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Diagonal cubic equations, II

by

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1. Introduction. Let $\lambda_1, \ldots, \lambda_7$ be positive integers, with $\Pi = \lambda_1 \ldots \lambda_7$. The object of this paper is to prove

THEOREM 1. *The equation*

$$\lambda_1 x_1^3 + \ldots + \lambda_7 x_7^3 = 0 \tag{1.1}$$

has a solution in nonzero integers satisfying

$$\lambda_1 |x_1|^3 + \ldots + \lambda_7 |x_7|^3 \ll \Pi^{61}. \tag{1.2}$$

Here and subsequently, implied constants are absolute. (We use $C_1, C_2, \ldots$ to denote absolute constants satisfying various conditions specified below.)

The number of variables here cannot be reduced, as we see by considering the equation

$$x_1^3 - Ax_2^3 + p(x_3^3 - Ax_4^3) + p^2(x_5^3 - Ax_6^3) = 0.$$

This has only the trivial solution in integers when p is a prime, $p \equiv 1 \pmod 3$, and A is a cubic non-residue modulo p.

The proof of Theorem 1 is based on the important ideas of Vaughan [8] on Waring's problem. The constant 61 could be made smaller, but only at the cost of lengthening the very complicated argument. The theorem may well be true with 61 replaced by 1.

The ordered set $\lambda_1, \ldots, \lambda_7$ is said to be *reduced* if

$$\lambda_1, \ldots, \lambda_7 \text{ are cube-free;} \tag{1.3}$$

(1.4) no prime divides λ_j for more than four values of j; and

$$\lambda_5 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_6 \geqslant \lambda_7 \geqslant \lambda_1. \tag{1.5}$$

After introducing some more notation, a result on solutions of (1.1) that applies only to reduced sets will be stated as Theorem 2. This will be used as a stepping stone to reach Theorem 1.

Throughout the paper, the symbols $\omega, p, p_1, \ldots$ are reserved for primes. Let

$$N = C_1 \Pi^{61} \tag{1.6}$$

where C_1 is sufficiently large. Let δ denote a sufficiently small positive absolute constant, and let $\varepsilon = \delta^2$.

Let $R(\lambda_1, \ldots, \lambda_7)$ denote the number of solutions of the equation

$$\lambda_1 x^3 - \lambda_2 p^3 y^3 + \lambda_3 p^3 p_3^3 z_1^3 + \lambda_4 p^3 p_4^3 z_2^3 + \lambda_5 u^3 + \lambda_6 p_6^3 v_1^3 + \lambda_7 p_7^3 v_2^3 = 0 \tag{1.7}$$

in integers $x, y, z_1, z_2, u, v_1, v_2$ and primes p, p_3, p_4, p_6, p_7 satisfying

$$P < x \leqslant 2P, \quad W < y \leqslant 2W, \quad R < z_i \leqslant 2R, \quad U < u \leqslant 2U, \quad V < v_i \leqslant 2V, \tag{1.8}$$

$$p \nmid x, \quad p_3 \nmid y, \quad p_4 \nmid y, \quad p_6 \nmid u, \quad p_7 \nmid u, \tag{1.9}$$

$$Y < p \leqslant 2Y, \quad p \equiv 2 \pmod 3, \tag{1.10}$$

$$Z_i < p_i \leqslant 2Z_i, \quad p_i \equiv 2 \pmod 3) \quad (i = 3, 4, 6, 7). \tag{1.11}$$

Here

$$P = N^{1/3} \lambda_1^{-1/3}, \quad Y = N^{17/345} \lambda_1^{-17/345}, \quad R = (1/40) N^{84/345}, \tag{1.12}$$

$$W = N^{98/345} \lambda_1^{17/345} \lambda_2^{-1/3}, \quad U = (1/40) N^{1/3} \lambda_5^{-1/3}, \quad V = (1/40) N^{2/7}, \tag{1.13}$$

$$Z_3 = N^{14/345} \lambda_1^{17/345} \lambda_3^{-1/3}, \quad Z_4 = 2N^{14/345} \lambda_1^{17/345} \lambda_4^{-1/3}, \tag{1.14}$$

$$Z_6 = N^{1/21} \lambda_6^{-1/3}, \quad Z_7 = 2N^{1/21} \lambda_7^{-1/3}. \tag{1.15}$$

The inequalities

$$N \ll \lambda_1 P^3, \lambda_2 Y^3 W^3, \lambda_3 Y^3 Z_3^3 R^3, \lambda_4 Y^3 Z_4^3 R^3, \lambda_5 U^3, \lambda_6 Z_6^3 V^3, \lambda_7 Z_7^3 V^3 \ll N \tag{1.16}$$

are easily verified. It is also easy to see that (1.10), (1.11) imply

$$p > p_4 > p_3 > \Pi, \quad p > p_7 > p_6 > \Pi \tag{1.17}$$

using (1.5) and (1.6). In particular

$$(pp_3p_4p_6p_7, \Pi) = 1. \tag{1.18}$$

Here (a_1, a_2), or more generally $(a_1, \ldots, a_n)$, denotes greatest common divisor; while $[a_1, \ldots, a_n]$ denotes least common multiple.

THEOREM 2. *In the above notation, we have*

$$R(\lambda_1, \ldots, \lambda_7) > 0$$

for reduced sets $\lambda_1, \ldots, \lambda_7$.

It follows from Theorem 2 and (1.6), (1.16) that (1.1) has solutions satisfying (1.2) whenever $\lambda_1, \ldots, \lambda_7$ is a reduced set. Thus the assertion of Theorem 1 holds whenever $\lambda_1, \ldots, \lambda_7$ satisfy (1.3) and (1.4).

To extend this result to general sets $\lambda_1, \ldots, \lambda_7$ we use the case $k = 7$ of the following lemma. The cases $k = 8, 9$ will be used in a subsequent paper [2]. Lemma 1 is essentially due to Pitman and Ridout [4].

LEMMA 1. *For given* $K > 0$, $C \geqslant 1$, $k \geqslant 7$, *let* $A(K, C, k)$ *be the following proposition.*

$A(K, C, k)$: *There is a solution of*

$$\lambda_1 x_1^3 + \ldots + \lambda_k x_k^3 = 0 \tag{1.19}$$

in nonzero integers such that

$$\sum_{i \leqslant k} \lambda_i x_i^3 \leqslant K(\lambda_1 \ldots \lambda_k)^C.$$

Suppose that $A(K, C, k)$ *holds provided that the* λ_i *are cube-free positive integers such that no prime divides more than* $k-3$ *of them. Then* $A(K, C, k)$ *holds for every set of positive integers* $\lambda_1, \ldots, \lambda_k$.

Proof. We first show that $A(K, C, k)$ holds for every set of cube-free positive integers λ_i. Let $p_0 = 1$ and let p_r be the rth prime ($r = 1, 2, \ldots$). It will be sufficient to prove that A_r holds for all r, where A_r is the proposition:

A_r: *If the* λ_i *are cube-free nonzero integers such that no prime greater than* p_r *divides more than* $k-3$ *of them, then* $A(K, C, k)$ *holds.*

By the hypotheses of Lemma 1, A_0 holds. We now show that A_r implies A_{r+1}. Let us assume that A_r holds and that the λ_i are cube-free positive integers such that no prime greater than $q = p_{r+1}$ divides more than $k-3$ of them. Then we can write

$$\sum_{i \leqslant k} \lambda_i x_i^3 = a_1 y_1^3 + \ldots + a_s y_s^3 + q(b_1 z_1^3 + \ldots + b_t z_t^3) + q^2(c_1 w_1^3 + \ldots + c_u w_u^3),$$

$$\lambda_1 \ldots \lambda_k = q^{t+2u} \gamma.$$

Here $s+t+u = k$, $q \nmid \gamma$, and no prime greater than p_r divides more than $k-3$ of the a_i, b_i, c_i. If $s \geqslant 3$, then $A(K, C, k)$ follows from A_r. If $s \leqslant 2$, $t \geqslant 3$, then by A_r the equation

$$q^2(a_1 y_1^3 + \ldots) + (b_1 z_1^3 + \ldots) + q(c_1 w_1^3 + \ldots) = 0 \tag{1.20}$$

has a solution in nonzero integers with

$$q^2(a_1|y_1|^3 + \ldots) + (b_1|z_1|^3 + \ldots) + q(c_1|w_1|^3 + \ldots) \leqslant K(q^{2s+u}\gamma)^C.$$

Multiplying (1.20) by q, we obtain a solution of (1.19) in nonzero integers with

$$\sum_i \lambda_i |x_i|^3 \leqslant Kq(q^{2s+u}\gamma)^C \leqslant K(q^{t+2u}\gamma)^C = K(\lambda_1 \ldots \lambda_k)^C.$$

To obtain the second inequality, note that $t+u = k-s \geqslant k-2 \geqslant 5$, so that

$$1 + C(2s+u) \leqslant C(1+2s+u) \leqslant C(5+u) \leqslant C(t+2u).$$

Similarly, if $s \leqslant 2$, $t \leqslant 2$, then by applying A_r to the equation

$$q(a_1 y_1^3 + \ldots) + q^2(b_1 z_1^3 + \ldots) + (c_1 w_1^3 + \ldots) = 0$$

and multiplying by q^2, we obtain a solution of (1.19) with

$$\sum_{i \leqslant k} \lambda_i |x_i|^3 \leqslant K q^2 (q^{s+2t} \gamma)^C \leqslant K (q^{t+2u} \gamma)^C.$$

Here we use $u = k-s-t \geqslant k-4 \geqslant 3$, and

$$2 + C(s+2t) \leqslant C(2+s+2t) \leqslant C(6+t) \leqslant C(t+2u).$$

Thus our assumptions imply that $A(K, C, k)$ holds in all cases, and we have shown that A_r implies A_{r+1}. By induction, A_r holds for all r.

We now remove the restriction to cube-free coefficients. Suppose that, for each i,

$$\lambda_i = \mu_i^3 \nu_i$$

where the positive integer ν_i is cube-free. Thus $\lambda_1 \ldots \lambda_k = \mu^3 \nu$, where

$$\mu = \prod_{i=1}^{k} \mu_i, \quad \nu = \prod_{i=1}^{k} \nu_i.$$

By what has just been proved, there exist nonzero integers $z_1, \ldots, z_k$ such that

$$\sum_{i \leqslant k} \nu_i z_i^3 = 0, \qquad \sum_{i \leqslant k} \nu_i |z_i|^3 \leqslant K \nu^C.$$

Multiplying by μ^3 we see that the nonzero integers

$$x_i = \mu \mu_i^{-1} z_i \qquad (i = 1, \ldots, k)$$

satisfy (1.19) with

$$\sum_{i \leqslant k} \lambda_i |x_i|^3 \leqslant K \mu^3 \nu^C \leqslant K (\mu^3 \nu)^C = K(\lambda_1 \ldots \lambda_k)^C,$$

and $A(K, C, k)$ holds. This completes the proof of Lemma 1. In particular, Theorem 1 follows from Theorem 2.

2. Outline of the proof of Theorem 2. Let $e(\theta) = \exp(2\pi i \theta)$. We write

(2.1) $$f_d(X, \alpha) = \sum_{\substack{X < x \leqslant 2X \\ (x,d)=1}} e(\alpha x^3),$$

and use the notation f in place of f_1.

Let $\boldsymbol{p}$ denote the ordered set p, p_3, p_4, p_6, p_7 and let

(2.2) $$F(\boldsymbol{p}; \alpha) = f_p(P, \lambda_1 \alpha) \overline{f_{p_3 p_4}(W, \lambda_2 p^3 \alpha)} f(R, \lambda_3 p^3 p_3^3 \alpha) \times f_{p_6 p_7}(U, \lambda_5 \alpha) f(V, \lambda_6 p_6^3 \alpha) f(V, \lambda_7 p_7^3 \alpha),$$

(2.3) $$F(\alpha) = \sum_{\substack{\boldsymbol{p} \\ (1.10),(1.11)}} F(\boldsymbol{p}; \alpha).$$

The summation here is over ordered sets $\boldsymbol{p}$ satisfying (1.10), (1.11); we often use this type of notation below.

Let $\mathscr{U}$ be the unit interval $(LN^{-1}, 1+LN^{-1}]$; then clearly

(2.4) $$R(\lambda_1, \ldots, \lambda_7) = \int_{\mathscr{U}} F(\alpha)\, d\alpha.$$

Here

(2.5) $$L = \lambda_1 \lambda_2 \lambda_3 N^{78/345}.$$

When $1 \leqslant a \leqslant q \leqslant L$ and $(a, q) = 1$, we take $\mathfrak{M}(q,a)$ to be the interval $\{\alpha: |\alpha - a/q| \leqslant q^{-1} L N^{-1}\}$, and let $\mathfrak{M}$ denote the union of all such $\mathfrak{M}(q, a)$. The $\mathfrak{M}(q, a)$ are disjoint subsets of $\mathscr{U}$, as we easily verify using (1.6), (2.5). Let

(2.6) $$\mathfrak{m} = \mathscr{U} \setminus \mathfrak{M}.$$

We shall introduce below an approximation $F^*(\alpha)$ to $F(\alpha)$ on $\mathfrak{M}$. We shall proceed to show that

(2.7) $$\int_{\mathfrak{M}} F^*(\alpha)\, d\alpha \gg \lambda_1^{34/345} \Pi^{-1/3-\varepsilon} N^{426/345} (\log N)^{-5}$$

and that

(2.8) $$\int_{\mathfrak{M}} |F^*(\alpha) - F(\alpha)|\, d\alpha \ll \lambda_1^{34/345} \Pi^{-1/3} N^{426/345-\varepsilon}$$

and

(2.9) $$\int_{\mathfrak{m}} F(\alpha)\, d\alpha \ll \lambda_1^{34/345} \Pi^{-1/3} N^{426/345-\varepsilon}.$$

It follows from (2.6)–(2.9) and (2.4) that

(2.10) $$R(\lambda_1, \ldots, \lambda_7) \gg \lambda_1^{34/345} \Pi^{-1/3-\varepsilon} N^{426/345} (\log N)^{-5}.$$

Thus Theorem 2 will follow once (2.7)–(2.9) have been proved.

