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Irrationality results for theta functions by Gel'fond–Schneider's method

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0. Introduction. Let τ be a fixed complex number with positive imaginary part, and let q denote the number $e^{i\pi\tau}$ of absolute value less than one. Then the series

$$\sum_{n\in\mathbf{Z}} q^{n^2} e^{2i\pi nu}$$

defines an entire function of u, denoted by $\theta(u)$ or $\theta(u,q)$, satisfying the functional equation

$$\theta(u+\lambda+\mu\tau) = \theta(u)\exp(-2i\pi\mu u - i\pi\tau\mu^2)$$

for all λ, $\mu\in\mathbf{Z}$; in particular

$$\theta(u+1) = \theta(u) \quad \text{and} \quad \theta(u+\tau) = q^{-1}e^{-2i\pi u}\theta(u). \tag{0.1}$$

Therefore θ is a special theta function with respect to the lattice $\mathbf{Z}+\tau\mathbf{Z}$. This theta function (as well as three closely related ones) was introduced by Jacobi in 1829 in his famous Fundamenta Nova Theoriae Functionum Ellipticarum [8].

Seemingly the first non-trivial investigation of arithmetic properties of theta functions goes back to Bernstein and Szász [1]. Using a criterion of Eisenstein concerning irregular continued fractions, they showed in 1915: for non-zero rational numbers v and $q = r/s$ with r, $s\in\mathbf{Z}$ and $|s|\geqslant\max(2,|r|^3)$, the right half

$$\sum_{n\geqslant 0} q^{n^2} v^n$$

of the theta series $\theta((\log v)/2i\pi, q)$ is irrational.

Some few years later Tschakaloff [14 I] studied arithmetically the following entire function

$$T(z) = T(z,a) = \sum_{n\geqslant 0} a^{-n(n-1)/2} z^n \tag{0.2}$$

satisfying the functional equation

$$T(az) = 1 + azT(z), \tag{0.3}$$

where $a \in C$ is fixed with $|a| > 1$. Applying essentially the Padé approximation method used by Hermite for his classical transcendence proof of the number e, he could show the irrationality of $T(\xi, a)$ for non-zero rational ξ and $a = s/r$ with $r, s \in Z$ and $|s| > |r|^{(3+\sqrt{5})/2}$. Because of $T(vq, q^{-2}) = \sum_{n \geq 0} q^{n^2} v^n$, this last result implied a slight improvement on that of Bernstein and Szász.

In his paper, Tschakaloff pointed out that his theorem remains true if the rational field is replaced by any imaginary quadratic number field with unique factorization. About 50 years later one of us [4] showed the unique factorization property to be unnecessary. The proof in [4] used the method of Newton interpolation series, and had the further advantage to lead immediately to quantitative refinements of these irrationality type results, which have been axiomatized in [3] and [16] by the same method. Especially in [16] the values of entire transcendental functions satisfying Poincaré's functional equation

$$F(az) = P(z)F(z) + Q(z)$$

are investigated, where P and Q are polynomials with coefficients in some fixed imaginary quadratic number field. It is just this kind of functional equation generalizing (0.3) which plays an important rôle in our present work.

For the sake of completeness we should include here the remark that some of the irrationality theorems indicated until now have been generalized to results on linear independence over Q or over imaginary quadratic number fields; see [14 II], [10], [13], [11], [12], [2].

Three years ago, in the survey paper [5], one of us announced the following theorem concerning the Tschakaloff function T from (0.2): *if $a > 1$ and d are positive integers, then the set of rational numbers ξ with $a^{-1} < |\xi| \leq 1$ such that $T(\xi, a)$ is algebraic of degree not greater than d has less than $16d^2$ elements.*

Whereas a less precise bound was obtained by Senkon [9] using again Newton's interpolation series, we applied Schneider's method from transcendence theory for the first time to the topic under consideration. This idea led us to a proof of the following much more general result, which depends on a generalization of the first main result (Theorem 2.1) of [7]. We denote by h the absolute logarithmic height (see § 1 below).

THEOREM 0.4. *Let K be a number field of degree δ embedded in C; let $a \in K$, with $|a| > 1$ and let P and Q be polynomials in $K[X]$, with $\deg P = \Delta$. Let F be a transcendental entire function satisfying the functional equation*

$$F(az) = P(z)F(z) + Q(z).$$

Then for each positive integer d, the set of algebraic numbers ζ with $|a|^{-1} < |\zeta| \leq 1$ such that $F(\zeta)$ is algebraic with

$$[K(\zeta, F(\zeta)) : K] \leq d$$

is finite with at most

$$(0.5) \qquad 700 \Delta d^2 \delta h(a)/\log|a|$$

elements.

When applying this result to the above function T in the special case $K = Q$, we get the bound $700d^2$ instead of $16d^2$ as quoted earlier. We did not try to get a sharp absolute constant, and the value 700 can be decreased without much effort.

In Section 1 we introduce some notations and state a few preliminary results; in Section 2, we apply Gel'fond–Schneider's method and give a refinement of Theorem 2.1 in [7]. In Section 3, we study the functional equation $F(az) = P(z)F(z) + Q(z)$ and deduce Theorem 0.4 (more generally we include derivatives). In Section 4, we perform the change of variables $z = e^{2i\pi u}$ and give irrationality results on values of theta functions.

1. Preliminaries. When $\beta_1, \ldots, \beta_m$ are algebraic numbers in a number field K, we define the absolute logarithmic height of the $(m+1)$-tuple $(1, \beta_1, \ldots, \beta_m)$ by

$$h(1, \beta_1, \ldots, \beta_m) = \frac{1}{[K:Q]} \sum_v \log \max\{1, |\beta_1|_v, \ldots, |\beta_m|_v\},$$

where v runs over the set of places of K, with the usual normalisation:

$$\prod_v |\alpha|_v = 1 \quad \text{for all } \alpha \in K,\ \alpha \neq 0.$$

For $m = 1$, we write $h(\beta)$ instead of $h(1, \beta)$.

When P is a polynomial in one or several variables with complex coefficients, we denote by $L(P)$ (= length of P) the sum of the absolute values of the coefficients of P.

