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Additive bases with many representations

by

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In additive number theory, the set A of nonnegative integers is an *asymptotic basis of order* 2 if every sufficiently large integer can be written as the sum of two elements of A. Let $r_A(n)$ denote the number of representations of n in the form $n = a+a'$, where $a, a' \in A$ and $a \leqslant a'$. An asymptotic basis A of order 2 is *minimal* if no proper subset of A is an asymptotic basis of order 2. Erdös and Nathanson [2] proved that if A is an asymptotic basis of order 2 such that $r_A(n) \geqslant c \cdot \log n$ for some constant $c > 1/\log(4/3)$ and every sufficiently large integer n, then some subset of A is a minimal asymptotic basis of order 2.

It is an open problem to determine whether the set A must contain a minimal asymptotic basis of order 2 if $r_A(n)$ merely tends to infinity as n tends to infinity. This paper contains several results connected with this question. Let $|S|$ denote the cardinality of the set S. For any set A of nonnegative integers, let

$$S_A(n) = \{a \in A \mid n-a \in A\}$$

be the *solution set* of n in A. Erdös and Nathanson [3] proved that there exists a probability measure on the space of all sets of positive integers such that, with probability 1, a random set A has the properties that $r(n) \to \infty$ and $|S_A(m) \cap S_A(n)|$ is bounded for all $m \neq n$. We shall show that the following weaker condition suffices to prove the existence of a minimal asymptotic basis: If $r_A(n) \to \infty$ and if $|S_A(m) \cap S_A(n)| < (1/2-\delta)|S_A(n)|$ for some $\delta > 0$ and all sufficiently large integers m and n with $m \neq n$, then A contains a minimal asymptotic basis. On the other hand, we shall prove that for any integer t there exists an asymptotic basis A of order 2 such that every sufficiently large integer has at least t distinct representations as a sum of two elements of A, but A contains no minimal asymptotic basis of order 2. The proof will use a refinement of a method applied previously by the authors to construct an asymptotic basis A of order 2 with the property that $A \setminus S$ is an asymptotic basis of order 2 if and only if the set $A \cap S$ is finite [1].

Erdös and Nathanson [4] have recently written a survey of results and open problems concerning minimal asymptotic bases.

Notation. Let A and B be sets of integers. Denote by $A+B$ the set of all integers n of the form $n = a+b$, with $a \in A$ and $b \in B$. Let $2A = A+A$. Let $S_A(n) = \{a \in A |\ n-a \in A\}$, and let $S'_A(n) = \{a \in S_A(n) |\ a \geqslant n/2\}$. Then $r_A(n) = |S'_A(n)| = [(|S_A(n)|+1)/2]$. Let S be any subset of A. We write that "S destroys n" if, whenever $n = a+a'$ with $a, a' \in A$, then either $a \in S$ or $a' \in S$. For any real numbers a and b, let $[a, b]$ denote the set of integers n such than $a \leqslant n \leqslant b$.

LEMMA 1. *Let A be a set of nonnegative integers. If*

$$|S_A(n) \cap S_A(u)| < (1/2)|S_A(n)|,$$

then $n \in 2(A \setminus S_A(u))$.

Proof. If $n \notin 2(A \setminus S_A(u))$, then $S_A(u)$ destroys n, and so $S_A(u)$ contains at least one element of each pair $\{a, a'\}$ of elements of A such that $a+a' = n$. It follows that

$$|S_A(n) \cap S_A(u)| \geqslant r_A(n) = [(|S_A(n)|+1)/2] \geqslant |S_A(n)|/2,$$

which contradicts the hypothesis of the lemma.

THEOREM 1. *Let A be an asymptotic basis of order 2 such that*
(i) $r_A(n) \to \infty$ *as* $n \to \infty$, *and*
(ii) *there exists* $\delta > 0$ *and* N_0 *such that for all* $m, n \geqslant N_0$, $m \neq n$,

$$|S_A(n) \cap S_A(m)| < (1/2-\delta)|S_A(n)|.$$

Then A contains a minimal asymptotic basis of order 2.