We recall the standard notations

(2.11) $$S(q, b) = \sum_{r=1}^{q} e\left(\frac{br^3}{q}\right), \qquad J(\beta, A) = \int_A^{2A} e(\beta x^3)\, dx.$$

The estimate

(2.12) $$S(q, b) \ll q^{1+\varepsilon} \psi(q) \qquad \text{for } (q, b) = 1,$$

where $\psi(q)$ is the multiplicative function with

(2.13) $$\psi(p^{3h+r}) = p^{-h-r/2} \qquad (h = 0, 1, \ldots; 0 \leqslant r \leqslant 2)$$

follows from Lemmas 4.3 and 4.4 of [5]. By partial integration,

(2.14) $$J(\beta, A) \ll \frac{A}{1 + A^3 |\beta|}$$

for positive A and real β.

The following lemma enables us to approximate to sums $f_d(X, \lambda\alpha)$. This gives rise naturally to the approximation F^* mentioned above.

LEMMA 2. *Let d be an integer with $\ll 1$ divisors. Let*

$$s_d(q,c) = q^{-1}\sum_{b|d}\mu(b)\frac{S(q,b^3c)}{b}. \tag{2.15}$$

Let λ be a positive integer. Then

$$f_d(X,\lambda\alpha) = s_d(q,\lambda a)J\left(\lambda\left(\alpha-\frac{a}{q}\right),X\right)+O\left(q^{1/2+\varepsilon}\left(1+\lambda X^3\left|\alpha-\frac{a}{q}\right|\right)^{1/2}\right) \tag{2.16}$$

for any $X>0$, real α and rational number a/q.

Proof. It is an immediate consequence of Theorem 2 of [6] that

$$f(A,\alpha) = r^{-1}S(r,t)J\left(\alpha-\frac{t}{r},A\right)+O\left(r^{1/2+\varepsilon}\left(1+A^3\left|\alpha-\frac{t}{r}\right|\right)^{1/2}\right) \tag{2.17}$$

for any rational number t/r with $(t,r)=1$, any real α and $A>0$. We observe that

$$r^{-1}S(r,t) = s^{-1}S(s,tsr^{-1}) \tag{2.18}$$

for a multiple s of r.

The sum $f_d(X,\lambda\alpha)$ may be expressed in the form

$$f_d(X,\lambda\alpha) = \sum_{b|d}\mu(b)f(Xb^{-1},\lambda b^3\alpha). \tag{2.19}$$

We have

$$(Xb^{-1})^3\left|\lambda b^3\alpha-\lambda\frac{b^3a}{q}\right| = \lambda X^3\left|\alpha-\frac{a}{q}\right|. \tag{2.20}$$

Thus

$$f(Xb^{-1},\lambda b^3\alpha) = q^{-1}S(q,\lambda b^3 a)J\left(\lambda b^3\alpha-\frac{\lambda b^3 a}{q},Xb^{-1}\right) +O\left(q^{1/2+\varepsilon}\left(1+\lambda X^3\left|\alpha-\frac{a}{q}\right|\right)^{1/2}\right) \tag{2.21}$$

from (2.16), (2.18) and (2.20). Moreover,

$$J\left(\lambda b^3\alpha-\frac{\lambda b^3 a}{q},Xb^{-1}\right) = b^{-1}J\left(\lambda\left(\alpha-\frac{a}{q}\right),X\right) \tag{2.22}$$

on a simple change of variable. Combining (2.21), (2.22), (2.19) and (2.15), we obtain (2.16). This proves Lemma 2.

For $\alpha\in\mathfrak{M}(q,a)$, we introduce the notations

$$s(p,\alpha) = s_p(q,\lambda_1 a)J\left(\lambda_1\left(\alpha-\frac{a}{q}\right),P\right), \tag{2.23}$$

$$g(p,p_3,p_4,\alpha) = s_{p_3p_4}(q,\lambda_2 p^3 a)J\left(\lambda_2\left(\alpha-\frac{a}{q}\right)p^3,W\right), \tag{2.24}$$

$$h(p,p_i,\alpha) = s_1(q,\lambda_i p^3p_i^3 a)J\left(\lambda_i\left(\alpha-\frac{a}{q}\right)p^3p_i^3,R\right) \quad (i=3,4), \tag{2.25}$$

$$k(p_6,p_7,\alpha) = s_{p_6p_7}(q,\lambda_5 a)J\left(\lambda_5\left(\alpha-\frac{a}{q}\right),U\right), \tag{2.26}$$

$$l(p_i,\alpha) = s_1(q,\lambda_i p_i^3 a)J\left(\lambda_i\left(\alpha-\frac{a}{q}\right)p_i^3,V\right) \quad (i=6,7). \tag{2.27}$$

The natural approximation to $F(\alpha)$ on $\mathfrak{M}$ is now seen to be the function $F^*(\alpha)$, defined on $\mathfrak{M}$ in the following way. For $\alpha\in\mathfrak{M}(q,a)$, let

$$F^*(p,\alpha) = s(p,\alpha)\overline{g(p,p_3,p_4,\alpha)}h(p,p_3,\alpha)h(p,p_4,\alpha)k(p_6,p_7,\alpha) \times l(p_6,\alpha)l(p_7,\alpha), \tag{2.28}$$

and let

$$F^*(\alpha) = \sum_{\substack{p\\(1.10),(1.11)}} F^*(p,\alpha). \tag{2.29}$$

The proofs of (2.8) and (2.7) are along standard lines and are given in Sections 3–4. In Section 5 we state Lemma 12, and show that it implies (2.9). The proof of Lemma 12 occupies the remainder of the paper.

3. Proof of (2.8). It is convenient to write

$$\Delta = \Delta(\alpha,a,q) = 1+N\left|\alpha-\frac{a}{q}\right|. \tag{3.1}$$

LEMMA 3. *Let p satisfy (1.10), (1.11). Let $\alpha\in\mathfrak{M}(q,a)$ where $1\leqslant a\leqslant q\leqslant L$, $(a,q)=1$. Then*

$$f_p(P,\lambda_1\alpha)-s(p,\alpha)\ll q^{1/2+\varepsilon}\Delta^{1/2}, \tag{3.2}$$

$$f_{p_3p_4}(W,\lambda_2 p^3\alpha)-g(p,p_3,p_4,\alpha)\ll q^{1/2+\varepsilon}\Delta^{1/2}, \tag{3.3}$$

$$f(R,\lambda_i p^3p_i^3\alpha)-h(p,p_i,\alpha)\ll q^{1/2+\varepsilon}\Delta^{1/2}, \tag{3.4}$$

$$f_{p_6p_7}(U,\lambda_5\alpha)-k(p_6,p_7,\alpha)\ll q^{1/2+\varepsilon}\Delta^{1/2}, \tag{3.5}$$

and

$$f(V,\lambda_i p_i^3\alpha)-l(p_i,\alpha)\ll q^{1/2+\varepsilon}\Delta^{1/2}. \tag{3.6}$$

Proof. The bound (3.2) is an immediate consequence of (2.16) with $X=P$, $d=p$, $\lambda=\lambda_1$, in view of the definitions (2.1), (2.23), (3.1). Note that, from (1.16),

$$\lambda X^3 = \lambda_1 P^3 \ll N.$$

The inequalities (3.3)–(3.6) are proved similarly.

LEMMA 4. *For any $A > N^\varepsilon$ and natural number r, we have*

(3.7) $$\sum_{A < \omega \leqslant 2A} (r, \omega) \ll Ar^\varepsilon.$$

Proof. The sum in (3.7) is

$$\leqslant \sum_{A<\omega\leqslant 2A} 1 + \sum_{\substack{A<\omega\leqslant 2A \\ \omega | r}} \omega \ll Ar^\varepsilon.$$

LEMMA 5. *Let p be a prime satisfying* (1.10) *and let $\alpha \in \mathfrak{M}(q, a)$. Then*

(3.8) $$\sum_{\substack{p_4,p_5,p_6,p_7 \\ (1.11)}} |F(\mathbf{p}, \alpha) - F^*(\mathbf{p}, \alpha)|$$

$$\ll Z_3 Z_4 Z_6 Z_7 \Pi^{1/3} q^{-3/2+2\delta} (q, p^3) PWRUV^2 \Delta^{-5/2}.$$

Proof. For any natural numbers q and b we have the rather crude bound

$$\psi(q/(q, b^3)) \leqslant b\psi(q) \leqslant bq^{-1/3}$$

from (2.13). Hence, we may deduce from (2.15) and (2.12) that

(3.9) $$s_d(q, c) \ll q^\varepsilon \psi(q/(q, c)) \leqslant (q, c)^{1/3} q^{\varepsilon - 1/3}$$

for a positive integer d with $O(1)$ divisors.

Let

(3.10) $$A_1 = q^{-1/3} P\Delta^{-1}, \quad A_2 = (q, p^3)^{1/3} q^{-1/3} W\Delta^{-1}, \quad A_5 = q^{-1/3} U \Delta^{-1},$$

(3.11) $$A_i = q^{-1/3} (q, p^3)(q, p_i^3)^{1/3} R\Delta^{-1} \quad (i = 3, 4),$$

(3.12) $$A_i = q^{-1/3} (q, p_i^3)^{1/3} V\Delta^{-1} \quad (i = 7, 8).$$

The following estimates are a consequence of the definitions (2.23)–(2.27) and the bounds (3.9) and (2.14):

(3.13) $$s(p, \alpha) \ll \lambda_1^{1/3} q^\varepsilon A_1,$$

(3.14) $$g(p, p_3, p_4, \alpha) \ll \lambda_2^{1/3} q^\varepsilon A_2,$$

(3.15) $$h(p, p_i, \alpha) \ll \lambda_i^{1/3} A_i \quad (i = 3, 4),$$

(3.16) $$k(p_6, p_7, \alpha) \ll \lambda_5^{1/3} q^\varepsilon A_5,$$

(3.17) $$l(p_i, \alpha) \ll \lambda_i^{1/3} q^\varepsilon A_i \quad (i = 6, 7).$$

Here $\alpha \in \mathfrak{M}(q, a)$ with $1 \leqslant a \leqslant q \leqslant L$, $(a, q) = 1$.

It is easy to see from (1.12), (1.13), (1.6) that

(3.18) $$R = \min(P, W, R, U, V),$$

(3.19) $$q^{-1/3} R\Delta^{-1} \geqslant q^{1/2} \quad (1 \leqslant q \leqslant L).$$

For the last inequality we require the observation that

$$q\Delta R^{-1} \ll LR^{-1} \ll \Pi N^{-6/345} \ll N^{-\varepsilon}$$

that may be deduced from (2.5), (1.12), (1.6).

From (3.18), (3.19), (3.10)–(3.12) it follows that the right-hand sides of (3.13)–(3.17) are all $\gg q^{1/2+\varepsilon}$. In conjunction with (3.2)–(3.6) and (2.2), (2.29) this implies that

(3.20) $$F(\mathbf{p}, \alpha) - F^*(\mathbf{p}, \alpha) \ll \sum_{i=1}^{7} q^{1/2+\delta} \Pi^{1/3} \Delta^{7/2} \prod_{\substack{j \leqslant 7 \\ j \neq i}} A_j,$$

whenever $\alpha \in \mathfrak{M}(q, a)$ and $\mathbf{p}$ satisfies (1.10), (1.11).

Let $\varrho_1, \ldots, \varrho_6$ be any subsequence of six of the terms

$$q^{-1/3}, \quad (q, p^3)^{1/3} q^{-1/3}, \quad (q, p^3)^{1/3} (q, p_3^3)^{1/3} q^{-1/3},$$

$$(q, p^3)^{1/3} (q, p_4^3)^{1/3} q^{-1/3}, \quad q^{-1/3}, \quad (q, p_6^3)^{1/3} q^{-1/3}, \quad (q, p_7^3)^{1/3} q^{-1/3}.$$

From (3.7) we see that

(3.21) $$\sum_{\substack{p_3,p_4,p_6,p_7 \\ (1.11)}} \varrho_1 \ldots \varrho_6 \ll Z_3 Z_4 Z_6 Z_7 (q, p^3) q^{-2+\delta}.$$

Moreover, a subproduct of six of P, W, R, R, U, V, V is seen from (3.18) to be $\leqslant PWRUV^2$. Combining this observation with (3.20), (3.21) and (3.10)–(3.12) we obtain (3.8). This completes the proof of Lemma 5.

Proof of (2.8). In view of Lemma 5, we have

(3.22) $$\int_{\mathfrak{M}} |F^*(\alpha) - F(\alpha)| \, d\alpha$$

$$\ll \sum_{\substack{p \\ (1.10), (1.11)}} \sum_{q \leqslant L} \sum_{\substack{a \leqslant q \\ (a,q)=1}} \int_{-\infty}^{\infty} \frac{Z_3 Z_4 Z_6 Z_7 \Pi^{1/3} q^{-3/2+2\delta} (q, p^3) PWRUV^2}{(1+N|\alpha - a/q|)^{5/2}} d\alpha$$

$$\ll Z_3 Z_4 Z_6 Z_7 \Pi^{1/3} PWRUV^2 N^{-1} \sum_{\substack{p \\ (1.10)}} \sum_{q \leqslant L} (q, p^3) q^{-1/2+2\delta}.$$

Moreover,

(3.23) $$\sum_{\substack{p \\ (1.10)}} \sum_{q \leqslant L} (q, p^3) q^{-1/2+2\delta} \ll \sum_{j=0}^{3} \sum_{\substack{p \\ (1.10)}} p^j \sum_{\substack{q \leqslant L \\ p^j | q}} q^{-1/2+2\delta} \ll YL^{1/2+2\delta}.$$

From (2.5), (1.12) and (1.6),

(3.24) $$\Pi^{1/3} L^{1/2+2\delta} \ll RN^{-\varepsilon}.$$

Moreover,

(3.25) $$YZ_3 Z_4 Z_6 Z_7 PWR^2 UV^2 N^{-1} = \lambda_1^{34/345} \Pi^{-1/3} N^{426/345}$$

from (1.12)–(1.15). The bound (2.8) now follows from (3.22)–(3.25).