We shall use the following simple lemma (compare with [7], Lemme 0.1):

LEMMA 1.0. *Let P be a polynomial with coefficients in Z in the km indeterminates X_{ij} $(1 \leq i \leq m, 1 \leq j \leq k)$. Assume that for each j, $1 \leq j \leq k$, P is of degree at most L_j with respect to the m variables $X_{1j}, \ldots, X_{mj}$. Let β_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq k)$ be algebraic numbers; we write $\boldsymbol{\beta}$ for the mk-tuple $(\beta_{ij})_{1 \leq i \leq m, 1 \leq j \leq k}$. Then*

$$h(P(\boldsymbol{\beta})) \leq \log L(P) + \sum_{j=1}^{k} L_j h(1, \beta_{1j}, \ldots, \beta_{mj}).$$

We shall use a version of Siegel's lemma which is both a refinement of Lemme 1.1 in [7] and a special case of Lemma 1 in [6]:

LEMMA 1.1. *Let m, k, T, L be positive integers, $\alpha_{\mu j \tau}$ $(1 \leq \mu \leq m, 1 \leq j \leq k,$ $1 \leq \tau \leq T)$ be algebraic numbers, and $P_{\lambda\mu}$ $(1 \leq \lambda \leq L,\ 1 \leq \mu \leq m)$ be polyno-*

mials, with integer coefficients in the mkT variables $X_{\mu j\tau}$. We write $A_{\lambda\mu}$ for the value of $P_{\lambda\mu}$ at the point $(\alpha_{\mu j\tau})_{1\leqslant\mu\leqslant m,\,1\leqslant j\leqslant k,\,1\leqslant\tau\leqslant T}$, and we define

$$\Lambda_\mu = \sum_{\lambda=1}^{L} L(P_{\lambda\mu}) \quad (1\leqslant\mu\leqslant m).$$

For $1\leqslant\mu\leqslant m$ and $1\leqslant j\leqslant k$, we write $\theta_{\mu j}$ for the $(T+1)$-tuple $(1, \alpha_{\mu j1}, \ldots, \alpha_{\mu jT})$. For $1\leqslant\mu\leqslant m$, let K_μ be a number field containing the kT numbers $\alpha_{\mu j\tau}$ $(1\leqslant j\leqslant k, 1\leqslant\tau\leqslant T)$, and let d_μ be the degree of K_μ over $\mathbf{Q}$. Let m' be the number of those fields K_μ which are totally complex. Define $D \doteq \sum_{\mu=1}^{m} d_\mu$, and assume $L>D$.

Then there exist rational integers $x_1, \ldots, x_L$, not all zero, satisfying

$$\sum_{\lambda=1}^{L} x_\lambda A_{\lambda\mu} = 0 \quad (1\leqslant\mu\leqslant m)$$

and

$$\max_{1\leqslant\lambda\leqslant L} |x_\lambda| \leqslant \left[\left(2^{m'}\cdot\prod_{\mu=1}^{m}\left(\Lambda_\mu^{d_\mu}\prod_{j=1}^{k} e^{L_j d_\mu h(\theta_{\mu j})}\right)\right)^{1/(L-D)}\right].$$

The bracket denotes the integral part. Notice that the right-hand side is always at least 1 (even if all the $A_{\lambda\mu}$ vanish!).

Lemma 1.1 of [7] corresponds to the special case where each $P_{\lambda\mu}$ is a monomial (depending on λ) in k variables.

Proof of Lemma 1.1. We use Lemma 1 of [6]. If G_μ denotes the set of embeddings of K_μ into $\mathbf{C}$, we have for all (μ, σ) with $1\leqslant\mu\leqslant m$ and $\sigma\in K_\mu$:

$$\sum_{\lambda=1}^{L} |\sigma(A_{\lambda\mu})| \leqslant \Lambda_\mu\cdot\prod_{j=1}^{k}\max\{1, |\sigma(\alpha_{\mu j1})|, \ldots, |\sigma(\alpha_{\mu jT})|\}^{L_j},$$

hence

$$\prod_{\sigma\in G_\mu}\sum_{\lambda=1}^{L} |\sigma(A_{\lambda\mu})| \leqslant \Lambda_\mu^{d_\mu}\cdot\exp\left\{\sum_{j=1}^{k} L_j d_\mu h(\theta_{\mu j})\right\}.$$

Lemma 1.1 follows.

We need also to estimate derivatives; for functions of one complex variable z, we write $D = d/dz$, so that $Df = f'$ is the derivative of f.

LEMMA 1.2. *Let $f_1, \ldots, f_k$ be functions of one complex variable, which are analytic in some domain of $\mathbf{C}$. For $t, \lambda_1, \ldots, \lambda_k$ non-negative integers, the function*

$$\frac{1}{t!}D^t(f_1^{\lambda_1}\cdots f_k^{\lambda_k})$$

is a polynomial in the $k(t+1)$ functions $\frac{1}{\tau!}D^\tau f_j$ $(0\leqslant\tau\leqslant t,\ 1\leqslant j\leqslant k)$; for $1\leqslant j\leqslant k$, this polynomial is homogeneous of degree λ_j in the $t+1$ variables $\frac{1}{\tau!}D^\tau f_j$ $(0\leqslant\tau\leqslant t)$, and its length is at most

$$(t+1)^{\lambda_1+\cdots+\lambda_k}.$$

Proof. For t and λ non-negative integers, we define a polynomial $A_{t\lambda}$ in $\mathbf{Z}[X_0, \ldots, X_t]$ by

$$A_{0\lambda}(X_0) = X_0^\lambda \quad \text{for } \lambda\geqslant 0,$$
$$A_{t0}(X_0, \ldots, X_t) = 0 \quad \text{for } t>0,$$

and

$$A_{t\lambda}(X_0, \ldots, X_t) = \sum_{\tau_1+\cdots+\tau_\lambda=t}\prod_{i=1}^{\lambda} X_{\tau_i} \quad \text{for } \lambda>0 \text{ and } t>0.$$

In the summation, $(\tau_1, \ldots, \tau_\lambda)$ runs over the λ-tuples of non-negative integers of sum t. If $\lambda>0$, then this polynomial is homogeneous of degree λ, with non-negative coefficients; therefore its length is

$$L(A_{t\lambda}) = A_{t\lambda}(1, \ldots, 1) = \sum_{\tau_1+\cdots+\tau_\lambda=t} 1 = \binom{t+\lambda-1}{\lambda-1} \leqslant (t+1)^{\lambda-1}.$$