Proof. Choose $N_1 \geqslant N_0$ such that $n \in 2A$ for all $n \geqslant N_1$. Choose $a_1 \in A$ with $a_1 > N_1$. Choose $a'_1 \in A$ with $a'_1 > a_1$, and let $u_1 = a_1 + a'_1$. Then $u_1 > 2N_1$ and $a'_1 \in S'_A(u_1)$. We define the set A_1 by

$$A_1 = (A \setminus S'_A(u_1)) \cup \{a'_1\}.$$

Then $A_1 \subseteq A_0 = A$, and $u_1 = a_1 + a'_1$ is the unique representation of u_1 as the sum of two elements of A_1. Since $a \geqslant u_1/2 > N_1$ for all $a \in A \setminus A_1$, it follows that for $n \leqslant N_1$ we have $n \in 2A_1$ if and only if $n \in 2A$. Let $n > N_1$, $n \neq u_1$. Since

$$|S_A(n) \cap S_A(u_1)| < (1/2-\delta)|S_A(n)| < |S_A(n)|/2,$$

it follows from Lemma 1 that $n \in 2(A \setminus S_A(u_1)) \subseteq 2A_1$.

Let $k \geqslant 1$. Suppose that we have constructed a decreasing finite sequence of subsets $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_k$ such that $2A = 2A_k$. Suppose also that for $i = 1, \ldots, k$ we have constructed integers $a_i, a'_i \in A_k$ such that, if we define $u_i = a_i + a'_i$, then $u_1 < \ldots < u_k$ and $u_i = a_i + a'_i$ is the unique representation of u_i as the sum of two elements of A_k. Finally, we assume that

$$A_{i-1} \setminus A_i \subseteq S'_A(u_i)$$

for $i = 1, \ldots, k$.

Choose τ such that $0 < \tau < 2\delta$. Since $r_A(n) \to \infty$, there exists $M > u_k$ such that $r_A(n) > (1/\tau) \sum_{i=1}^{k} r_A(u_i)$ for all $n \geqslant M$. Choose $a_{k+1} \in A_k$ such that $a_{k+1} \leqslant u_k$. We shall shortly impose an additional condition on the choice of a_{k+1}. Choose $a'_{k+1} \in A_k$ such that $a'_{k+1} > 2M$, and define $u_{k+1} = a_{k+1} + a'_{k+1}$. Then $u_{k+1} > 2M > 2u_k$ and $a'_{k+1} \in S'_A(u_{k+1}) \cap A_k$. Define the set $A_{k+1} \subseteq A_k$ by

$$A_{k+1} = (A_k \setminus S'_A(u_{k+1})) \cup \{a'_{k+1}\}.$$

Then $u_{k+1} = a_{k+1} + a'_{k+1}$ is the unique representation of u_{k+1} as the sum of two elements of A_{k+1}.

We shall show that $2A_{k+1} = 2A$. Since $2A = 2A_k$, it suffices to show that $2A_{k+1} = 2A_k$. Note that $u_{k+1}/2 > M$, hence

$$A_k \setminus A_{k+1} \subseteq S'_A(u_{k+1}) \subseteq [M+1, u_{k+1}], \tag{1}$$

and so, if $n \leqslant M$, then $n \in 2A_{k+1}$ if and only if $n \in 2A_k$. Let $n > M$, $n \neq u_{k+1}$. Then $n \in 2A_k$. Let $R(n)$ (resp. $R'(n)$) denote the number of representations of n as a sum of two elements of A_k (resp. A_{k+1}). We must show that $R'(n) > 0$. Since

$$A \setminus A_k \subseteq \bigcup_{i=1}^{k} S'_A(u_i),$$

it follows that

$$r_A(n) \leqslant R(n) + \sum_{i=1}^{k} |S'_A(u_i)| = R(n) + \sum_{i=1}^{k} r_A(u_i) < R(n) + \tau r_A(n),$$