4. Proof of (2.7). We begin by noting that

$$S(q, cm^3) = S(q, c) \tag{4.1}$$

for $(q, m) = 1$; in particular,

$$S(q, -c) = S(q, c). \tag{4.2}$$

On the other hand, for $m|q$ it is easy to see that

$$S(q, cm^3) = m \sum_{\substack{x=1\\ m|x}}^{q} e\left(\frac{cx^3}{q}\right). \tag{4.3}$$

It is convenient to write

$$S(q, a, h) = \sum_{\substack{x=1\\ (x,q,h)=1}}^{q} e\left(\frac{ax^3}{q}\right) \tag{4.4}$$

where h, q are natural numbers and a is an integer. Note that

$$S(q, a, h) = S(q, a) \quad \text{for } (q, h) = 1. \tag{4.5}$$

Let

$$\chi(q, p) = \begin{cases} 1 & \text{if } p\nmid q, \\ 0 & \text{if } p|q. \end{cases} \tag{4.6}$$

LEMMA 6. *We have*

$$s_p(q, c) = \left(1-\frac{1}{p}\right)^{\chi(q,p)} \frac{S(q, c, p)}{q}. \tag{4.7}$$

For distinct primes p_3, p_4 *we have*

$$s_{p_3p_4}(q, c) = \left(1-\frac{1}{p_3}\right)^{\chi(q,p_3)} \left(1-\frac{1}{p_4}\right)^{\chi(q,p_4)} \frac{S(q, c, p_3p_4)}{q}. \tag{4.8}$$

Proof. We have

$$s_p(q, c) = \frac{1}{q}\left\{S(q, c) - \frac{S(q, cp^3)}{p}\right\}.$$

If $p\nmid q$ the last expression is

$$\frac{1}{q}S(q, c)(1-p^{-1}) = \frac{S(q, c, p)}{q}(1-p^{-1})$$

from (4.1), (4.5); otherwise it is

$$\frac{1}{q}S(q, c, p)$$

from (4.3). This proves (4.7).

Next,

$$s_{p_3p_4}(q, c) = \frac{1}{q}\left\{S(q, c) - \frac{S(q, cp_3^3)}{p_3} - \frac{S(q, cp_4^3)}{p_4} + \frac{S(q, cp_3^3 p_4^3)}{p_3p_4}\right\}. \tag{4.9}$$

If $p_3\nmid q$ the last expression is

$$\frac{1}{q}\left\{S(q, c)\left(1-\frac{1}{p_3}\right) - \frac{S(q, cp_4^3)}{p_4}\left(1-\frac{1}{p_3}\right)\right\} = \left(1-\frac{1}{p_3}\right)\left(1-\frac{1}{p_4}\right)^{\chi(q,p_4)} \frac{S(q, c, p_4)}{q}$$
$$= \left(1-\frac{1}{p_3}\right)\left(1-\frac{1}{p_4}\right)^{\chi(q,p_4)} \frac{S(q, c, p_3p_4)}{q}$$

from (4.7).

Now, let $p_3|q$. By the above, we need only consider the case $p_4|q$. The right-hand side of (4.9) is

$$\frac{1}{q}\left\{\sum_{\substack{x=1\\ p_3\nmid x}}^{q} e\left(\frac{cx^3}{q}\right) - \frac{1}{p_4}\sum_{\substack{x=1\\ p_3\nmid x}}^{q} e\left(\frac{cx^3p_4^3}{q}\right)\right\}$$
$$= \frac{1}{q}\left\{\sum_{\substack{x=1\\ p_3\nmid x}}^{q} e\left(\frac{cx^3}{q}\right) - \sum_{\substack{x=1\\ p_3\nmid x,\, p_4|x}}^{q} e\left(\frac{cx^3}{q}\right)\right\} = \frac{S(q, c, p_3p_4)}{q}$$

from (4.3). This proves Lemma 6.

LEMMA 7. *For coprime positive integers* q, r *we have*

$$S(q, a, h)S(r, b, h) = S(qr, ar+bq, h).$$

Proof. Let $(q, h) = d$, $(r, h) = d'$, so that $(qr, h) = dd'$. In the sum

$$S = \sum_{\substack{t=1\\ (t,d)=1}}^{q} \sum_{\substack{u=1\\ (u,d')=1}}^{r} e\left[\frac{(ar+bq)(tr+uq)^3}{qr}\right], \tag{4.10}$$

the values of $tr+uq$ run over $q\dfrac{\varphi(d)}{d}\dfrac{r\varphi(d')}{d'}$ distinct residue classes $x \pmod{qr}$, all of which have $(x, d) = (x, d') = 1$.

Hence these values x run once over the $qr\varphi(dd')/(dd')$ residue classes $\pmod{qr}$ with $(x, dd') = 1$, that is, with

$$(x, qr, h) = 1.$$

This proves that $S = S(qr, ar+bq, h)$.

Discarding cross terms in (4.10),

$$S(qr, ar+bq, h) = \sum_{\substack{t=1\\ (t,d)=1}}^{q} \sum_{\substack{u=1\\ (u,d')=1}}^{r} e\left[\frac{at^3r^3}{q} + \frac{bq^3u^3}{r}\right]. \tag{4.11}$$

In the last summation tr runs over $q\varphi(d)/d$ values $\pmod q$ coprime to d, and qu runs over $r\varphi(d')/d'$ values $\pmod r$ coprime to d'. The right-hand side of (4.11) is therefore equal to $S(q, a, h)\, S(r, b, h)$, and the lemma is proved.

For the remainder of this section, we write

$$(4.12)\quad S(\mathbf{p}, q) = \frac{1}{q^7} \sum_{\substack{a=1\\(a,q)=1}}^{q} S(q, \lambda_1 a, p) S(q, \lambda_2 a p^3, p_3 p_4) S(q, \lambda_3 a p_3^3) \times S(q, \lambda_4 a p_4^3) S(q, \lambda_5 a, p_6 p_7) S(q, \lambda_6 a p_6^3) S(q, \lambda_7 a p_7^3).$$

In view of (2.23)–(2.29), Lemma 6 and (4.2) we have, for $q \leqslant L, |\beta| \leqslant q^{-1} L N^{-1}$,

$$(4.13)\quad \sum_{a=1}^{q} F^*\left(\mathbf{p}, \beta + \frac{a}{q}\right) = \theta(q, \mathbf{p}) S(\mathbf{p}, q) \varphi(\mathbf{p}, \beta)$$

whenever $\mathbf{p}$ satisfies (1.10), (1.11). Here

$$(4.14)\quad \theta(q, \mathbf{p}) = \prod_{\substack{\omega \in \{p, p_3, p_4, p_6, p_7\}\\ \omega \nmid q}} \left(1 - \frac{1}{\omega}\right)$$

and

$$(4.15)\quad \varphi(\mathbf{p}, \beta) = J(\lambda_1 \beta, P) J(-\lambda_2 p^3 \beta, W) J(\lambda_3 p^3 p_3^3 \beta, R) J(\lambda_4 p^3 p_4^3, R) \times J(\lambda_5 \beta, U) J(\lambda_6 p_6^3 \beta, V) J(\lambda_7 p_7^3 \beta, V).$$

LEMMA 8. *$S(\mathbf{p}, q)$ is a multiplicative function of q.*

Proof. Let q, r be coprime. Then

$$S(\mathbf{p}, qr) = \frac{1}{(qr)^7} \sum_{\substack{a=1\\(a,q)=1}}^{q} \sum_{\substack{b=1\\(b,r)=1}}^{q} S(q, \lambda_1 (ar+bq), p) \ldots S(q, \lambda_7 (ar+bq) p_7^3)$$

$$= \frac{1}{q^7 r^7} \sum_{\substack{a=1\\(a,q)=1}}^{q} \sum_{\substack{b=1\\(b,r)=1}}^{q} S(q, \lambda_1 a, p) S(q, \lambda_1 b, p) \ldots S(q, \lambda_7 a p_7^3) S(q, \lambda_7 b p_7^3)$$

on decomposing each of the seven factors using Lemma 7. Lemma 8 now follows at once.

LEMMA 9. *We have, for any $\mathbf{p}$ satisfying* (1.10), (1.11),

$$(4.16)\quad S(\mathbf{p}, p_i^t) = 0 \quad (i = 3, 4, 6, 7;\ t \geqslant 1);$$

$$(4.17)\quad S(\mathbf{p}, p^t) = 0 \quad (t \geqslant 1).$$

Proof. Since $p_i \equiv 2 \pmod 3$, we have

$$(4.18)\quad S(p_i^{t_i}, a, p_i) = 0 \quad (t_i \geqslant 2,\ p_i \nmid a),$$

$$(4.19)\quad S(p_i, a) = 0 \quad (p_i \nmid a)$$

(cf. [8], p. 166). We recall that $p_i \nmid \lambda_j$ for all $i = 3, 4, 6, 7$ and $j = 1, \ldots, 7$; see (1.18).

Now let $t_i \geqslant 2$. Then we apply (4.18), (1.17) to deduce that the factor $S(p_i^{t_i}, \lambda_2 p^3 a, p_3 p_4)$ in (4.12) vanishes (for $i = 3, 4$) and the factor $S(p_i^{t_i}, \lambda_5 a, p_6 p_7)$ vanishes (for $i = 6, 7$). Hence (4.16) holds for $t_i \geqslant 2$.

As for $S(\mathbf{p}, p_i)$, we deduce from (4.19), (1.17) that $S(p_i, \lambda_j a p_j^3)$ vanishes, where $j \in \{6, 7\}$, $j \neq i$. This completes the proof of (4.16).

For (4.17), we again use (1.17), (4.18) and (4.19). This time,

$$S(p^t, \lambda_1 a, p) = 0 \quad (t \geqslant 2),$$

$$S(p, \lambda_6 a p_6^3) = 0.$$

This yields the desired result. The lemma now follows.

LEMMA 10. *We have*

$$(4.20)\quad \int_{\mathfrak{M}} F^*(\alpha)\, d\alpha = \sum_{\substack{\mathbf{p}\\(1.10),(1.11)}} \prod_{\omega \in \{p, p_3, p_4, p_6, p_7\}} \left(1 - \frac{1}{\omega}\right) \mathfrak{S}_L(\mathbf{p}) I(\mathbf{p}).$$

Here

$$(4.21)\quad \mathfrak{S}_L(\mathbf{p}) = \sum_{\substack{q \leqslant L\\(q, p p_3 p_4 p_6 p_7) = 1}} S(\mathbf{p}, q),$$

$$(4.22)\quad I(\mathbf{p}) = \int_{-LN^{-1}}^{LN^{-1}} \varphi(\mathbf{p}, \beta)\, d\beta.$$

Proof. From (2.29), (4.13) we have

$$(4.23)\quad \int_{\mathfrak{M}} F^*(\alpha)\, d\alpha = \sum_{\substack{\mathbf{p}\\(1.10),(1.11)}} \sum_{q \leqslant L} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{a/q - LN^{-1}}^{a/q + LN^{-1}} F^*(\mathbf{p}, \alpha)\, d\alpha$$

$$= \sum_{\substack{\mathbf{p}\\(1.10),(1.11)}} \sum_{q \leqslant L} \int_{-LN^{-1}}^{LN^{-1}} \sum_{\substack{a=1\\(a,q)=1}}^{q} F^*\left(\mathbf{p}, \beta + \frac{a}{q}\right) d\beta$$

$$= \sum_{\substack{\mathbf{p}\\(1.10),(1.11)}} \sum_{q \leqslant L} \theta(q, \mathbf{p}) S(\mathbf{p}, q) \int_{-LN^{-1}}^{LN^{-1}} \varphi(\mathbf{p}, \beta)\, d\beta.$$

In view of Lemmas 8 and 9, we may insert the summation condition

$$(q, p p_3 p_4 p_6 p_7) = 1$$

in the last summation over q. For these values of q we have

$$(4.24)\quad \theta(q, \mathbf{p}) = \prod_{\omega \in \{p, p_3, p_4, p_6, p_7\}} \left(1 - \frac{1}{\omega}\right).$$

The lemma now follows from (4.21)–(4.24).

LEMMA 11. *For $\mathbf{p}$ satisfying* (1.10), (1.11) *we have*

$$(4.25)\quad \mathfrak{S}_L(\mathbf{p}) \gg \Pi^{-\varepsilon}.$$

Proof. We have

$$\sum_{\substack{q > L\\(q, p p_3 p_4 p_6 p_7) = 1}} |S(\mathbf{p}, q)| \ll \sum_{q > L} \Pi^{1/3} q^{\varepsilon - 4/3} \ll \Pi^{1/3} L^{\varepsilon - 1/3}$$

from (4.12), (4.7), (4.8) and (3.9). Thus

(4.26) $$\mathfrak{S}_L(p) = \mathfrak{S}(p) + O(\Pi^{1/3} L^{\varepsilon-1/3}),$$

where

(4.27) $$\mathfrak{S}(p) = \sum_{q=1}^{\infty} S(p, q) = \prod_{\omega \notin \{p, p_3, p_4, p_6, p_7\}} \chi(\omega)$$

by a standard argument. Here

(4.28) $$\chi(\omega) = 1 + S(p, \omega) + S(p, \omega)^2 + \dots$$

Moreover, for $\omega \notin \{p, p_3, p_4, p_6, p_7\}$ we have

(4.29) $$S(p, \omega^l) = \sum_{\substack{a=1 \\ \omega \nmid a}}^{\omega^l} \frac{S(\omega^l, \lambda_1 a)}{\omega^l} \dots \frac{S(\omega^l, \lambda_7 a)}{\omega^l}$$

from (4.12), (4.4), (4.1). The last expression is, of course, independent of p.