If we define, for $\lambda=0$, $\binom{t+\lambda-1}{\lambda-1}$ to be 0 for $t>0$ and 1 for $t=1$, then

$$L(A_{t\lambda}) = \binom{t+\lambda-1}{\lambda-1} \quad \text{for all } \lambda\geqslant 0 \text{ and } t\geqslant 0.$$

From Leibniz rule of derivatives we have

$$\frac{1}{t!}D^t(f^\lambda) = A_{t\lambda}\left(f, f', \frac{1}{2}f'', \ldots, \frac{1}{t!}D^t f\right).$$

Next, for non-negative integers $t, \lambda_1, \ldots, \lambda_k$, we define a polynomial $B_{t\lambda_1\ldots\lambda_k}$ with non-negative coefficients in $\mathbf{Z}$ in the $k(t+1)$ variables $X_{\tau j}$ $(0\leqslant\tau\leqslant t,\ 1\leqslant j\leqslant k)$ by

$$B_{t\lambda_1\ldots\lambda_k}(X_{01}, \ldots, X_{tk}) = \sum_{\tau_1+\cdots+\tau_k=t}\prod_{j=1}^{k} A_{\tau_j\lambda_j}(X_{0j}, \ldots, X_{tj}).$$

If $t>0$ and $\lambda_1=\cdots=\lambda_k=0$, then this polynomial is the zero polynomial. Otherwise, for $1\leqslant j\leqslant k$, this polynomial is homogeneous in $X_{0j}, \ldots, X_{tj}$ of degree λ_j; the length of this polynomial is

$$L(B_{t\lambda_1\ldots\lambda_k}) = B_{t\lambda_1\ldots\lambda_k}(1, \ldots, 1) = \sum_{\tau_1+\cdots+\tau_k=t}\prod_{j=1}^{k}\binom{\tau_j+\lambda_j-1}{\lambda_j-1} \leqslant (t+1)^{\lambda_1+\cdots+\lambda_k},$$

because

$$\binom{\tau_j+\lambda_j-1}{\lambda_j-1} \leqslant (t+1)^{\lambda_j-1} \quad \text{for } \lambda_j \geqslant 1$$

and

$$\sum_{\tau_1+\ldots+\tau_k=t} 1 \leqslant (t+1)^k.$$

Finally,

$$\frac{1}{t!} D^t (f_1^{\lambda_1} \ldots f_k^{\lambda_k})$$

is the value of the polynomial $B_{t\lambda_1\ldots\lambda_k}$ where the variables $X_{\tau j}$ are replaced by $(1/\tau!) D^\tau f_j$ $(0 \leqslant \tau \leqslant t,\ 1 \leqslant j \leqslant k)$.

2. Gel'fond–Schneider's method. In this section we state and prove a variant of Theorem 2.1 in [7] concerning Schneider's method.

Let $k \geqslant 2$ be an integer, d, μ, $X_1, \ldots, X_k$, r, R, T be functions on the positive integers with positive real values. For each positive integer N, we define

$$\varphi(N) = \log\left\{\frac{R(N)^2 + r(N) r(N+1)}{R(N)\big(r(N) + r(N+1)\big)}\right\}.$$

We assume that there exists a positive integer N_0 such that for $N \geqslant N_0$, the following properties hold:

(2.1) $r(N+1) < R(N)$, and the function R is non-decreasing. We set $R_0 = \limsup_{N\to\infty} R(N)$, with $0 < R_0 \leqslant \infty$.

(2.2) The function $\mu\varphi/d$ is non-decreasing and tends to infinity when N tends to infinity.

(2.3) For $1 \leqslant j \leqslant k$, the function $\mu\varphi/X_j$ is non-decreasing and $X_j \geqslant d$.

(2.4) We assume

$$\limsup_{N\to\infty} \mu(N) = \infty,$$

$$\limsup_{N\to\infty} \frac{d(N) \log T(N)}{X_j(N)} = 0 \quad (1 \leqslant j \leqslant k)$$

and

$$\limsup_{N\to\infty} \frac{T(N)-1}{\mu(N)\varphi(N)} \log\big(1/r(N)\big) \leqslant 0.$$

(2.5) The function $N \to \mu(N+1)\varphi(N+1)/\mu(N)\varphi(N)$ is bounded from above; we set

$$B = \limsup_{N\to\infty} \mu(N+1)\varphi(N+1)/\mu(N)\varphi(N).$$

We choose two positive numbers A_1 and A_2 satisfying

$$A_1 > k\left(1 + 2B + \frac{1}{A_2} + \frac{B}{A_2}\right). \tag{2.6}$$

THEOREM 2.7. *Let K be a number field of degree δ, and $f_1, \ldots, f_k$ be meromorphic functions in the disk $\{|z| < R_0\}$ of the complex plane. We assume that $f_1, \ldots, f_k$ are algebraically independent over $\mathbf{Q}$. For $1 \leqslant j \leqslant k$, let g_j be an analytic function in the disk $\{|z| < R_0\}$ such that $g_j f_j$ is also analytic in the disk $\{|z| < R_0\}$; we assume*

$$\log\max\{|g_j|_{R(N)}, |g_j f_j|_{R(N)}\} \leqslant X_j(N). \tag{2.8}$$

For each $N \geqslant N_0$, let Γ_N be a non-empty finite subset of the disk $\{|z| \leqslant r(N)\}$; for each $N \geqslant N_0$ and each $\gamma \in \Gamma_N$, let $T(\gamma, N)$ be a positive integer, with $T(\gamma, N) \leqslant T(N)$ and

$$\sum_{\gamma\in\Gamma_N} T(\gamma, N) = \mu(N).$$

We assume that for all $N \geqslant N_0$, all $\gamma \in \Gamma_N$, and all integers j, t with $1 \leqslant j \leqslant k$, $0 \leqslant t < T(\gamma, N)$, we have

$$g_j(\gamma) \neq 0 \quad \text{and} \quad D^t f_j(\gamma) \in \bar{\mathbf{Q}};$$

we denote by $\alpha_j(\gamma, N)$ the $(T(\gamma, N)+1)$-tuple

$$\left(1, f_j(\gamma), f_j'(\gamma), \ldots, \frac{1}{t!} D^t f_j(\gamma), \ldots, \frac{1}{(T(\gamma, N)-1)!} D^{T(\gamma,N)-1} f_j(\gamma)\right).$$

We assume, for all N, γ, j as above,

$$\log|g_j(\gamma)| \geqslant -X_j(N), \tag{2.9}$$

$$d(N)\, h\big(\alpha_j(\gamma, N)\big) \leqslant X_j(N) \tag{2.10}$$

and

$$[K(\alpha_1(\gamma, N), \ldots, \alpha_k(\gamma, N)) : \mathbf{Q}] \leqslant d(N). \tag{2.11}$$

Then there exists $N_1 \geqslant N_0$ such that, for all $N \geqslant N_1$,

$$\delta\,\mu(N)^{k-1} \varphi(N)^k < c\, d(N) \prod_{j=1}^{k} X_j(N) \tag{2.12}$$

with $c = A_1^k (A_2 + 1)$.

