zeta(1+it)| \geqslant e^{\gamma}(\log\log H - \log\log\log H + O(1)).$$

Proof. Let $\delta = (\log X)^{-1}$, $s = \sigma + it$ where $T \leqslant t \leqslant 2T$ and $2 \geqslant \sigma \geqslant 1-\delta$. We start with

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\, w = 2} \zeta(s+w) X^{w} \operatorname{Exp}(w^2)\frac{dw}{w} = \sum_{n=1}^{\infty}\frac{1}{n^s}\Delta\left(\frac{X}{n}\right)$$

where for $u > 0$,

$$\Delta(u) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, w = 2} u^{w} \operatorname{Exp}(w^2)\frac{dw}{w}.$$

We note that $\Delta(u) = O(u^5)$ for $0 < u \leqslant 1$ and that $\Delta(u) = 1 + O(u^{-5})$ for $u \geqslant 1$. By moving the line of integration to $\mathrm{Re}\, w = \frac{3}{4} - \sigma$ and integrating we obtain the following lemma.

Lemma 1. *Let*

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}\Delta\left(\frac{X}{n}\right) + E(s, X).$$

Then

$$\iint_{\substack{1-\delta \leqslant \sigma \leqslant 2\\ T \leqslant t \leqslant 2T}} |E(s, X)|^2\, d\sigma dt = O(TX^{-1/2}).$$

We next record another lemma.

Lemma 2. *For* $1-\delta \leqslant \sigma \leqslant 2$ *and* $T \leqslant t \leqslant 2T$, *we have uniformly*

$$\sum_{n=1}^{\infty}\frac{1}{n^s}\Delta\left(\frac{X}{n}\right) = \sum_{n \leqslant X}\frac{1}{n^s} + O(1) = O(\log X).$$

Proof. Follows from the properties of $\Delta(u)$ stated above.

Theorem 9 follows from Theorem 5 on using Lemmas 1 and 2.

5. Concluding remarks. 1. Nearly all the bounds (both upper and lower) have their analogues for $|\zeta(1+it)|^{-1}$. The constant e^{γ} will have to be replaced by $(6/\pi^2)e^{\gamma}$ for this purpose.

2. Theorem 5 can be used to prove that both $|\zeta(s)|$ and $|\zeta(s)|^{-1}$ are unbounded in $\sigma > 1$, e.g. on $\sigma = 1 + \dfrac{1}{\log t}$, $t \geqslant 10$. Precise theorems similar to those of this paper can also be proved.

6. An announcement. Recently K. Ramachandra has proved the following two results.

Theorem 10. (a) *In Theorem* 8(b) *let* $\log\log\log T$ *be replaced by* $\log\log\log\log T$. *Then the conclusion of Theorem* 8(b) *still holds where* $K = |I| + 10\,000$.

(b) *In Theorem* 9 *let* $\log\log\log T$ *be replaced by* $\log\log\log\log T$ *and* $X = \operatorname{Exp}\operatorname{Exp}\operatorname{Exp}\operatorname{Exp}(\alpha H)$. *Then the conclusion of Theorem* 9 *still holds.*

Theorem 11. (a) *In Theorem* 8(b) *let* $\log\log\log T$ *be replaced by* $\log\log\log\log\log T$. *Then the conclusion of Theorem* 8(b) *still holds where* $K = |I| + 10\,000$ *provided we assume that the least upper bound of the real parts of the zeros of* $\zeta(s)$ *is less than* 1.

(b) *In Theorem* 9 *let* $\log\log\log T$ *be replaced by* $\log\log\log\log\log T$ *and* $X = \operatorname{Exp}\operatorname{Exp}\operatorname{Exp}\operatorname{Exp}\operatorname{Exp}(\alpha H)$. *Then the conclusion of Theorem* 9 *still holds.*

These will be published as the next paper with the same title.*

Acknowledgement. The authors are thankful to the referee for pointing out an oversight.

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Received on 5.1.1988
and in revised form on 15.11.1988 (1776)

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