From (1.4), for given ω there are at least three values of $i \leqslant 7$ for which λ_i is not divisible by ω. Thus, for $l = 1, 2$, we deduce from (4.29) and Lemmas 4.3 and 4.4 of [5] that

(4.30) $$S(p, \omega^l) \ll \omega^l \prod_{i=1}^{7} \{\omega^l/(\omega^l, \lambda_i)\}^{-1/2} \ll \omega^l \omega^{-3l/2} \ll \omega^{-1/2}.$$

For $l \geqslant 3$ we appeal to both (1.4) and (1.3). Much as above,

$$S(p, \omega^l) \ll \omega^l \prod_{i=1}^{7} \{\omega^l/(\omega^l, \lambda_i)\}^{-1/3} \ll \omega^{-4l/3} \prod_{i=1}^{7} (\omega^l, \lambda_i)^{1/3} \ll \omega^{-4l/3} \omega^{8/3};$$

$$\sum_{l \geqslant 3} S(p, \omega^l) \ll \omega^{-4/3}.$$

Hence

(4.31) $$|\chi(\omega) - 1| \ll \omega^{-1/2}.$$

This may be strengthened for ω not dividing Π to

(4.32) $$|\chi(\omega) - 1| \ll \omega^{-4/3} \quad (\omega \nmid \Pi)$$

by an obvious variant of (4.30). Combining (4.31) and (4.32), we have, for a suitable absolute constant C_2, the lower bound

(4.33) $$\prod_{\omega \geqslant C_2} \chi(\omega) \gg \Pi^{-\varepsilon}.$$

For the remaining factors ω with $\omega < C_2$, we use the easily verified formula

(4.34) $$1 + S(p, \omega) + \dots + S(p, \omega^l) = \frac{M(\omega^l)}{\omega^{6l}},$$

where $M(\omega^l)$ denotes the number of solutions of the congruence

$$\lambda_1 x_1^3 + \dots + \lambda_7 x_7^3 \equiv 0 \pmod{\omega^l}.$$

Using the work of Lewis [3] almost exactly as in [4], Lemma 10, we find that

(4.35) $$\omega^{-6l} M(\omega^l) \geqslant \omega^{-6} \quad (\omega \neq 3,\ l \geqslant 1)$$

while

(4.36) $$3^{-6l} M(3^l) \geqslant 3^{-36} \quad (l \geqslant 6).$$

Combining (4.33), (4.34)–(4.36) we see that

(4.37) $$\mathfrak{S}(p) \gg \Pi^{-\varepsilon}$$

for all p satisfying (1.10), (1.11). Lemma 11 now follows from (4.26) and (4.37), on taking into account (2.5) and (1.6).

Proof of (2.7). From Lemmas 10 and 11, we have

(4.38) $$\int_{\mathfrak{M}} F^*(\alpha)\, d\alpha \gg \Pi^{-\varepsilon} \sum_{\substack{p \\ (1.10),(1.11)}} I(p).$$

We now show that $\min(I(p), 0) \ll N^{-1} L^{-5} PWR^2 UV^2$, and

(4.39) $$I(p) \gg N^{-1} PWR^2 UV^2$$

for all p satisfying (1.11) and

(4.40) $$6Y/5 < p \leqslant 4Y/3.$$

The lower bound (2.7) follows at once from (4.38) and (4.39), on taking into account the identity (3.25).

To begin with, we apply the upper bound

$$\varphi(p, \beta) \ll PWR^2 UV^2 (1 + N|\beta|)^{-6}$$

which is a consequence of (4.15), (2.14). This yields

(4.41) $$I(p) = \int_{-\infty}^{\infty} \varphi(p, \beta)\, d\beta + O(N^{-1} L^{-5} PWR^2 UV^2).$$

By a change of variables in each of the J-factors in (4.15),

(4.42) $$\varphi(p, \beta) = \frac{1}{\Pi^{1/3} p^3 p_3 p_4 p_6 p_7} K(\beta, N^{1/3}) K(\beta, -B_2) \prod_{j=3}^{7} K(\beta, B_j).$$

Here

(4.43) $$K(\beta, B) = \int_{B^3}^{8B^3} \tfrac{1}{3} y^{-2/3} e(\beta y)\, dy,$$

$$B_2 = \lambda_2^{1/3} pW,\ B_3 = \lambda_3^{1/3} pp_3 R,\ B_4 = \lambda_4^{1/3} pp_4 R,\ B_5 = \lambda_5^{1/3} U,$$
$$B_6 = \lambda_6^{1/3} p_6 V, \quad B_7 = \lambda_7^{1/3} p_7 V. \tag{4.44}$$

Let $B(v)$ denote the set of points $(\alpha_1, \ldots, \alpha_6)$ in $\boldsymbol{R}^6$ with

$$N \leqslant v+\alpha_1-\alpha_2- \ldots -\alpha_6 \leqslant 8N, \tag{4.45}$$

$$B_{j+1}^3 \leqslant \alpha_j \leqslant 8B_{j+1}^3 \quad (j = 1, \ldots, 6). \tag{4.46}$$

Using Fubini's theorem, we find that

$$\Pi^{1/3} p^3 p_3 p_4 p_6 p_7 \varphi(\boldsymbol{p}, \beta) = \int_{-\infty}^{\infty} \psi(v) e(\beta v)\, dv \tag{4.47}$$

where

$$\psi(v) = \int_{B(v)} \frac{1}{3^7} (v+\alpha_1-\alpha_2- \ldots -\alpha_6)^{-2/3} (\alpha_1 \ldots \alpha_6)^{-2/3}\, d\alpha_1 \ldots d\alpha_6$$

is a continuous function with compact support. In view of Fourier's inversion theorem, it follows from (4.47) that

$$\int_{-\infty}^{\infty} \Pi^{1/3} p^3 p_3 p_4 p_6 p_7 \varphi(\boldsymbol{p}, \beta)\, d\beta = \psi(0) \gg (N^7)^{-2/3} \mu(B(0)). \tag{4.48}$$

Here μ denotes Lebesgue measure in $\boldsymbol{R}^6$; we have used (1.16) in the last step.

Let $\boldsymbol{p}$ satisfy (1.11), (4.40). For all $(\alpha_1, \ldots, \alpha_6)$ satisfying (4.46) and

$$B_2^3 \leqslant \alpha_1 \leqslant 9B_2^3/8,$$

it is clear from (1.12)–(1.15) that

$$0 \leqslant \alpha_2 + \ldots + \alpha_6 \leqslant N/5, \quad \left(\frac{6}{5}\right)^3 N \leqslant \alpha_1 \leqslant \frac{9}{8}\left(\frac{4}{3}\right)^3 N,$$

hence

$$N \leqslant \alpha_1 - (\alpha_2 + \ldots + \alpha_6) \leqslant 8N.$$

Thus it is clear that

$$\mu(B(0)) \gg B_2^3 \ldots B_7^3 \gg N^6. \tag{4.49}$$

Combining (4.48), (4.49), (4.41) we have

$$I(\boldsymbol{p}) \gg \Pi^{-1/3} Y^{-3} (Z_3 Z_4 Z_6 Z_7)^{-1} N^{4/3} \gg N^{-1} PWR^2 UV^2$$

in view of (3.25) and (1.12). This establishes (4.39) and completes the proof of (2.7).

5. The minor arcs: preliminary reduction. In the remainder of the paper we use the notations

$$H = C_3 P Y^{-3}, \quad Q = PY^{-1} \tag{5.1}$$

where C_3 is a sufficiently large absolute constant; also

$$M = L(2Y)^{-3}. \tag{5.2}$$

Further, let

$$S(\alpha) = \Bigg| \sum_{\substack{Z_3 < p_3 < 2Z_3 \\ p_3 \equiv 2 \ (\mathrm{mod}\ 3)}} \sum_{\substack{Z_4 < p_4 \leqslant 2Z_4 \\ p_4 \equiv 2 \ (\mathrm{mod}\ 3)}} f_{p_3 p_4}(W, \lambda_2 \alpha) f(R, \lambda_3 p_3^3 \alpha) f(R, \lambda_4 p_4^3 \alpha) \Bigg|^2 \tag{5.3}$$

and

$$\Phi_p(\alpha) = \sum_{\substack{P < y \leqslant 2P \\ p \mid y}} 1 + 2\,\mathrm{Re} \sum_{h \leqslant H} \sum_{\substack{2P + hp^3 < y \leqslant 4P - hp^3 \\ p \nmid y,\ y \equiv h \ (\mathrm{mod}\ 2)}} e\left[\frac{\lambda_1 \alpha}{4}(3hy^2 + h^3 p^6)\right]. \tag{5.4}$$

Let $\mathfrak{n}$ denote the set of real numbers in (0, 1] with the property that whenever $|\alpha - a/q| \leqslant q^{-1} LN^{-1}$ and $(a, q) = 1$ we have $q > M$. Let

$$T = \int_{\mathfrak{n}} \sum_{\substack{p \\ (1.10)}} \Phi_p(\alpha) S(\alpha)\, d\alpha. \tag{5.5}$$

LEMMA 12. *We have*

$$T \ll \{\lambda_1^{-5/138} \lambda_2^{-1/3} \lambda_3^{1/3} \lambda_4^{-2/3} + \lambda_1^{263/230} \lambda_3^{-2/3} \lambda_4^{-2/3}\} N^{454/345 + 6\delta}. \tag{5.6}$$

In the remainder of this section we shall show that (2.9) follows from Lemma 12. Consequently the proof of Theorem 2 will be complete, once we have established this lemma.

Given positive integers μ_1, μ_2, we write $S(\mu_1, \mu_2, B, Z, E)$ for the number of solutions of the equation

$$\mu_1 x_1^3 + \mu_2 p^3 (y_1^3 + y_2^3) = \mu_1 x_2^3 + \mu_2 p^3 (y_3^3 + y_4^3) \tag{5.7}$$

in integers $x_1, x_2, y_1, y_2, y_3, y_4$ and primes p satisfying

$$B < x_1, x_2 \leqslant 2B, \quad p \nmid \mu_1 x_1 x_2, \quad E < y_1, y_2, y_3, y_4 \leqslant 2E, \tag{5.8}$$

$$Z < p \leqslant 2Z, \quad p \equiv 2 \ (\mathrm{mod}\ 3). \tag{5.9}$$

LEMMA 13. *Let $B \geqslant 1$, $Z \geqslant 1$, $E \geqslant 1$. Let μ_1, μ_2 be positive integers,*

$$\mu_2 Z^3 E^3 \ll \mu_1 B^3 \ll \mu_2 Z^3 E^3; \quad \mu_1 \ll B^{10} \tag{5.10}$$

and write $A = BZ^{-3}$. Then

$$S(\mu_1, \mu_2, B, Z, E) \ll ZE^2 B^{1+\varepsilon} + AB^{13/6+\varepsilon} X^{-1/6} + ZB^{5/2+\varepsilon} \mu_1^2 X^{-7/2}$$
$$+ X^{1/2} A^{1/2} B^{1/2+\varepsilon} E (A^{1/2} Z^{1/2} E + ZA^{1/8} E^{9/8} + ZA^{3/8} E). \tag{5.11}$$

Here the parameter X, $1 \leqslant X \leqslant B$, is at our disposal.

Proof. This is Proposition A of [1], with a change of notation.

LEMMA 14. *Let*

$$(5.12)\quad J_1=\int_0^1\Bigg|\sum_{\substack{Z_6<p_6\le 2Z_6\\ p_6\equiv 2 \pmod 3}}\ \sum_{\substack{Z_7<p_7\le 2Z_7\\ p_7\equiv 2 \pmod 3}} f_{p_6p_7}(U,\lambda_5\alpha)f(V,\lambda_6p_6^3\alpha)f(V,\lambda_7p_7^3\alpha)\Bigg|^2 d\alpha.$$

Then

$$(5.13)\qquad J_1\ll \lambda_7^{-2/3}\lambda_5^{-1/3}\lambda_6^{1/3}N^{23/21+\varepsilon}.$$

Proof. From the inequality $|zw|\le\frac12|z|^2+\frac12|w|^2$, we have

$$J_1\le\sum_{k=6}^{7}\int_0^1\Bigg(\sum_{\substack{Z_6<p_6\le 2Z_6\\ p_6\equiv 2 \pmod 3}}\ \sum_{\substack{Z_7<p_7\le 2Z_7\\ p_7\equiv 2 \pmod 3}}\Big|f_{p_6p_7}(U,\lambda_5\alpha)f(V,\lambda_kp_k^3\alpha)^2\Big|\Bigg)^2 d\alpha.$$

Hence, by Cauchy's inequality,

$$(5.14)\quad J_1\ll Z_6Z_7\sum_{k=6}^{7}\int_0^1\sum_{\substack{Z_6<p_6\le 2Z_6\\ p_6\equiv 2 \pmod 3}}\ \sum_{\substack{Z_7<p_7\le 2Z_7\\ p_7\equiv 2 \pmod 3}}|f_{p_6p_7}(U,\lambda_5\alpha)^2f(V,\lambda_kp_k^3\alpha)^4|\,d\alpha$$

$$\ll Z_7^2Z_6S(\lambda_5,\lambda_6,U,Z_6,V)+Z_6^2Z_7S(\lambda_5,\lambda_7,U,Z_7,V).$$

We may apply Lemma 13 to $S(\lambda_5,\lambda_j,U,Z_j,V)$, since (5.10) is a consequence of (1.16). Taking $X=1$ in (5.11), we have $A=UZ_j^{-3}\ll N^{4/21}\lambda_j\lambda_5^{-1/3}$,

$$(5.15)\qquad Z_jV^2U\ll N^{20/21}\lambda_j^{-1/3}\lambda_5^{-1/3},$$

$$(5.16)\qquad AU^{13/6}\ll N^{115/126}\lambda_j\lambda_5^{-19/18},$$

$$(5.17)\qquad Z_jU^{5/2}\lambda_5^2\ll N^{37/42}\lambda_j^{-1/3}\lambda_5^{7/6},$$

$$(5.18)\qquad AU^{1/2}V^2Z_j^{1/2}\ll N^{20/21}\lambda_j^{5/6}\lambda_5^{-1/2},$$

$$(5.19)\qquad A^{5/8}U^{1/2}V^{17/8}Z_j\ll N^{79/84}\lambda_j^{7/24}\lambda_5^{-3/8},$$

$$(5.20)\qquad A^{7/8}U^{1/2}Z_jV^2\ll N^{20/21}\lambda_j^{13/24}\lambda_5^{-11/24}.$$

If $\{k,j\}=\{6,7\}$, it is easy to see that

$$(5.21)\quad \lambda_k^{-2/3}\lambda_j^{-1/3}(\lambda_j^{-1/3}\lambda_5^{-1/3}+\lambda_j^{5/6}\lambda_5^{-1/2}+\lambda_j^{13/24}\lambda_5^{-11/24})$$

$$\ll\lambda_k^{-2/3}\{(\lambda_j\lambda_5^{-1})^{1/3}+(\lambda_j\lambda_5^{-1})^{1/2}+(\lambda_j\lambda_5^{-1})^{11/24}\}$$

$$\ll\lambda_k^{-2/3}(\lambda_j\lambda_5^{-1})^{1/3}\ll(\lambda_6\lambda_5^{-1})^{1/3}\lambda_7^{-2/3}$$

using (1.5). Appealing to (1.5) and (1.6), we also find that

$$(5.22)\quad \lambda_k^{-2/3}\lambda_j^{-1/3}(\lambda_j\lambda_5^{-19/18}N^{-5/126}+\lambda_j^{-1/3}\lambda_5^{7/6}N^{-1/14}+\lambda_j^{7/24}\lambda_5^{-3/8}N^{-1/84})$$

$$\ll\lambda_k^{-2/3}\lambda_j^{1/3}\lambda_5^{-1/3}(\lambda_j^{1/3}N^{-5/126}+\lambda_5^{3/2}N^{-1/14}+1)$$

$$\ll\lambda_k^{-2/3}\lambda_j^{1/3}\lambda_5^{-1/3}\ll\lambda_6^{1/3}\lambda_5^{-1/3}\lambda_7^{-2/3}.$$

Combining (5.11), (5.15)–(5.20) and (5.21)–(5.22), we obtain

$$Z_k^2Z_jS(\lambda_5,\lambda_j,U,Z_j,V)\ll N^{23/21+\varepsilon}\lambda_6^{1/3}\lambda_5^{-1/3}\lambda_7^{-2/3}.$$

In view of (5.14), this completes the proof of Lemma 13.