Remark. The optimal choice of A_2 is the positive root of the quadratic equation

$$(2B+1)x^2 - (B+1)(k-1)x - (B+1)k = 0;$$

on the other hand, if $B = 1$, one may choose $A_2 = 2$ and $A_1 = 4k + \varepsilon$ (with $\varepsilon > 0$ sufficiently small), and one gets the conclusion (2.12) with $c = 3(4k)^k + 1$.

Theorem 2.1 in [7] corresponds to the special case where $f_1,\ldots,f_k$ are analytic rather than meromorphic (and $g_j = 1$), with no derivatives (which means $T(N) = T(\gamma, N) = 1$; this is Schneider's method), and the constant c in (2.12) was unspecified (only it does not depend on N). For our application here it is essential to know that c depends only on k and B. In most (all ?) applications, $B = 1$.

Our Theorem 2.7 contains most of the results which have been derived so far using Gel'fond's or Schneider's method. However an important exception worth mentioning is [15].

Proof. Let $\mathscr{E}$ be the set of the integers $N \geqslant N_0$ such that

$$(2.13)\qquad \delta\,\mu(N)^{k-1}\,\varphi(N)^k \geqslant c\,d(N)\prod_{j=1}^{k} X_j(N);$$

we assume that $\mathscr{E}$ is infinite, and we will deduce a contradiction.

Define

$$A = \min_{1\leqslant j\leqslant k}\ \liminf_{N\in\mathscr{E}} \mu(N)\,\varphi(N)/X_j(N).$$

We first prove, by induction on k, that there is no loss of generality to assume $A > A_1$.

If $k = 1$, the assumption (2.13) reads

$$\delta\,\varphi(N) \geqslant c\,d(N)\,X_1(N) \quad \text{for } N\in\mathscr{E},$$

with $c = A_1(A_2+1)$, and we have

$$\mu(N)\,\varphi(N)/X_1(N) \geqslant c\mu(N)\,d(N)/\delta = A_1(A_2+1)\,\mu(N)\,d(N)/\delta;$$

by (2.4) we have $\mu(N) \geqslant 1$, and by (2.11) we have $d(N) \geqslant \delta$; therefore

$$\mu(N)\,\varphi(N)/X_1(N) \geqslant A_1(A_2+1) \quad \text{for } N\in\mathscr{E},$$

and consequently $A > A_1$.

Let $k \geqslant 2$ be such that (2.13) holds for $N\in\mathscr{E}$, while (induction hypothesis) for all sufficiently large integer N, (2.12) holds with k replaced by $k-1$. Then for $N\in\mathscr{E}$ sufficiently large,

$$\frac{1}{A_1}\cdot\frac{\mu(N)\,\varphi(N)}{X_k(N)} \geqslant A_1^{k-1}(A_2+1)\frac{d(N)\,X_1(N)\ldots X_{k-1}(N)}{\delta\,\mu(N)^{k-2}\,\varphi(N)^{k-1}}.$$

From the induction hypothesis, for sufficiently large N, the right-hand side is > 1; because of (2.3) the left-hand side is a non-decreasing function of N; hence

$$\liminf_{N\in\mathscr{E}} \frac{\mu(N)\,\varphi(N)}{X_k(N)} > A_1.$$

Therefore we will assume $A > A_1$.

According to (2.6), we can choose $\varepsilon > 0$ sufficiently small, so that, if we set

$$\frac{1}{A_3} = \frac{1}{A_1A_2}(k+\varepsilon(k+3)),\qquad \frac{1}{A_4} = \frac{k}{A_1}+\frac{1}{A_3}+\varepsilon$$

and

$$\frac{1}{A_5} = \frac{2k}{A_1}+\frac{1}{A_3}+3\varepsilon,$$

we have

$$(2.14)\qquad \frac{1}{A_4}+\frac{B+\varepsilon}{A_5} < 1.$$

Now we take a sufficiently large integer $N_1 \geqslant N_0$, and we take $N\in\mathscr{E}$, $N \geqslant N_1$.

First step. We define

$$L_j = \mu(N)\,\varphi(N)/A_1X_j(N) \qquad (1 \leqslant j \leqslant k).$$

Since N is sufficiently large and $A_1 < A$, we have $L_j \geqslant 1$; hence the integral part $[L_j]$ of L_j satisfies

$$L_j < [L_j]+1 \leqslant L_j+1 \leqslant 2L_j,$$

and therefore the number

$$L = \delta\prod_{j=1}^{k}([L_j]+1)$$

satisfies

$$\delta\prod_{j=1}^{k} L_j < L \leqslant 2^k\delta\prod_{j=1}^{k} L_j.$$

We will use the upper bounds

$$(2.15)\qquad d(N)\sum_{j=1}^{k} L_j h(\alpha_j(\gamma,N)) \leqslant \sum_{j=1}^{k} L_jX_j(N) \leqslant \frac{k}{A_1}\mu(N)\,\varphi(N),$$

and

$$(2.16)\qquad (A_2+1)\log L+\frac{1}{d(N)}\log 2+\delta\,h(\xi)+(L_1+\ldots+L_k)\log T(N) < \varepsilon\frac{k+3}{A_1}\cdot\frac{\mu(N)\,\varphi(N)}{d(N)};$$

the later comes from the observation that

$$\log L_j \leqslant \log\left(\frac{\mu(N)\,\varphi(N)}{A_1\,d(N)}\right) \leqslant \frac{\varepsilon}{A_1(A_2+1)}\cdot\frac{\mu(N)\,\varphi(N)}{d(N)}$$

which follows from (2.2) and (2.3) for sufficiently large N, while

$$\log\delta+k\log 2<\frac{\varepsilon}{A_1(A_2+1)}\cdot\frac{\mu(N)\varphi(N)}{d(N)}$$

also because $\mu\varphi/d$ tends to infinity with N by (2.2); hence

$$\log L\leqslant\frac{\varepsilon(k+1)}{A_1(A_2+1)}\cdot\frac{\mu(N)\varphi(N)}{d(N)};$$

moreover

$$\frac{1}{d(N)}\log 2+\delta h(\xi)<\frac{\varepsilon}{A_1}\cdot\frac{\mu(N)\varphi(N)}{d(N)}$$

and

$$(L_1+\ldots+L_k)\log T(N)<\frac{\varepsilon}{A_1}\cdot\frac{\mu(N)\varphi(N)}{d(N)}$$

because of (2.4). This completes the proof of (2.16).