and so $R(n) > (1-\tau) r_A(n)$ for $n > M$. By (1), the number of representations of n as a sum of two elements of A_k that are not representations of n as a sum of two elements of A_{k+1} is at most

$$\begin{aligned} |S_A(n) \cap (A_k \setminus A_{k+1})| &\leqslant |S_A(n) \cap S'_A(u_{k+1})| \leqslant |S_A(n) \cap S_A(u_{k+1})| \\ &< (1/2-\delta)|S_A(n)| \\ &\leqslant (1/2-\delta)\, 2r_A(n) = (1-2\delta) r_A(n). \end{aligned}$$

This implies that

$$\begin{aligned} R'(n) &\geqslant R(n) - (1-2\delta) r_A(n) \\ &> (1-\tau) r_A(n) - (1-2\delta) r_A(n) = (2\delta - \tau) r_A(n) > 0 \end{aligned}$$

and so $n \in 2A_{k+1}$ for all $n > M$. This completes the induction.

Let $A^* = \bigcap_{k=0}^{\infty} A_k$. Then $2A^* = 2A$ and so A^* is an asymptotic basis of order 2. Moreover, $u_k = a_k + a_k'$ is the unique representation of u_k as the sum of two elements of the set A^*.

In order for A^* to be a minimal asymptotic basis of order 2, we impose the following additional condition on the choice of the integers a_k: If $a \in A^*$, then $a = a_k$ for infinitely many k. This means that for any $a \in A^*$ there will be infinitely many integers u_k such that $u_k \notin 2(A^* \setminus \{a\})$. Thus, A^* is minimal. This completes the proof.

LEMMA 2. *Let $I = [a, b]$ and $J = [c, d]$, where $b \leqslant c$. Let $k \geqslant 1$. If $m \in [a+c+k-1, b+d-k+1]$, then m has at least k representations in the form $m = x+y$, where $x \in I$, $y \in J$, and $x \leqslant y$. If $n \in [2a+2k-2, 2b-2k+2]$, then n has at least k representations in the form $n = x+y$, where $x, y \in I$, and $x \leqslant y$.*

Proof. Since $[a+c+k-1, b+d-k+1] = [a+k-1, b]+[c, d-k+1]$, it follows that $m = x+y$, where $x \in [a+k-1, b]$ and $y \in [c, d-k+1]$, hence $x \leqslant y$. Then $m = (x-i)+(y+i)$, where $x-i \in I = [a, b]$, $y+i \in J = [c, d]$, and $x-i \leqslant y+i$ for $i = 0, 1, \ldots, k-1$.

Since $[2a+2k-2, 2b-2k+2] = [a+k-1, b-k+1]+[a+k-1, b-k+1]$, it follows that $n = x+y$, where $x, y \in [a+k-1, b-k+1]$ and $x \leqslant y$, hence $n = (x-i)+(y+i)$, where $x-i$, $y+i \in I$ and $x-i \leqslant y+i$ for $i = 0, 1, \ldots, k-1$. This completes the proof.

LEMMA 3. *Let $n_0 \leqslant n_1 \leqslant n_2 \leqslant \ldots$ be a sequence of positive integers such that $n_{k-1} \geqslant 3k^2+6k+1$ and $n_k \geqslant 8n_{k-1}$ for $k \geqslant 1$. Let $N_k = 2n_k+1$. For each $k \geqslant 1$, define the following sets of integers:*

$$P_k = [N_{k-1}+1, n_k - N_{k-1}],$$

$$Q_k = \{n_k - n_{k-1} - 3ku+1 \mid u = 1, 2, \ldots, k+1\},$$

$$R_k = [n_k+1, n_k+N_{k-1}] \setminus \{n_k+n_{k-1}+3ku \mid u = 1, 2, \ldots, k+1\}.$$

Let $B_k = P_k \cup Q_k \cup R_k$ and $B = \bigcup_{k=1}^{\infty} B_k$. Then

(i) $N_k \notin 2B$ for $k \geqslant 0$, and

(ii) If $k \geqslant 3$ and $n \in [N_{k-1}+1, N_k-1]$, then n has at least k representations in the form $n = u+v$, where $u, v \in B_k \cup B_{k-1} \cup B_{k-2}$.