The next lemma is only slightly different from Lemma 10 of [8].

LEMMA 15. *Let*

$$(5.23)\quad J_2=\int_{\mathfrak m}\Bigg|\sum_{\substack{p\\(1.10)}}\ \sum_{\substack{Z_3<p_3\le 2Z_3\\ p_3\equiv 2 \pmod 3}}\ \sum_{\substack{Z_4<p_4\le 2Z_4\\ p_4\equiv 2 \pmod 3}} f_p(P,\lambda_1\alpha)$$

$$\times f_{p_3p_4}(W,\lambda_2p^3\alpha)f(R,\lambda_3p^3p_3^3\alpha)f(R,\lambda_4p^3p_4^3\alpha)\Bigg|^2 d\alpha.$$

Then

$$(5.24)\qquad J_2\ll YT.$$

Proof. Since the integrand in (5.23) has period 1, we may suppose that $\mathfrak m$ is the set of α in $(0,1]$ such that whenever $|\alpha-a/q|\le q^{-1}LP^{-3}$ and $(a,q)=1$ we have $q>L$.

By Cauchy's inequality and (5.3) we have

$$(5.25)\qquad J_2\ll Y\sum_{\substack{p\\(1.10)}}\int_{\mathfrak m}|f_p(P,\lambda_1\alpha)|^2S(p^3\alpha)\,d\alpha.$$

For a given p satisfying (1.10), we write $D=p^3$. Let

$$\mathfrak n_k=\{\alpha:\ \alpha-k\in\mathfrak n\},\qquad \mathscr B_D=\bigcup_{k=0}^{D-1}\mathfrak n_k,\qquad \mathscr A_D=\{\alpha:\ \alpha D\in\mathscr B_D\}.$$

We show that $\mathfrak m\subset\mathscr A_D$. Let $\alpha\in\mathfrak m$ and choose k so that $0\le k<\alpha D\le k+1\le D$. Then we have $\alpha D-k\in\mathfrak n$. To see this, suppose the contrary. Since $0<\alpha D-k\le 1$ there exist a, q with $(a,q)=1$, $q\le M$, $|\alpha D-k-a/q|\le q^{-1}LN^{-1}$. Thus

$$\left|\alpha-\frac{qk+a}{qD}\right|\le(qD)^{-1}LN^{-1}.$$

Hence there exist b, r with $(b,r)=1$, $|\alpha-b/r|\le r^{-1}LN^{-1}$, and

$$r\le qD\le M(2Y)^3=L.$$

This contradicts the definition of $\mathfrak m$. So $\alpha D-k\in\mathfrak n$, and $\alpha\in\mathscr A_D$. This proves that $\mathfrak m\subset\mathscr A_D$.

It follows that

$$(5.26)\qquad \int_{\mathfrak m}|f_p(P,\lambda_1\alpha)|^2S(p^3\alpha)\,d\alpha\le\int_{\mathscr A_D}|f_p(P,\lambda_1\alpha)|^2S(D\alpha)\,d\alpha.$$

By the change of variables $\beta = D\alpha$ we obtain

$$\int_{\mathscr{A}_D} |f_p(P, \lambda_1\alpha)|^2 S(D\alpha)\,d\alpha = \frac{1}{D}\int_{\mathscr{B}_D} |f_p(P, \lambda_1\beta/D)|^2 S(\beta)\,d\beta$$
$$= \frac{1}{D}\sum_{k=0}^{D-1}\int_{\mathfrak{n}_k} |f_p(P, \lambda_1\beta/D)|^2 S(\beta)\,d\beta$$
$$= \frac{1}{D}\sum_{k=0}^{D-1}\int_{\mathfrak{n}} |f_p(P, \lambda_1(\alpha+k)/D)|^2 S(\alpha+k)\,d\alpha.$$

Thus

$$(5.27)\qquad \int_{\mathscr{A}_D} |f_p(P, \lambda_1\alpha)|^2 S(D\alpha)\,d\alpha = \int_{\mathfrak{n}} \frac{1}{D}\sum_{k=0}^{D-1} |f_p(P, \lambda_1(\alpha+k)/D)|^2 S(\alpha)\,d\alpha.$$

By (2.1),

$$\frac{1}{D}\sum_{k=0}^{D-1}|f_p(P, \lambda_1(\alpha+k)/D)|^2 = \sum_{\substack{P<x_1\leqslant 2P\\ p\nmid x_1}}\ \sum_{\substack{P<x_2\leqslant 2P\\ p\nmid x_2\\ \lambda_1x_1^3\equiv\lambda_1x_2^3 \pmod D}} e\left(\frac{\alpha\lambda_1}{D}(x_1^3-x_2^3)\right).$$

Since $D = p^3$ and $p \equiv 2 \pmod 3$, $p \nmid \lambda_1$ (by (1.18)) the conditions $p\nmid x_1$, $p\nmid x_2$, $\lambda_1 x_1^3 \equiv \lambda_1 x_2^3 \pmod D$ are equivalent to $p\nmid x_1$, $p\nmid x_2$, $x_1 \equiv x_2 \pmod D$. Let $h = (x_2-x_1)/D$, $y = x_2+x_1$. Then the summation conditions are equivalent to $2P < y+hp^3 \leqslant 4P$, $2P < y-hp^3 \leqslant 4P$, $p\nmid y$, $y \equiv h \pmod 2$. Thus the double sum becomes

$$\sum_{h}\ \sum_{\substack{2P+|h|P^3<y\leqslant 4P-|h|p^3\\ p\nmid y,\ y\equiv h \pmod 2}} e\left[\frac{\alpha\lambda_1}{4D}(Dh(3y^2+D^2h^2))\right]$$
$$= \sum_{\substack{P<y\leqslant 2P\\ p\nmid y}} 1 + 2\mathrm{Re}\sum_{h>0}\ \sum_{\substack{2P+hp^3<y\leqslant 4P+hp^3\\ p\nmid y,\ y\equiv h \pmod 2}} e\left[\frac{\lambda_1}{4}(3\alpha hy^2+\alpha h^3p^6)\right].$$

Clearly the innermost summation conditions imply that $h \leqslant H$. Therefore, by (5.4),

$$\frac{1}{D}\sum_{k=0}^{D-1}\left|f_p\left(P, \frac{\lambda_1(\alpha+k)}{D}\right)\right|^2 = \Phi_p(\alpha).$$

Thus, by (5.25), (5.26), (5.27) and (5.5),

$$J_2 \ll Y\int_{\mathfrak{n}}\ \sum_{\substack{p\\ (1.10)}} \Phi_p(\alpha)S(\alpha)\,d\alpha = YT.$$

This completes the proof of Lemma 15.

Proof of (2.9). Suppose for the moment that Lemma 12 is true. We apply Cauchy's inequality, recalling (2.2), (2.3), (5.12) and (5.23); and then use the bounds (5.13) and (5.24). This yields

$$(5.28)\qquad \int_{\mathfrak{m}} F(\alpha)\,d\alpha \ll J_1^{1/2}J_2^{1/2} \ll J_1^{1/2}\,Y^{1/2}\,T^{1/2}$$
$$\ll N^{23/42+17/690+\varepsilon}\lambda_5^{-1/6}\lambda_6^{1/6}\lambda_7^{-1/3}\lambda_1^{-17/690}T^{1/2}.$$

By (5.6) the last expression is

$$(5.29)\qquad \ll N^{2971/2415+4\delta}\lambda_5^{-1/6}\lambda_6^{1/6}\lambda_7^{-1/3}\lambda_1^{-17/690}$$
$$\times\{\lambda_1^{-5/276}\lambda_2^{-1/6}\lambda_3^{1/6}\lambda_4^{-1/3} + \lambda_1^{263/460}\lambda_3^{-1/3}\lambda_4^{-1/3}\}.$$

Two separate calculations are now needed to show that the expression in (5.29) is $O(\lambda_1^{34/345}\Pi^{-1/3}N^{426/345-\varepsilon})$. Firstly, (1.6) yields

$$(5.30)\qquad N^{426/345-2971/2415-5\delta}\Pi^{-1/3} = N^{11/2415-5\delta}\Pi^{-1/3}$$
$$\gg \Pi^{671/2415-1/3-400\delta} \gg \Pi^{-1/18} \gg \lambda_5^{-1/6}\lambda_6^{1/6}\lambda_7^{-1/3}\lambda_2^{-1/6}\lambda_3^{1/6}\lambda_4^{-1/3}\lambda_1^{-13/92}$$

since

$$\lambda_5^{-1/6}\lambda_2^{-1/6}\lambda_3^{1/6} \leqslant (\lambda_5\lambda_2\lambda_3)^{-1/18}, \qquad \lambda_4^{-1/3}\lambda_6^{1/6} \leqslant (\lambda_4\lambda_6)^{-1/18},$$
$$\lambda_7^{-1/3}\lambda_1^{-13/92} \leqslant (\lambda_7\lambda_1)^{-1/18}$$

from (1.5). Similarly, the first expression in (5.30) is

$$(5.31)\qquad \gg \Pi^{-1/18} \gg \lambda_5^{-1/6}\lambda_6^{1/6}\lambda_7^{-1/3}\lambda_1^{619/1380}\lambda_3^{-1/3}\lambda_4^{-1/3},$$

since

$$\lambda_5^{1/6-1/18}\lambda_2^{-1/18}\lambda_3^{1/3-1/18}\lambda_4^{1/3-1/18}\lambda_6^{-1/6-1/18}\lambda_7^{1/3-1/18}\lambda_1^{-619/1380-1/18}$$
$$\geqslant \lambda_2^{1/18}\lambda_3^{5/18}\lambda_6^{1/18}\lambda_7^{5/18}\lambda_1^{-619/1380-1/18} \geqslant \lambda_1^{11/18-619/1380} \geqslant 1.$$

Combining (5.29)–(5.31), we obtain the desired bound (2.9) for the left-hand side of (5.28).

6. Manipulation of the sum T. This section follows §5 of [8] quite closely. Let

$$(6.1)\qquad F(\beta, \gamma; h) = \sum_{\substack{2P<y\leqslant 4P\\ y\equiv h \pmod 2}} e(\tfrac{3}{4}\beta y^2 - \gamma y),$$

$$(6.2)\qquad G_h(\varrho, \sigma) = \sum_{\substack{Y<p\leqslant 2Y\\ p\equiv 2 \pmod 3\\ p\leqslant (2P/h)^{1/3}}} e(\tfrac{1}{4}\varrho p^6 + \sigma p^3),$$

$$(6.3)\qquad F_p(\alpha; h) = \sum_{\substack{2P+hp^3<y\leqslant 4P-hp^3\\ y\equiv h \pmod 2}} e(\tfrac{3}{4}\lambda_1\alpha hy^2),$$

$$(6.4)\qquad \Psi_p(\alpha) = 2\mathrm{Re}\sum_{h\leqslant H} F_p(\alpha; h)\,e(\tfrac{1}{4}\lambda_1\alpha h^3p^6),$$

$$D_p(\alpha; h) = \sum_{\substack{2P/p+hp^2<y\leqslant 4P/p-hp^2 \\ y\equiv h \,(\mathrm{mod}\, 2)}} e\left(\tfrac{3}{4}\lambda_1 \alpha h p^2 y^2\right), \tag{6.5}$$

$$\Xi_p(\alpha) = 2\mathrm{Re} \sum_{h\leqslant H} D_p(\alpha; h)\, e\left(\tfrac{1}{4}\lambda_1 \alpha h^3 p^6\right). \tag{6.6}$$

We form the integrals

$$T_1(p) = \int_{\mathfrak{n}} \Psi_p(\alpha) S(\alpha)\, d\alpha, \tag{6.7}$$

$$T_2(p) = \int_{\mathfrak{n}} \Xi_p(\alpha) S(\alpha)\, d\alpha \tag{6.8}$$

and

$$T_3 = \int_0^1 S(\alpha)\, d\alpha. \tag{6.9}$$

By (5.4),

$$\Phi_p(\alpha) = \Psi_p(\alpha) - \Xi_p(\alpha) + O(P)$$

whence

$$T = \sum_{\substack{p \\ (1.10)}} (T_1(p) - T_2(p)) + O(PYT_3) \tag{6.10}$$

from (5.5).