We choose a generator ξ from K over $\boldsymbol{Q}$, and we construct a non-zero polynomial

$$P(X_1,\ldots,X_k)=\sum_{\lambda}\sum_{i=0}^{\delta-1}p_{\lambda i}\xi^i\prod_{j=1}^{k}X_j^{\lambda_j}$$

in $K[X_1,\ldots,X_k]$, of degree at most L_j in X_j, $1\leqslant j\leqslant k$, such that the function $F=P(f_1,\ldots,f_k)$ vanishes on each $\gamma\in\Gamma_N$ with multiplicity $\geqslant T(\gamma,N)$. We have written λ for $(\lambda_1,\ldots,\lambda_k)$ with $0\leqslant\lambda_j\leqslant L_j$ $(1\leqslant j\leqslant k)$.

The system of linear equations we have to solve is

$$\sum_{\lambda}\sum_{i=0}^{\delta-1}p_{\lambda i}\xi^i\prod_{j=1}^{k}B_{t\lambda_1\ldots\lambda_k}(\theta_{t\gamma})=0\qquad(\gamma\in\Gamma_N),$$

where $B_{t\lambda_1\ldots\lambda_k}$ is a polynomial (given by Lemma 1.2) in $k(t+1)$ variables, and $\theta_{t\gamma}$ is the $k(t+1)$-tuple of components $(1/\tau!)D^\tau f_j(\gamma)$ $(0\leqslant\tau\leqslant t,\ 1\leqslant j\leqslant k)$.

We use Lemma 1.1 where the D appearing there is not greater than $d(N)\mu(N)$. Since $N\in\mathscr{E}$, the inequality (2.13) with $c=A_1^k(A_2+1)$ ensures that $d(N)\mu(N)\leqslant L/(A_2+1)$; thus

$$\frac{\mu(N)}{L-D}\leqslant\frac{1}{A_2 d(N)}.$$

Therefore, using Lemmas 1.0, 1.1 and 1.2 together with (2.15) and (2.16), we get a solution $(p_{\lambda i})$ in $\boldsymbol{Z}$ with

$$\log\max_{\lambda,i}|p_{\lambda i}|$$

$$\leqslant\frac{\mu(N)}{L-D}\Big\{\log 2+d(N)\big(\log L+\delta h(\xi)+(L_1+\ldots+L_k)\log T(N)\big)+\sum_{j=1}^{k}L_jX_j(N)\Big\}$$

and

(2.17)
$$\log\sum_{\lambda}\sum_{i=0}^{\delta-1}|p_{\lambda i}|\leqslant\frac{1}{A_3}\cdot\frac{\mu(N)\varphi(N)}{d(N)}.$$

Second step. We introduce the analytic function

$$\Phi=F\prod_{j=1}^{k}g_j^{[L_j]}.$$

Let M be an integer, $M\geqslant N$, such that F vanishes at each $\gamma\in\Gamma_M$ with multiplicity $\geqslant T(\gamma,M)$. We prove:

(2.18)
$$\log|\Phi|_{r(M+1)}\leqslant-\Big(1-\frac{1}{A_4}\Big)\mu(M)\varphi(M).$$

Indeed, from the maximum principle applied to the analytic function

$$\Phi(z)\prod_{\gamma\in\Gamma_M}\left(\frac{R(M)^2-z\bar\gamma}{R(M)(z-\gamma)}\right)^{T(\gamma,M)}$$

on the disks $\{|z|\leqslant r(M+1)\}$ and $\{|z|\leqslant R(M)\}$, we deduce from (2.1):

$$\log|\Phi|_{r(M+1)}\leqslant\log|\Phi|_{R(M)}-\mu(M)\varphi(M).$$

From (2.8) and (2.17) we obtain

$$\log|\Phi|_{R(M)}\leqslant\frac{1}{A_3}\cdot\frac{\mu(N)\varphi(N)}{d(N)}+\delta\log\max\{1,|\xi|\}+\sum_{j=1}^{k}L_jX_j(M).$$

Now our assumption (2.3) yields

(2.19)
$$L_jX_j(M)\leqslant\frac{1}{A_1}\mu(M)\varphi(M).$$

Our claim (2.18) follows at once.

Third step. Let $M\geqslant N$ be an integer such that

(2.20)
$$\log|\Phi|_{r(M)}<-\frac{1}{A_5}\mu(M)\varphi(M).$$

We prove that F vanishes at each $\gamma\in\Gamma_M$ with multiplicity $\geqslant T(\gamma,M)$.

Otherwise, there is a $\gamma\in\Gamma_M$ and a $t\in\boldsymbol{Z}$, $0\leqslant t<T(\gamma,M)$, such that $D^tF(\gamma)\neq 0$. We choose firstly such a γ, and next a t minimal for this property; therefore

$$D^t\Phi(\gamma)=D^tF(\gamma)\prod_{j=1}^{k}g_j(\gamma)^{[L_j]}\neq 0.$$

By Liouville inequality, Lemmas 1.0 and 1.2, using (2.2), (2.10), (2.17) and (2.19), we have

$$-\log\left|\frac{1}{t!}D^t F(\gamma)\right|$$

$$\leqslant \frac{1}{A_3}\mu(N)\varphi(N)+d(M)\delta h(\xi)+(L_1+\ldots+L_k)\log T(M)+\sum_{j=1}^{k} d(M)L_j h(\alpha_j(\gamma, M))$$

$$\leqslant \left(\frac{1}{A_3}+\frac{k}{A_1}+2\varepsilon\right)\mu(M)\varphi(M).$$

Next we use (2.9):

$$\log\left|\frac{1}{t!}D^t \Phi(\gamma)\right| \geqslant -\left(\frac{1}{A_3}+\frac{2k}{A_1}+2\varepsilon\right)\mu(M)\varphi(M).$$

From Cauchy's inequalities and (2.4) we deduce:

$$\log|\Phi|_{r(M)} \geqslant t\log r(M)+\log\left|\frac{1}{t!}D^t \Phi(\gamma)\right| \geqslant -\frac{1}{A_5}\mu(M)\varphi(M),$$

which gives a contradiction with (2.20).