We estimate T_3 *via* Lemma 13. Just as in (5.14), we have

$$T_3 \ll Z_4^2 Z_3 S(\lambda_2, \lambda_3, W, Z_3, R) + Z_3^2 Z_4 S(\lambda_2, \lambda_4, W, Z_4, R). \tag{6.11}$$

We may apply Lemma 13 to $S(\lambda_2, \lambda_j, W, Z_j, R)$, since (5.10) is a consequence of (1.16). Taking $X = 1$ in (5.11) as before, and recalling (1.12)–(1.14), we now have

$$A = WZ_j^{-3} \ll N^{56/345} \lambda_1^{-34/345} \lambda_2^{-1/3} \lambda_j.$$

Moreover,

$$Z_j R^2 W \ll N^{56/69} \lambda_1^{34/345} \lambda_2^{-1/3} \lambda_j^{-1/3}, \tag{6.12}$$

$$AW^{13/6} \ll N^{7/9} \lambda_1^{17/2070} \lambda_2^{-19/18} \lambda_j, \tag{6.13}$$

$$Z_j W^{5/2} \lambda_2^2 \ll N^{259/345} \lambda_1^{119/690} \lambda_2^{7/6} \lambda_j^{-1/3}, \tag{6.14}$$

$$AW^{1/2} R^2 Z_j^{1/2} \ll N^{56/69} \lambda_1^{-17/345} \lambda_2^{-1/2} \lambda_j^{5/6}, \tag{6.15}$$

$$A^{5/8} W^{1/2} R^{17/8} Z_j \ll N^{553/690} \lambda_1^{17/1380} \lambda_2^{-3/8} \lambda_j^{7/24}, \tag{6.16}$$

$$A^{7/8} W^{1/2} Z_j R^2 \ll N^{56/69} \lambda_1^{-17/1380} \lambda_2^{-11/24} \lambda_j^{13/24}. \tag{6.17}$$

Arguing much as in (5.21), (5.22) we find that

$$Z_k^2 Z_j S(\lambda_2, \lambda_j, W, Z_j, R) \ll N^{322/345+\varepsilon} \lambda_1^{17/69} \lambda_2^{-1/3} \lambda_3^{1/3} \lambda_4^{-2/3}$$

for $\{k, j\} = \{3, 4\}$, and consequently

$$PYT_3 \ll N^{454/345+\varepsilon} \lambda_1^{-47/345} \lambda_2^{-1/3} \lambda_3^{1/3} \lambda_4^{-2/3} \tag{6.18}$$

from (6.11), (1.12).

Let

$$T_5(\gamma, \theta) = \int_{\mathfrak{n}} \sum_{h\leqslant H} \left|F(\alpha\lambda_1 h, \gamma; h)\, G_h(\alpha\lambda_1 h^3, \theta\gamma h)\right| S(\alpha)\, d\alpha. \tag{6.19}$$

By a very minor adaptation of the proof of (5.26) of [8], one finds that

$$\sum_{\substack{p \\ (1.10)}} T_1(p) \ll (\log P) \sup_{0\leqslant\theta\leqslant 1} T_5(\gamma, \theta) + N^{454/345} \Pi^{-1}. \tag{6.20}$$

We omit the details.

In the next section we estimate $T_5(\gamma, \theta)$ and in Section 8 we give a bound for $\sum_{\substack{p \\ (1.10)}} T_2(p)$. We will then be able to combine these results with those of this section to prove Lemma 12.

7. The estimation of $T_5(\gamma, \theta)$. Although this section and the next are very similar to §§ 6 and 7 of [8], it is necessary to give full details because of the powers of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ that enter into the estimates.

LEMMA 16. *Suppose that* $\alpha \in \mathbf{R}$,

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{24qHP}, \quad (a, q) = 1. \tag{7.1}$$

Then

$$\sum_{h\leqslant H} |F(\alpha h, \gamma; h)|^2 \ll P^\delta \left[\frac{q^{-1}HP^2}{1+Q^3|\alpha - a/q|} + HP + q\right]. \tag{7.2}$$

Proof. As in the proof of the Lemma of [7] we have

$$\sum_{h\leqslant H} |F(\alpha h, \gamma; h)|^2 \ll P^\varepsilon \sum_{j\leqslant 12HP} \min(P, \|\alpha j\|^{-1}), \tag{7.3}$$

where $\|\ldots\|$ denotes distance from the nearest integer.

By (7.1), when $j \leqslant 12HP$ and $q \nmid j$ we have

$$\|\alpha j\| = \left\|\frac{aj}{q} + j\left(\alpha - \frac{a}{q}\right)\right\| \geqslant \left\|\frac{aj}{q}\right\| - 12HP\left|\alpha - \frac{a}{q}\right| \geqslant \left\|\frac{aj}{q}\right\| - \frac{1}{2q} \geqslant \frac{1}{2}\left\|\frac{aj}{q}\right\|.$$

Moreover, when $q \mid j$ we have

$$\|\alpha j\| = \left\|j\left(\alpha - \frac{a}{q}\right)\right\| = j\left|\alpha - \frac{a}{q}\right|.$$

Hence

(7.4) $$\sum_{j\leqslant 12HP}\min(P,\|\alpha j\|^{-1})\ll\sum_{r=1}^{q-1}\left(\frac{HP}{q}+1\right)\left\|\frac{ar}{q}\right\|^{-1}+\sum_{k\leqslant 12HP/q}\min\left[P,\frac{1}{kq|\alpha-a/q|}\right]$$
$$\ll P^{\varepsilon}\left[HP+q+\min\left[\frac{HP^2}{q},\frac{1}{q|\alpha-a/q|}\right]\right].$$

Since $HP^2=C_3Q^3$, the lemma now follows on combining (7.3) and (7.4).

LEMMA 17. *Let $\alpha\in\mathbf{R}$, $\gamma\in\mathbf{R}$. Suppose that*
$$\left|\alpha-\frac{a}{q}\right|\leqslant q^{-1}Q^{-3}H^{3/4},\quad (a,q)=1,\quad q\leqslant Q^3H^{-3/4}.$$
Then

(7.5) $$\sum_{h\leqslant H}|G_h(\alpha h^3,\gamma h)|^2\ll P^{\varepsilon}\left[\frac{HY^2q^{-1/3}}{(1+Q^3|\alpha-a/q|)^{1/3}}+H^{3/4}Y^2\right].$$

Proof. This is essentially Lemma 8 of [8]. We are able to quote directly, because

(7.6) $$Y=P^{17/115},\quad Q=P^{98/115},\quad H=C_3P^{64/115}$$

from (1.12), (5.1); and this is just as in [8]. An insignificant difference is that the condition $p\leqslant(P/(2h))^{1/3}$ in (5.25) of [8] has been corrected to $p\leqslant(2P/h)^{1/3}$ in (6.2) of the present paper.

It is convenient to write

(7.7) $$\Gamma=\Gamma(\alpha,a,q)=q^{1/2+\varepsilon}\left(1+NY^{-3}\left|\alpha-\frac{a}{q}\right|\right)^{1/2}.$$

LEMMA 18. *Let $q\geqslant 1$, $(a,q)=1$ and $\alpha\in\mathbf{R}$. Let p_3, p_4 be primes, $Z_j<p_j\leqslant 2Z_j$ $(j=3,4)$. Then*

(7.8) $$S(\alpha)\ll q^{\delta}Z_3^2Z_4^2\left\{\frac{\lambda_2\lambda_3^2\psi(q)^6W^2R^4}{(1+NY^{-3}|\alpha-a/q|)^6}+\lambda_2q^2\psi(q)^2W^2+\Gamma^6\right\}.$$

Proof. By Lemma 2, we have
$$f_{p_3p_4}(W,\lambda_2\alpha)=s_{p_3p_4}(q,\lambda_2a)J\left(\lambda_2\left(\alpha-\frac{a}{q}\right),W\right)+O(\Gamma)$$
and
$$f(R,\lambda_ip_i^3\alpha)=s_1(q,\lambda_ip_i^3\alpha)J\left(\lambda_i\left(\alpha-\frac{a}{q}\right)p_i^3,R\right)+O(\Gamma).$$

(Note that
$$NY^{-3}\ll\lambda_2W^3,\quad \lambda_iZ_i^3R^3\ll NY^{-3}$$
from (1.16).) Applying (3.9), (2.13) and (2.14), we find that

(7.9) $$f_{p_3p_4}(W,\lambda_2\alpha)\ll\frac{q^{\varepsilon}\psi(q/(q,\lambda_2))W}{1+NY^{-3}|\alpha-a/q|}+\Gamma\ll\frac{q^{\varepsilon}\lambda_2^{1/2}\psi(q)W}{1+NY^{-3}|\alpha-a/q|}+\Gamma$$

and

(7.10) $$f(R,\lambda_ip_i^3\alpha)\ll\frac{q^{\varepsilon}\psi(q/(q,\lambda_ip_i^3))R}{1+NY^{-3}|\alpha-a/q|}+\Gamma\ll\frac{q^{\varepsilon}\lambda_i^{1/2}(q,p_i)\psi(q)R}{1+NY^{-3}|\alpha-a/q|}+\Gamma$$

for $i=3,4$.

Combining (5.3), (7.9) and (7.10), we have

(7.11) $S(\alpha)^{1/2}$
$$\ll q^{3\varepsilon}\sum_{Z_3<p_3\leqslant 2Z_3}\sum_{Z_4<p_4\leqslant 2Z_4}\left[\frac{\lambda_2^{1/2}\psi(q)W}{1+NY^{-3}|\alpha-a/q|}+\Gamma\right]\left[\frac{\lambda_3^{1/2}(q,p_3)\psi(q)R}{1+NY^{-3}|\alpha-a/q|}+\Gamma\right]$$
$$\times\left[\frac{\lambda_4^{1/2}(q,p_4)\psi(q)R}{1+NY^{-3}|\alpha-a/q|}+\Gamma\right]$$
$$\ll q^{5\varepsilon}Z_3Z_4\left[\frac{\lambda_2^{1/2}\psi(q)W}{1+NY^{-3}|\alpha-a/q|}+\Gamma\right]\left[\frac{\lambda_3\psi(q)^2R^2}{(1+NY^{-3}|\alpha-a/q|)^2}+\Gamma^2\right].$$

For the last step we require (3.7), (1.5).

Suppose for a moment that

(7.12) $$\Gamma>\frac{\lambda_2^{1/2}\psi(q)W}{1+NY^{-3}|\alpha-a/q|}.$$

Then clearly, from (1.5), (3.18),
$$\Gamma^2>\frac{\lambda_3\psi(q)^2R^2}{(1+NY^{-3}|\alpha-a/q|)^2}.$$

Thus (7.11) yields
$$S(\alpha)^{1/2}\ll q^{5\varepsilon}Z_3Z_4\Gamma^3,$$

which implies (7.8).

Now suppose (7.12) is false. Then
$$S(\alpha)^{1/2}\ll\frac{q^{5\varepsilon}Z_3Z_4\lambda_2^{1/2}\psi(q)W}{1+NY^{-3}|\alpha-a/q|}\left\{\frac{\lambda_3\psi(q)^2R^2}{(1+NY^{-3}|\alpha-a/q|)^2}+\Gamma^2\right\}.$$

Again, (7.8) follows. This completes the proof of Lemma 18.

LEMMA 19. *Let $\gamma\in\mathbf{R}$, $\theta\in\{-1,1\}$. Let*

$$X = \lambda_1^{1/5}. \tag{7.13}$$

Then

$$T_5(\gamma, \theta) \ll \{X^{1/2}\lambda_1^{-47/345}\lambda_2^{-1/3}\lambda_3^{1/3}\lambda_4^{-2/3} + X^{-9/2}\lambda_1^{47/23}\lambda_3^{-2/3}\lambda_4^{-2/3}\} N^{454/345+5\delta}. \tag{7.14}$$

Proof. Let $\mathfrak{n}_1$ denote the set of α in $\mathfrak{n}$ with the property that whenever

$$\left|\lambda_1\alpha - \frac{b}{r}\right| \leqslant r^{-1}H^{7/4}Q^{-3}X^{-1}, \quad (b, r) = 1, \tag{7.15}$$

one has $r > X^{-1}H^{7/4}$. Let $\alpha \in \mathfrak{n}_1$. We apply Lemma 16 with $\lambda_1\alpha$ in place of α. By Dirichlet's theorem there exist integers b, r satisfying (7.15) with $r \leqslant Q^3H^{-7/4}X$; here we must have $r > X^{-1}H^{7/4}$. Now the condition (7.1) is easily verified, since $Q^3H^{-7/4}X \gg HP^{1+\varepsilon}$ from (7.6). Thus

$$\sum_{h \leqslant H} |F(\alpha h\lambda_1, \gamma; h)|^2 \ll P^\delta(P^2H^{-3/4}X + HP + Q^3H^{-7/4}X) \ll P^{2+\delta}H^{-3/4}X \tag{7.16}$$

by (7.6).

We may also choose c, s so that $(c, s) = 1$, $s \leqslant Q^3H^{-3/4}$ and

$$\left|\lambda_1\alpha - \frac{c}{s}\right| \leqslant s^{-1}H^{3/4}Q^{-3}.$$

Then $|\lambda_1\alpha - c/s| \leqslant s^{-1}H^{7/4}Q^{-3}X^{-1}$, so that $s > H^{7/4}X^{-1} > H^{3/4}$. Here we appeal to (5.1), (1.12), (1.5). By Lemma 17, applied to $\lambda_1\alpha$ in place of α,

$$\sum_{h \leqslant H} |G_h(\lambda_1\alpha h^3, \theta\gamma h)|^2 \ll P^\varepsilon H^{3/4}Y^2.$$

Hence by Cauchy's inequality, (7.16), (6.9) and (6.18),

$$\int_{\mathfrak{n}} \sum_{h \leqslant H} |F(\alpha h\lambda_1, \gamma; h)\, G_h(\lambda_1\alpha h^3, \theta\gamma h)|\, S(\alpha)\, d\alpha \ll X^{1/2}P^{1+\delta}YT_3 \ll X^{1/2}N^{454/345+2\delta}\lambda_1^{-47/345}\lambda_2^{-1/3}\lambda_3^{1/3}\lambda_4^{-2/3}. \tag{7.17}$$

It remains to consider $\mathfrak{n}\backslash\mathfrak{n}_1$. Let $\alpha \in \mathfrak{n}\backslash\mathfrak{n}_1$. There are integers b, r satisfying

$$\left|\lambda_1\alpha - \frac{b}{r}\right| \leqslant r^{-1}H^{7/4}Q^{-3}X^{-1}, \quad (b, r) = 1, \quad r \leqslant X^{-1}H^{7/4}. \tag{7.18}$$

We write $b/\lambda_1 r = a/q$ in lowest terms. Clearly $q = rd$ where $d = \lambda_1/(\lambda_1, b)$. Also

$$\left(r, \frac{\lambda_1}{d}\right) = (r, (\lambda_1, b)) = 1.$$

Because $\alpha \in \mathfrak{n}$, we know that either

$$\left|\alpha - \frac{a}{rd}\right| > r^{-1}d^{-1}LN^{-1},$$

or $rd > M$, or both.

Let d be a given divisor of λ_1. Let

$$\mathfrak{N}_1(d, r, a) = \left\{\alpha\colon \left|\alpha - \frac{a}{rd}\right| \leqslant r^{-1}d^{-1}LN^{-1}\right\}. \tag{7.19}$$

It is clear from (2.5), (5.1), (1.12), (1.6) that the interval

$$\mathfrak{N}_2(d, r, a) = \left\{\alpha\colon \left|\alpha - \frac{a}{rd}\right| \leqslant r^{-1}H^{3/4}Q^{-3}\lambda_1^{-1}\right\} \tag{7.20}$$

contains $\mathfrak{N}_1(d, r, a)$. Similarly, the interval

$$\mathfrak{N}_3(d, r, a) = \left\{\alpha\colon \left|\alpha - \frac{a}{rd}\right| \leqslant r^{-1}H^{7/4}Q^{-3}\lambda_1^{-1}X^{-1}\right\} \tag{7.21}$$

contains $\mathfrak{N}_2(d, r, a)$.

Let $\mathfrak{N}_1(d)$ denote the union of the $\mathfrak{N}_1(d, r, a)$ with

$$1 \leqslant a \leqslant rd, \quad (a, rd) = \left(r, \frac{\lambda_1}{d}\right) = 1, \tag{7.22}$$

$$r \leqslant Md^{-1}. \tag{7.23}$$

Let $\mathfrak{N}_2(d)$ denote the union of the $\mathfrak{N}_2(d, r, a)$ with (7.22) and $r \leqslant H^{3/4}$. Let $\mathfrak{N}_3(d)$ denote the union of the $\mathfrak{N}_3(d, r, a)$ with (7.22) and $r \leqslant X^{-1}H^{7/4}$. By the discussion following (7.18) we have, modulo one,

$$\mathfrak{n}\backslash\mathfrak{n}_1 \subset \bigcup_{d|\lambda_1} \{\mathfrak{N}_3(d)\backslash\mathfrak{N}_1(d)\}.$$

Since $Md^{-1} \leqslant LY^{-3} \leqslant H^{3/4}$ from (5.2), (2.5), (1.12), (5.1), (1.6), we have $\mathfrak{N}_1(d) \subset \mathfrak{N}_2(d)$. Similarly $\mathfrak{N}_2(d) \subset \mathfrak{N}_3(d)$. Thus, modulo one,

$$\mathfrak{n}\backslash\mathfrak{n}_1 \subset \bigcup_{d|\lambda_1} \{\mathfrak{N}_3(d)\backslash\mathfrak{N}_2(d)\} \cup \{\mathfrak{N}_2(d)\backslash\mathfrak{N}_1(d)\}. \tag{7.24}$$

Let $\mathscr{L}(d, r, a)$ denote $\mathfrak{N}_3(d, r, a)$ when $H^{3/4} < r \leqslant X^{-1}H^{7/4}$ and (7.22) holds; and denote $\mathfrak{N}_3(d, r, a)\backslash\mathfrak{N}_2(d, r, a)$ when $r \leqslant H^{3/4}$ and (7.22) holds. For fixed d, we have

$$\mathfrak{N}_3(d)\backslash\mathfrak{N}_2(d) = \bigcup_{\substack{r \leqslant X^{-1}H^{7/4};\, a \\ (7.22)}} \mathscr{L}(d, r, a). \tag{7.25}$$

Let $\alpha \in \mathscr{L}(d, r, a)$. The fraction $\lambda_1 a/rd$ may be written in the form b/r with $(b, r) = 1$, since

$$(\lambda_1 a, rd) = d\left(\frac{\lambda_1}{d}. a, r\right) = d$$

from (7.22). Lemma 16, with $\lambda_1\alpha$ in place of α, is applicable since $(b, r) = 1$ and

$$(7.26)\qquad \left|\lambda_1\alpha-\frac{b}{r}\right| = \lambda_1\left|\alpha-\frac{a}{rd}\right| \leqslant r^{-1}H^{7/4}Q^{-3}X^{-1} \leqslant 1/(24rHP)$$

from (7.6). Thus

$$(7.27)\qquad \sum_{h\leqslant H}|F(\lambda_1\alpha h,\gamma;h)|^2 \ll P^{\delta}\left\{\frac{r^{-1}HP^2}{1+Q^3\left|\lambda_1\alpha-\frac{b}{r}\right|}+HP+r\right\} \ll \frac{r^{-1}HP^{2+\delta}}{1+NY^{-3}\left|\alpha-\frac{a}{rd}\right|}.$$

Here we require the observations that

$$Q^3\lambda_1 = P^3Y^{-3}\lambda_1 = NY^{-3},\qquad HP+r \ll HP \ll r^{-1}HP^2,$$

$$(HP+r)rQ^3\left|\lambda_1\alpha-\frac{b}{r}\right| \ll HPH^{7/4} \ll HP^2.$$

Choose c, s so that $|\lambda_1\alpha-c/s| \leqslant s^{-1}H^{3/4}Q^{-3}$, $(c, s) = 1$, $s \leqslant Q^3H^{-3/4}$. If $b/r = c/s$, then by the definition of $\mathscr{L}(d, r, a)$, we have $s > H^{3/4}$. If $b/r \neq c/s$, then

$$\frac{1}{rs} \leqslant \left|\frac{b}{r}-\frac{c}{s}\right| \leqslant (H^{3/4}s^{-1}+H^{7/4}r^{-1})Q^{-3} \leqslant \frac{1}{2sr}+H^{7/4}r^{-1}Q^{-3}$$

from (7.26), whence $s > \frac{1}{2}Q^3H^{-7/4} > H^{3/4}$ once more. Now we may apply Lemma 17 with $\lambda_1\alpha$ in place of α, obtaining

$$\sum_{h\leqslant H}|G_h(\lambda_1\alpha h^3, \theta\gamma h)|^2 \ll Y^2H^{3/4}P^{\varepsilon}.$$

Therefore, by (7.27) and Cauchy's inequality,

$$\sum_{h\leqslant H}|F(\lambda_1\alpha h, \gamma; h)\,G_h(\lambda_1\alpha h^3, \theta\gamma h)| \ll \frac{YH^{7/8}P^{1+\delta}r^{-1/2}}{[1+NY^{-3}|\alpha-a/rd|]^{1/2}}.$$

It follows that

$$(7.28)\qquad \int_{\mathfrak{N}_3(d)\setminus\mathfrak{N}_2(d)}\sum_{h\leqslant H}|F(\lambda_1\alpha h, \gamma; h)\,G_h(\lambda_1\alpha h^3, \theta\gamma h)|\,S(\alpha)\,d\alpha$$

$$\ll \sum_{\substack{r\leqslant X^{-1}H^{7/4}\\ (7.22)}}\ \sum_{a\leqslant rd}\ \int_{\mathscr{L}(d,r,a)}\frac{YH^{7/8}P^{1+\delta}r^{-1/2}S(\alpha)}{[1+NY^{-3}|\alpha-a/rd|]^{1/2}}\,d\alpha.$$

By Lemma 18, for $(a, rd) = 1$ we have

$$\int_{\mathscr{L}(d,r,a)}\frac{S(\alpha)}{(1+NY^{-3}|\alpha-a/rd|)^{1/2}}\,d\alpha$$

$$\ll (rd)^{2\delta}Z_3^2Z_4^2\int_{\mathscr{L}(d,r,a)}\left\{\frac{\lambda_2\lambda_3^2\,\psi(r)^6\,W^2R^4}{(1+NY^{-3}|\alpha-a/rd|)^{13/2}}+\frac{\lambda_2W^2r^2d^2\,\psi(r)^2}{(1+NY^{-3}|\alpha-a/rd|)^{1/2}}+r^3d^3\left(1+NY^{-3}\left|\alpha-\frac{a}{rd}\right|\right)^{5/2}\right\}d\alpha.$$

The integral here is

$$\ll \lambda_2\lambda_3^2N^{-1}Y^3W^2R^4\,\psi(r)^6\min\left(1,\ (rH^{-3/4})^{11/2}\right)$$
$$+\lambda_2W^2\,\psi(r)^2r^2d^2N^{-1}Y^3(r^{-1}X^{-1}H^{7/4})^{1/2}+r^3d^3N^{-1}Y^3(r^{-1}X^{-1}H^{7/4})^{7/2}.$$

Hence the left-hand side of (7.28) is

$$\ll YZ_3^2Z_4^2P^{1+4\delta}H^{7/8}\Big\{\lambda_2\lambda_3^2N^{-1}Y^3W^2R^4d\sum_{r\leqslant X^{-1}H^{7/4}}r^{1/2}\psi(r)^6\times\min\left(1,\ (rH^{-3/4})^{11/2}\right)$$
$$+d^3\lambda_2W^2N^{-1}Y^3X^{-1/2}H^{7/8}\sum_{r\leqslant X^{-1}H^{7/4}}r^2\,\psi(r)^2+d^4N^{-1}Y^3(X^{-1}H^{7/4})^{9/2}\Big\}.$$

From [8], (6.24)–(6.26) we have

$$(7.29)\qquad \sum_{q>\lambda}q^{1/2+\theta}\,\psi(q)^6 \ll \lambda^{\theta-7/6}\quad \text{when } \theta < 7/6,$$

$$(7.30)\qquad \sum_{q\leqslant\lambda}q^{1/2+\theta}\,\psi(q)^6 \ll \lambda^{\theta-7/6}\quad \text{when } \theta > 7/6,$$

$$(7.31)\qquad \sum_{q\leqslant\mu}q^2\,\psi(q)^2 \ll \mu^2.$$

Thus the left-hand side of (7.28) is

$$(7.32)\qquad \ll YZ_3^2Z_4^2P^{1+4\delta}H^{7/8}\{\lambda_1\lambda_2\lambda_3^2N^{-1}Y^3W^2R^4(H^{3/4})^{-7/6}$$
$$+\lambda_1^3\lambda_2W^2N^{-1}Y^3X^{-5/2}H^{35/8}+\lambda_1^4N^{-1}Y^3X^{-9/2}H^{63/8}\}$$
$$\ll \lambda_1^{47/23}\lambda_3^{-2/3}\lambda_4^{-2/3}X^{-9/2}N^{454/345+4\delta}.$$

The third term is dominant here, as one easily verifies by an appeal to (7.6), (1.12)–(1.14), (1.6).

Let $\mathscr{R}(d, r, a)$ denote $\mathfrak{N}_2(d, r, a)$ when $Md^{-1} < r \leqslant H^{3/4}$ and (7.22) holds; and denote $\mathfrak{N}_2(d, r, a)\setminus\mathfrak{N}_1(d, r, a)$ when $r \leqslant Md^{-1}$ and (7.22) holds. For fixed d,

$$(7.33)\qquad \mathfrak{N}_2(d)\setminus\mathfrak{N}_1(d) = \bigcup_{\substack{r\leqslant H^{3/4};a\\ (7.22)}}\mathscr{R}(d, r, a).$$

Let $\alpha\in\mathscr{R}(d, r, a)$. The inequality (7.27) holds, because (7.26) is evidently satisfied with $(b, r) = 1$. Moreover, by Lemma 17 applied to $\lambda_1\alpha$ in place of α,

$$\sum_{h\leqslant H}|G_h(\lambda_1\alpha h^3, \theta\gamma h)|^2 \ll \frac{P^{\varepsilon}HY^2r^{-1/3}}{(1+NY^{-3}|\alpha-a/rd|)^{1/3}}.$$

Therefore, by Cauchy's inequality,

$$\sum_{h\leqslant H}|F(\lambda_1\alpha h, \gamma; h)\,G_h(\lambda_1\alpha h^3, \theta\gamma h)| \ll \frac{YHP^{1+\delta}r^{-2/3}}{(1+NY^{-3}|\alpha-a/rd|)^{2/3}}.$$

Hence

$$\int_{\mathfrak{N}_2(d)\setminus\mathfrak{N}_1(d)} \sum_{h\leqslant H} |F(\lambda_1\alpha h, \gamma; h)\, G_h(\lambda_1\alpha h^3, \theta\gamma h)|\, S(\alpha)\, d\alpha \tag{7.34}$$

$$\ll \sum_{\substack{r\leqslant H^{3/4}\\ (7.22)}} \sum_{a\leqslant rd} \int_{\mathfrak{R}(d,r,a)} \frac{YHP^{1+\delta} S(\alpha) r^{-2/3}}{(1+NY^{-3}|\alpha - a/rd|)^{2/3}}\, d\alpha.$$

By Lemma 18,

$$\int_{\mathfrak{R}(d,r,a)} \frac{S(\alpha)}{(1+NY^{-3}|\alpha - a/rd|)^{2/3}}\, d\alpha$$

$$\ll (rd)^{2\delta} Z_3^2 Z_4^2 \int_{\mathfrak{R}(d,r,a)} \left\{ \frac{\lambda_2\lambda_3^2 \psi(r)^6 W^2 R^4}{(1+NY^{-3}|\alpha - a/rd|)^{20/3}} + \frac{\lambda_2 r^2 d^2 \psi(r)^2 W^2}{(1+NY^{-3}|\alpha - a/rd|)^{2/3}} + r^3 d^3 \left(1+NY^{-3}\left|\alpha - \frac{a}{rd}\right|\right)^{7/3} \right\} d\alpha.$$

The integral here is

$$\ll \lambda_2\lambda_3^2 N^{-1} Y^3 W^2 R^4 \psi(r)^6 \min(1, (rdM^{-1})^{17/3})$$
$$+ d^2 \lambda_2 W^2 N^{-1} Y^3 r^2 \psi(r)^2 (r^{-1}H^{3/4})^{1/3} + r^3 d^3 N^{-1} Y^3 (r^{-1}H^{3/4})^{10/3}.$$

Therefore the left-hand side of (7.34) is

$$\ll YZ_3^2 Z_4^2 HP^{1+4\delta} \times \{ d\lambda_2\lambda_3^2 N^{-1} Y^3 W^2 R^4 \sum_{r\leqslant H^{3/4}} r^{1/3} \psi(r)^6 \min(1, (rd/M)^{17/3}) + d^3\lambda_2 W^2 N^{-1} Y^3 H^{1/4} \sum_{r\leqslant H^{3/4}} r^2 \psi(r)^2 + d^4 N^{-1} Y^3 H^{13/4} \} \tag{7.35}$$

$$\ll YZ_3^2 Z_4^2 HP^{1+4\delta} \{ d^{7/3} \lambda_2\lambda_3^2 N^{-1} Y^3 W^2 R^4 M^{-4/3} + d^3\lambda_2 W^2 N^{-1} Y^3 H^{7/4} + d^4 N^{-1} Y^3 H^{13/4} \}.$$

The second and third summands in the last expression are easily seen to contribute

$$\ll \lambda_3^{-2/3} \lambda_4^{-2/3} \lambda_1^{47/23} X^{-9/2} N^{454/345+4\delta} \tag{7.36}$$

by an appeal to (7.6), (1.12)–(1.14), (1.6). The first summand contributes

$$\ll \lambda_1^{132/69} \lambda_2^{1/3} \lambda_3^{4/3} \lambda_4^{-2/3} N^{490/345+4\delta} M^{-4/3} \tag{7.37}$$
$$\ll \lambda_3^{-2/3} \lambda_4^{-2/3} \lambda_1^{47/23} X^{-9/2} N^{454/345+4\delta}$$

by the definitions of M, L in (5.2), (2.5).