Conclusion. By (2.5), for sufficiently large N, and for each $M \geqslant N$, we have

$$\mu(M+1)\varphi(M+1) < (B+\varepsilon)\mu(M)\varphi(M).$$

From (2.4) we know that $\mu(M)$ in unbounded. Consider the inequalities (2.14), (2.18) and (2.20); we claim that the function F is the zero function in the disk $\{|z| < R_0\}$. This is plain if $\limsup_{M\to\infty} r(M) > 0$, while if $r(M)\to 0$, this follows from the assumption $\limsup_{M\to\infty}\mu(M) = \infty$, together with the fact that F vanishes at each point $\gamma\in\Gamma_M$ with multiplicity $\geqslant T(\gamma, M)$. Therefore we get a contradiction with our assumption that the functions $f_1,\ldots,f_k$ are algebraically independent. This proves Theorem 2.7.

3. The functional equation $F(az) = P(z)F(z)+Q(z)$.

(a) *Formal case.* Let Δ, Δ' be two integers with $\Delta \geqslant 0$ and $\Delta' \geqslant -1$; we consider the ring $\mathscr{A}$ of polynomials with coefficients in $\mathbf{Z}$ and $\Delta+\Delta'+5$ unknowns; it will be convenient to write these unknowns as follows:

$$X,\ Y,\ a,\ a_0,\ \ldots,\ a_\Delta,\ b_0,\ \ldots,\ b_{\Delta'}.$$

We define

$$P(X) = \sum_{i=0}^{\Delta} a_{\Delta-i}X^i \quad\text{and}\quad Q(X) = \sum_{j=0}^{\Delta'} b_{\Delta'-j}X^j.$$

Let E be the set $\{X, aX, a^2X, \ldots\}$, and F be a map from E into $\mathscr{A}$ satisfying $F(X) = Y$ and

$$F(az) = P(z)F(z)+Q(z) \quad \text{for all } z\in E.$$

LEMMA 3.1. *For each $n \geqslant 0$ and each $z\in E$, we have*

$$F(a^n z) = P_n(z)F(z)+Q_n(z)$$

where $P_n(X)$ and $Q_n(X)$ are the elements of $\mathscr{A}$ which are defined by $P_0 = 1$, $Q_0 = 0$, and, for $n \geqslant 1$,

$$P_n(X) = \prod_{\nu=0}^{n-1} P(a^\nu X)$$

and

$$Q_n(X) = \sum_{\nu=0}^{n-1} Q(a^\nu X)P_{n-\nu-1}(a^{\nu+1}X).$$

Proof. Easy induction.

LEMMA 3.2. *With the hypotheses of Lemma 3.1, for each $n \geqslant 1$, the polynomial P_n satisfies*

$$\deg_X P_n = n\Delta \quad \text{and} \quad \deg_a P_n = \binom{n}{2}\Delta,$$

where $\binom{n}{2}$ is the binomial coefficient $n(n-1)/2$ (with $\binom{1}{2} = 0$); further P_n is homogeneous in $a_0, \ldots, a_\Delta$ of degree n, and does not depend on $b_0, \ldots, b_{\Delta'}$; furthermore the coefficients of P_n are non-negative integers of sum $L(P_n)$ given by:

$$L(P_n) = (\Delta+1)^n.$$

If $\Delta' = -1$, then $Q_n = 0$ for all $n \geqslant 0$. If $\Delta' \geqslant 0$, then for each $n \geqslant 0$ the polynomial Q_n satisfies

$$\deg_X Q_n = (n-1)\Delta+\Delta' \quad \text{and} \quad \deg_a Q_n \leqslant \binom{n}{2}\Delta+(n-1)\Delta';$$

moreover, Q_n is of degree $n-1$ in $a_0, \ldots, a_\Delta$, and is homogeneous of degree 1 in $b_0, \ldots, b_{\Delta'}$, and finally the coefficients of Q_n are non-negative integers of sum $L(Q_n)$ given by:

$$L(Q_n) = (\Delta'+1)\sum_{\nu=0}^{n-1}(\Delta+1)^\nu.$$

Proof. Considered as a polynomial either in X, a or a_0, the polynomial P_n has for leading term (= term of highest degree) $a_0^n a^{\Delta n(n-1)/2} X^{n\Delta}$. Considered as a polynomial in X, Q_n has for leading term $a_0^{n-1} b_0 a^{\Delta n(n-1)/2} X^{(n-1)\Delta}$. Since $\binom{n-\nu-1}{2}+(n-\nu-1)(\nu+1)=\binom{n}{2}-\binom{\nu+1}{2}$, the degree of Q_n in a is not greater than

$$\max_{0\leqslant \nu\leqslant n-1}\left\{\left[\binom{n}{2}-\binom{\nu+1}{2}\right]\Delta+\nu\Delta'\right\}.$$

The coefficients being non-negative, their sum is the value of the polynomial where all the indeterminates are replaced by 1.

LEMMA 3.3. *With the hypotheses of Lemma* 3.1, *for each* $n\geqslant 1$, $F(a^n X)$ *is an element of* $\mathscr{A}$ *satisfying*

$$\deg_Y F(a^n X)=1,$$
$$\deg_X F(a^n X)=(n-1)\Delta+\max\{\Delta;\Delta'\},$$
$$\deg_a F(a^n X)\leqslant\binom{n}{2}\Delta+(n-1)\max\{\Delta',0\},$$
$$\deg_{a_i} F(a^n X)=n \quad (0\leqslant i\leqslant\Delta),$$
$$\deg_{b_j} F(a^n X)=1 \quad (0\leqslant j\leqslant\Delta').$$

Moreover, the coefficients of $F(a^n X)$ *are non-negative integers of sum*

$$(\Delta+1)^{n-1}(\Delta+n\Delta'+n+1).$$

Proof. This follows from Lemmas 3.1 and 3.2.