We combine the estimates (7.32), (7.36) and (7.37) and sum over all divisors d of λ_1. In view of (7.24) we have

$$\int_{\mathfrak{n}\setminus\mathfrak{n}_1} \sum_{h\leqslant H} |F(\lambda_1\alpha h, \gamma; h)\, G_h(\lambda_1\alpha h^3, \theta\gamma h)|\, S(\alpha)\, d\alpha \ll \lambda_1^{47/23} \lambda_3^{-2/3} \lambda_4^{-2/3} X^{-9/2} N^{454/345+5\delta}.$$

In conjunction with (7.17) this establishes (7.14), and the proof of Lemma 19 is complete.

8. The estimation of T_2. Let

$$\Omega_p(\beta) = \sum_{h\leqslant H} \Big| \sum_{\substack{2P/p + hp^2 < y \leqslant 4P/p - hp^2\\ y\equiv h \pmod 2}} e(\tfrac{3}{4}\beta h y^2) \Big|^2. \tag{8.1}$$

By (6.5), (6.6) and Cauchy's inequality,

$$|\Xi_p(\alpha)|^2 \leqslant 4H\Omega_p(\lambda_1\alpha p^2). \tag{8.2}$$

LEMMA 20. *Suppose that* $\alpha\in\mathbf{R}$, *that* p *satisfies* (1.10), *and that*

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{p}{24qHP}, \quad (a, q) = 1. \tag{8.3}$$

Then

$$\Omega_p(\alpha) \ll P^\delta \left(\frac{q^{-1}HQ^2}{1+HQ^2|\alpha - a/q|} + HQ + q \right). \tag{8.4}$$

Proof. This is a variant of Lemma 16. We have

$$\Omega_p(\alpha) \ll P^\varepsilon \sum_{j\leqslant 12HPp^{-1}} \min(P, \|\alpha j\|)^{-1}.$$

The proof now goes through as before with P replaced by Pp^{-1}, so that (8.3) plays the role of the inequality (7.1). Since $Pp^{-1} \ll Q$ from (1.10), (5.1), we obtain the bound (8.4) in place of (7.2). This completes the proof of Lemma 20.

LEMMA 21. *We have*

$$\sum_{\substack{p\\(1.10)}} T_2(p) \ll \{ \lambda_1^{-47/345} \lambda_2^{-1/3} \lambda_3^{1/3} \lambda_4^{-2/3} + X^{-9/2} \lambda_1^{47/23} \lambda_3^{-2/3} \lambda_4^{-2/3} \} N^{454/345+5\delta}. \tag{8.5}$$

Proof. Let $\alpha\in\mathfrak{n}$ and suppose p satisfies (1.10). Choose $u = u(p)$, $v = v(p)$ so that

$$\left|\lambda_1\alpha p^2 - \frac{u}{v}\right| \leqslant v^{-1}HP^{-2}, \quad (u, v) = 1, \quad v \leqslant P^2H^{-1}. \tag{8.6}$$

Let $\mathscr{S}(\alpha)$ be the set of p satisfying (1.10) for which

$$v = v(p) > H^2 Y^{-2}.$$

We observe that

$$HP^{-2} \leqslant p/(24HP) \tag{8.7}$$

from (7.6), (1.10). Thus we may apply Lemma 20 with $\lambda_1\alpha p^2$ in place of α. In view of (7.6), this gives

$$(8.8)\quad \Omega_p(\lambda_1\alpha p^2) \ll P^\delta\left[\frac{HQ^2}{H^2Y^{-2}}+HQ+P^2H^{-1}\right] \ll P^{2+\delta}H^{-1} \qquad (p\in\mathscr{S}(\alpha)).$$

Combining (8.2), (8.8), (6.9), (6.18) we have

$$(8.9)\qquad \int_{\mathfrak{n}}\sum_{p\in\mathscr{S}(\alpha)} \Xi_p(\alpha)S(\alpha)d\alpha \ll \lambda_1^{-47/345}\lambda_2^{-1/3}\lambda_3^{1/3}\lambda_4^{-2/3}N^{454/345+2\delta}.$$

Now let $\alpha\in\mathfrak{n}$ and consider a prime p with

$$(8.10)\qquad Y<p\leqslant 2Y,\quad p\equiv 2\ (\mathrm{mod}\ 3),\quad p\notin\mathscr{S}(\alpha).$$

Then $v=v(p)$, $u=u(p)$ satisfy

$$(8.11)\qquad \left|\lambda_1\alpha p^2-\frac{u}{v}\right|\leqslant v^{-1}HP^{-2},\quad (u,v)=1,\quad v\leqslant H^2Y^{-2}.$$

Choose $b=b(\alpha)$, $r=r(\alpha)$ so that

$$\left|\lambda_1\alpha-\frac{b}{r}\right|\leqslant r^{-1}(8H^2)^{-1},\quad (b,r)=1,\quad r\leqslant 8H^2.$$

Then

$$|ur-bvp^2| = rvp^2\left|\frac{u}{vp^2}-\frac{b}{r}\right|$$
$$\leqslant rHP^{-2}+vp^2(8H^2)^{-1}\leqslant 8H^3P^{-2}+4vY^2(8H^2)^{-1}<1$$

from (7.6), (8.11). Thus

$$(8.12)\qquad \frac{u}{vp^2}=\frac{b}{r}$$

whence

$$(8.13)\qquad r=vt\quad\text{with } t\,|\,p^2.$$

Moreover, applying Lemma 20 as before,

$$(8.14)\qquad \Omega_p(\lambda_1\alpha p^2)\ll P^\delta\left[\frac{v^{-1}HQ^2}{1+HQ^2|\lambda_1\alpha p^2-u/v|}+HQ+v\right]$$
$$\ll P^\delta\frac{v^{-1}HQ^2}{1+HQ^2|\lambda_1\alpha p^2-u/v|}$$

by (8.11), (7.6). Combining (8.14), (8.2) we have

$$\Xi_p(\lambda_1\alpha p^2)\ll\frac{P^\delta HQ}{(v(1+HQ^2|\lambda_1\alpha p^2-u/v|))^{1/2}}.$$

In view of (8.12), (8.13) we have

$$(8.15)\qquad \sum_{\substack{p\\(8.10)}}\Xi_p(\lambda_1\alpha p^2)\ll\frac{P^\delta HQ}{\{r(1+HP^2|\lambda_1\alpha-b/r|)\}^{1/2}}\times\Big\{Y+\sum_{\substack{p\\(8.10);\,t(p)=p}}t^{1/2}+\sum_{\substack{p\\(8.10);\,t(p)=p^2}}t^{1/2}\Big\}$$
$$\ll\frac{P^{2\delta}HQY}{\{r(1+NY^{-3}|\alpha-b/r\lambda_1|)\}^{1/2}}.$$

For the second bound in (8.15) we observe that if $t(p)>1$, then $p\,|\,r$.

Let $\mathfrak{n}_2$ denote the set of α in $\mathfrak{n}$ with the property that

$$r(\alpha)\left(1+NY^{-3}\left|\alpha-\frac{b}{r(\alpha)\lambda_1}\right|\right)>H^2Q^2P^{-2}.$$

It is clear from (8.15), (6.9), (6.18) that

$$(8.16)\qquad \int_{\mathfrak{n}_2}\sum_{\substack{p\\(8.10)}}\Xi_p(\lambda_1\alpha p^2)S(\alpha)d\alpha\ll\lambda_1^{-47/345}\lambda_2^{-1/3}\lambda_3^{1/3}\lambda_4^{-2/3}N^{454/345+2\delta}.$$

Let $\mathscr{L}(d,r,a)$, $\mathscr{R}(d,r,a)$ be defined as in the proof of Lemma 19 for all $d\,|\,\lambda_1$. For $\alpha\in\mathfrak{n}\backslash\mathfrak{n}_2$ we evidently have (7.18) since

$$r\left(1+Q^3\left|\lambda_1\alpha-\frac{b}{r}\right|\right)\ll H^2Q^2P^{-2}\ll X^{-1}H^{7/4}$$

from (7.6), (7.13), (1.12), (1.6). Arguing just as in that proof, it follows that

$$(8.17)\qquad \mathfrak{n}\backslash\mathfrak{n}_2\subset\bigcup_{d|\lambda_1}\Big\{\Big(\bigcup_{\substack{r\leqslant X^{-1}H^{7/4};a\\(7.22)}}\mathscr{L}(d,r,a)\Big)\cup\Big(\bigcup_{\substack{r\leqslant H^{3/4};a\\(7.22)}}\mathscr{R}(d,r,a)\Big)\Big\}.$$

Moreover, $HQ<H^{7/8}P^{1-\delta}$ from (7.6). Thus (8.15) yields

$$(8.18)\qquad \sum_{\substack{p\\(8.10)}}\Xi(\lambda_1\alpha p^2)\ll\frac{P^{2\delta}HQY}{\{r(1+NY^{-3}|\alpha-a/rd|)\}^{1/2}}\ll\frac{H^{7/8}P^{1+\delta}Y}{\{r(1+NY^{-3}|\alpha-a/rd|)\}^{1/2}}$$

on any of the intervals $\mathscr{L}(d,r,a)$, $\mathscr{R}(d,r,a)$ in (8.16). Hence

$$(8.19)\qquad \sum_{\substack{r\leqslant X^{-1}H^{7/4}\\(7.22)}}\sum_{a\leqslant rd}\int_{\mathscr{L}(d,r,a)}\sum_{\substack{p\\(8.10)}}\Xi_p(\lambda_1\alpha p^2)S(\alpha)d\alpha$$
$$\ll\sum_{\substack{r\leqslant X^{-1}H^{7/4}\\(7.22)}}\sum_{a\leqslant rd}\int_{\mathscr{L}(d,r,a)}\frac{H^{7/8}P^{1+\delta}YS(\alpha)}{\{r(1+NY^{-3}|\alpha-a/rd|)\}^{1/2}}d\alpha.$$

On the set $\mathscr{R}(d,r,a)$ we have

$$r\leqslant H^{3/4},\qquad\left|\alpha-\frac{a}{rd}\right|\leqslant r^{-1}H^{3/4}N^{-1}Y^3,$$

whence

$$\left\{r\left(1+NY^{-3}\left|\alpha-\frac{a}{rd}\right|\right)\right\}^{1/6} \ll H^{1/8} \ll Y$$

and (8.18) yields

$$(8.20) \qquad \sum_{\substack{r \leqslant H^{3/4} \\ (7.22)}} \sum_{a \leqslant rd} \int_{\mathfrak{R}(d,r,a)} \sum_{\substack{p \\ (8.10)}} \Xi_p(\lambda_1 \alpha p^2) S(\alpha)\, d\alpha$$

$$\ll \sum_{\substack{r \leqslant H^{3/4} \\ (7.22)}} \sum_{a \leqslant rd} \int_{\mathfrak{R}(d,r,a)} \frac{YHP^{1+\delta}}{\{r(1+NY^{-3}|\alpha-a/rd|)\}^{2/3}}\, d\alpha.$$

We have already obtained the bound (7.36) for the expressions on the right-hand sides of (8.19) and (8.20), in the course of the proof of Lemma 19. Therefore the bound (8.5) follows on combining (8.9), (8.16), (8.17), (8.19) and (8.20). This completes the proof of Lemma 21.

Lemma 12 now follows on combining (6.10), (6.9), (6.18), (6.20), (7.13), (7.14) and (8.5). As explained in Section 5, with the completion of this step we have finished the proof of Theorem 2.

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Linear forms in two logarithms and Schneider's method, II

by

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Introduction. We consider an homogenous linear combination of two logarithms of algebraic numbers with integer coefficients

$$b_1 \log \alpha_1 - b_2 \log \alpha_2.$$

We refine the lower bound which was obtained in our previous paper [7] by using the assumption that b_1, b_2 are rational integers. Our result will be very sharp as far as the dependence on the heights of α_1 and α_2 is concerned. We pay also a special attention to the absolute constant, which is important in numerical applications (e.g. [4] and also [3]).

1. A lower bound for linear forms in two logarithms. Our main result is Theorem 5.11 in Section 5. The hypotheses are a bit technical, and we give here a simpler statement. However for concrete applications where the value of the constant is important, our estimates of Sections 5 and 6 below will give better numerical values than Corollary 1.1.

Here we consider the absolute logarithmic height $h(\alpha)$ of algebraic numbers. Namely, if α is algebraic of degree d over $\boldsymbol{Q}$, with conjugates $\sigma_1\alpha, \ldots, \sigma_d\alpha$, and minimal polynomial

$$c_0 X^d + \ldots + c_d = c_0 \prod_{i=1}^{d} (X - \sigma_i \alpha) \qquad (c_0 > 0)$$

then

$$h(\alpha) = d^{-1}\left(\operatorname{Log} c_0 + \sum_{i=1}^{d} \operatorname{Log} \max(1, |\sigma_i \alpha|)\right).$$

The measure of α is defined by

$$M(\alpha) = |c_0| \prod_{i=1}^{d} \max\{1, |\sigma_i \alpha|\} = \exp\{d \cdot h(\alpha)\}.$$

Let α_1, α_2 be two non-zero algebraic numbers of exact degrees D_1, D_2. Let D denote the degree over $\boldsymbol{Q}$ of the field $\boldsymbol{Q}(\alpha_1, \alpha_2)$. For $j = 1, 2$, let $\log \alpha_j$ be any non-zero determination of the logarithm of α_j.