(b) *Complex case.* We now consider a complex number a, with $|a|>1$, two polynomials P and Q in $\boldsymbol{C}[X]$, of degrees Δ and Δ', and an entire function F in $\boldsymbol{C}$ which satisfies the functional equation

$$F(az)=P(z)F(z)+Q(z) \quad \text{for all } z\in\boldsymbol{C}.$$

LEMMA 3.4. *There exists a positive number* $c_1>0$ *such that, for all* $R\geqslant 2$,

$$\log|F|_R\leqslant\frac{\Delta}{2\log|a|}(\log R)^2+c_1\log R.$$

Proof. Let $t\in\boldsymbol{C}$ satisfy $|t|=R$ and $|F(t)|=|F|_R$. We define $n=\left[\dfrac{\log R}{\log|a|}\right]$ (integral part), so that $|a|^n\leqslant R<|a|^{n+1}$. Let $z=t\cdot a^{-n}$. We have

$$F(t)=P_n(z)F(z)+Q_n(z).$$

Since F is bounded on $1\leqslant|z|\leqslant|a|$, we deduce from Lemma 3.2:

$$\log|F(t)|\leqslant\binom{n}{2}\Delta\log|a|+c_2 n$$

for some constant $c_2>0$ independent of n and R; Lemma 3.4 then follows from the inequalities

$$n\leqslant\frac{\log R}{\log|a|} \quad\text{and}\quad n^2\log|a|\leqslant\frac{(\log R)^2}{\log|a|}.$$

Remarks. 1. Combined with Cauchy's inequalities, Lemma 3.4 shows that for $\Delta=0$, the function F is a polynomial. Notice that, conversely, if F and P are any polynomials, then $F(az)-P(z)F(z)$ is also a polynomial, and therefore F satisfies a functional equation of the form $F(az)=P(z)F(z)+Q(z)$.

2. Let P, Q and a be given. Considering Taylor expansions at the origin, it is easy to solve the functional equation $F(az)=P(z)F(z)+Q(z)$. The result is as follows:

if, for all $\mu\in\boldsymbol{Z}$, $\mu\geqslant 0$, we have $P(0)\neq a^\mu$, then there exists a unique power series $F(z)$ satisfying this functional equation. Moreover F has its coefficients in the field $\boldsymbol{Q}(a,a_0,\ldots,a_\Delta;\,b_0,\ldots,b_{\Delta'})$, and defines an entire function in $\boldsymbol{C}$;

if there exists an integer $\mu\geqslant 0$ such that $P(0)=a^\mu$, then the set of entire functions G satisfying $G(az)=G(z)P(z)$ is a $\boldsymbol{C}$-vector space of dimension 1, which contains non-zero elements whose Taylor expansions at the origin have coefficients in the field $\boldsymbol{Q}(a,a_0,\ldots,a_\Delta)$. If the given functional equation has a solution F_0, then the general solution is F_0+G, and in this case there are solutions F whose Taylor expansions at the origin have coefficients in the field $\boldsymbol{Q}(a,a_0,\ldots,a_\Delta;\,b_0,\ldots,b_{\Delta'})$.

(c) *Arithmetic case.* Let K be a number field of degree δ over $\boldsymbol{Q}$; we fix an embedding of K into $\boldsymbol{C}$. Let $a\in K$, $|a|>1$, and let P and Q be two elements of $K[X]$ with P of degree Δ.

LEMMA 3.5. *Let F be an entire function satisfying the functional equation*

$$F(az)=F(z)P(z)+Q(z).$$

Let ζ be an algebraic number such that $F(\zeta)$ is algebraic. Then there exists a constant $c_3>0$ such that for all $n\in\boldsymbol{Z}$, either $n<0$ and $P_{-n}(a^n\zeta)=0$, or else the number $F(a^n\zeta)$ belongs to the field $K(\zeta,F(\zeta))$ and satisfies

$$h(F(a^n\zeta))\leqslant\frac{|n|^2}{2}\Delta h(a)+c_3(|n|+1).$$

Proof. For $n\geqslant 0$, this follows readily from Lemmas 1.0 and 3.3. Let $n=-m$ be a negative integer. We write

$$F(a^{-m}X)=\frac{F(X)}{P_m(a^{-m}X)}-\frac{Q_m(a^{-m}X)}{P_m(a^{-m}X)}.$$

But

$$\frac{Q_m(a^{-m}X)}{P_m(a^{-m}X)}=\sum_{\nu=0}^{m-1}Q(a^{-m+\nu}X)\frac{P_{m-\nu-1}(a^{-m+\nu+1}X)}{P_m(a^{-m}X)}$$

and

$$\frac{P_{m-\nu-1}(a^{-m+\nu+1}X)}{P_m(a^{-m}X)}=\frac{\prod_{h=0}^{m-\nu-2}P(a^{h-m+\nu+1}X)}{\prod_{k=0}^{m-1}P(a^{-m+k}X)}=\frac{1}{\prod_{k=0}^{\nu}P(a^{-m+k}X)}.$$

Therefore

$$F(a^{-m}X)=\frac{F(X)}{P_m(a^{-m}X)}-\sum_{\nu=0}^{m-1}\frac{Q(a^{-m+\nu}X)}{\prod_{k=0}^{\nu}P(a^{-m+k}X)}.$$

From the relation

$$P_m(a^{-m}X)=\prod_{\nu=0}^{m-1}P(a^{-m+\nu}X)$$

one deduces that

$$F(a^{-m}\zeta)=\frac{U_m}{V_m}(a^{-1},a_0,\dots,a_\Delta,b_0,\dots,b_{\Delta'},\zeta,F(\zeta)),$$

where U_m and V_m are polynomials with rational integer coefficients, of degree $\leqslant\binom{m}{2}\Delta$ in a^{-1} and $\leqslant c_4 m$ in the other variables, and of length $\leqslant\exp(c_5 m)$. One gets the conclusion of Lemma 3.5 using a version of Lemma 1.0 for rational functions in place of polynomials.

Remarks. From Lemma 3.1, one deduces that in the annulus $|a|^{-1}<|z|\leqslant 1$, there are at most Δ points α (counting multiplicities) which are zero of one of the polynomials $P_n(a^{-n}X)$, $n\in\mathbf{N}$.

We will use Lemma 3.5 only with $n\geqslant 0$; but using the case $n\leqslant 0$, it is easy to improve the constant 700 to 100 in (0.5).

(d) *Derivatives.* By taking derivatives one deduces from the functional equation

$$F(az)=P(z)F(z)+Q(z)$$

and from Lemma 3.1, that for each t, n non-negative integers,

$$a^{nt}\frac{1}{t!}D^tF(a^nz)=\sum_{\tau_1+\tau_2=t}\frac{1}{\tau_1!}D^{\tau_1}P_n(z)\frac{1}{\tau_2!}D^{\tau_2}F(z)+\frac{1}{t!}D^tQ_n(z). \tag{3.6}$$

LEMMA 3.7. *With the assumptions of Lemma 3.5, let $t\geqslant 1$ be an integer such that the t numbers $F(\zeta)$, $F'(\zeta)$, …, $D^{t-1}F(\zeta)$ are algebraic. Then there exists a constant $c_6>0$ such that, for all $n\in\mathbf{Z}$, with $n\geqslant 0$, the t numbers $D^\tau F(a^n\zeta)$ belong to the field*

$$K(\zeta,F(\zeta),F'(\zeta),\dots,D^{t-1}F(\zeta)),$$

and satisfy

$$h\left(1,F(a^n\zeta),F'(a^n\zeta),\dots,\frac{1}{(t-1)!}D^{t-1}F(a^n\zeta)\right)\leqslant\frac{n^2}{2}\Delta h(a)+c_6(n+1).$$

Proof. Lemma 3.7 follows readily from (3.6) and Lemma 3.5.

(e) *Irrationality of the values of the function F and of its derivatives.* The following result extends Theorem 0.4 to values of derivatives of F.

THEOREM 3.8. *Let K be a number field of degree δ embedded in $\mathbf{C}$; let $a\in K$, with $|a|>1$, and let P and Q be polynomials in $K[X]$, with $\deg P=\Delta$. Let F be a transcendental entire function satisfying the functional equation*

$$F(az)=P(z)F(z)+Q(z).$$

Let d, s, t_1, …, t_s be positive integers; assume that there exist s distinct algebraic numbers $\zeta_1,\dots,\zeta_s$ in the annulus $|a|^{-1}<|\zeta|\leqslant 1$ such that

$$D_\tau F(\zeta_\sigma)\ \text{is algebraic for}\ 0\leqslant\tau<t_\sigma,\ 1\leqslant\sigma\leqslant s,$$

with

$$[K(\zeta_\sigma,F(\zeta_\sigma),F'(\zeta_\sigma),\dots,D^{t_\sigma-1}F(\zeta_\sigma)):K]\leqslant d\quad\text{for}\quad 1\leqslant\sigma\leqslant s.$$

Then

$$\sum_{\sigma=1}^{s}t_\sigma\leqslant 700\,\Delta d^2\delta h(a)/\log|a|. \tag{3.9}$$

We will prove Theorem 3.8 with the bound (3.9) replaced by the sharper one

$$94\,\Delta d\inf_{\varrho>\log|a|}\{\max\{d\delta h(a);\varrho\}\max\{d\delta h(a);\varrho^2/\log|a|\}/(\varrho-\log|a|)^2\}. \tag{3.10}$$

We get the conclusion of Theorem 3.8 by taking $\varrho=3d\delta h(a)$, because $\delta h(a)\geqslant\log|a|$ hence $2d\delta h(a)\leqslant\varrho-\log|a|$, and $\varrho^2\geqslant 3d\delta h(a)\log|a|$, while $94(27/4)<635$.

Let ϱ satisfy $\varrho>\log|a|$. We choose a sufficiently large constant c_7, and we use Theorem 2.7 with

$$k=2,\quad f_1(z)=z,\quad f_2(z)=F(z),\quad g_1(z)=g_2(z)=1,\quad d(N)=d\delta,$$

$$r(N)=|a|^N,\quad R(N)=\tfrac12\left[e^{\varrho N}(1+|a|)+(e^{2\varrho N}(1+|a|)^2-4|a|^{2N+1})^{1/2}\right],$$

so that

$$\varphi(N)=(\varrho-\log|a|)N,\quad R_0=\infty,\quad\text{and}\quad r(N)<R(N)<e^{\varrho N};$$

$$\Gamma_N = \{\zeta_\sigma a^n;\ 1 \leqslant \sigma \leqslant s,\ 0 \leqslant n < N\}, \qquad \mu(N) = N \sum_{\sigma=1}^{s} t_\sigma,$$

$$X_1(N) = N \max\{d\delta h(a); \varrho\} + c_7,$$

$$X_2(N) = \tfrac{1}{2} \Delta N^2 \max\{d\delta h(a); \varrho^2/\log|a|\} + c_7 N.$$

We have $B = 1$, and we choose $A_2 = 3/2$, $A_1 = 8.67$; hence $c < 188$. The assumption (2.8) follows from Lemma 3.4, while (2.10) and (2.11) follow from Lemmas 3.5 and 3.7.

Finally (3.10) is a consequence of (2.12).

4. Theta function (additive point of view). Let K be a number field of degree δ embedded in $\boldsymbol{C}$, τ be a complex number with positive imaginary part such that $q = e^{i\pi\tau} \in K$, P and Q two polynomials in $K[X]$ where P is of degree Δ, and f be an entire function in $\boldsymbol{C}$ satisfying

$$f(u+1) = f(u) \quad \text{and} \quad f(u+\tau) = P(e^{-2i\pi u}) f(u) + Q(e^{-2i\pi u}).$$

The fact that f is periodic of period 1 is equivalent to the fact that there exists a function F, analytic in $\boldsymbol{C}^*$, such that $F(e^{-2i\pi u}) = f(u)$. We perform the change of variables $z = e^{-2i\pi u}$ and we apply Theorem 0.4.

COROLLARY 4.1. *We assume that F is analytic at 0 and that the two functions $e^{2i\pi u}$ and $f(u)$ are algebraically independent over $\boldsymbol{Q}$. Let d be a positive integer; let $u_1, \ldots, u_s$ be complex numbers, which are pairwise distinct modulo $\boldsymbol{Z} + \boldsymbol{Z}\tau$, such that for $1 \leqslant \sigma \leqslant s$, the two numbers $\exp(2i\pi u_\sigma)$ and $f(u_\sigma)$ are algebraic, with*

$$[K(\exp(2i\pi u_\sigma), f(u_\sigma)) : K] \leqslant d.$$

Then

$$s < \frac{700}{\pi} \Delta d^2 \delta\, h(q)/\operatorname{Im}\tau.$$

Of course one may include values of derivatives (with respect to the differential operator $(1/2i\pi)d/du$) by using Theorem 3.8 in place of Theorem 0.4.

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