Proof. (i) Since the smallest element of B is N_0+1, it is clear that $N_0 \notin 2B$. Let $k \geqslant 1$. Note that

$$B \cap [N_{k-1}+1, n_k] = P_k \cup Q_k$$

and

$$B \cap [n_k+1, N_k] = B \cap [n_k+1, n_k+N_{k-1}] = R_k.$$

If $N_k = 2n_k+1 = c+d$, where $0 \leqslant c \leqslant d$, then $c \leqslant n_k$ and $d \geqslant n_k+1$. If $c \in B$ and $c \notin Q_k$, then $c \leqslant n_k - N_{k-1}$ and so $N_k \geqslant d = N_k - c \geqslant n_k + N_{k-1}+1$. Since $B \cap [n_k+N_{k-1}+1, N_k] = \emptyset$, it follows that $d \notin B$. If $c \in Q_k$, then $c = n_k - n_{k-1} - 3ku+1$ for some $u \in [1, k+1]$, hence $d = N_k - c = n_k + n_{k-1} + 3ku \in [n_k+1, N_k]$. Since $d \notin R_k$, it follows that $d \notin B$ and so $N_k \notin 2B$.

(ii) Let $k \geqslant 3$. We apply Lemma 2 to the set P_k. If

$$n \in [2N_{k-1}+2k, N_k - 2N_{k-1} - 2k+1], \tag{2}$$

then n has at least k distinct representations as the sum of two elements of P_k.

Define the sets S_k and T_k by

$$S_k = [n_k+1, n_k+n_{k-1}+k+1], \quad T_k = [n_k+n_{k-1}+3k(k+1)+1, n_k+N_{k-1}].$$

Then $S_k \cup T_k \subseteq R_k$. Since

$$N_{k-1}+n_k+n_{k-1}+3k(k+1)+k+1 \leqslant N_k - 2N_{k-1} - 2k+2,$$

it follows from Lemma 2, applied to the sets P_k and T_k, that if

$$n \in [N_k - 2N_{k-1} - 2k+2, N_k - k] \tag{3}$$

then n has at least k distinct representations in the form $n = x+y$, where $x \in P_k$ and $y \in T_k \subseteq R_k$. Similarly, Lemma 2, applied to the set S_{k-1}, implies that if

$$n \in [N_{k-1}+2k-1, N_{k-1}+N_{k-2}] \tag{4}$$

then n has at least k distinct representations as the sum of two elements of S_{k-1}. Finally, Lemma 2, applied to the sets P_k and P_{k-2}, shows that if

$$n \in [N_{k-1}+N_{k-2}+1, 2N_{k-1}+2k-1] \subseteq [N_{k-1}+N_{k-3}+k+1, n_k - N_{k-1}+n_{k-2}-N_{k-3}-k+1] \tag{5}$$

then n has at least k distinct representations in the form $n = x+y$, where $x \in P_k$, $y \in P_{k-2}$. From (2)–(5), we conclude that if $n \in [N_{k-1}+2k-1, N_k-k]$, then n has at least k distinct representations as a sum of two elements of $B_k \cup B_{k-1} \cup B_{k-2}$.

Let $n \in [N_k-k+1, N_k-1]$. Then $n = N_k - w$ for some $w \in [1, k-1]$ and

$$n = (n_k - n_{k-1} - 3ku+1)+(n_k+n_{k-1}+3ku-w) \in Q_k + R_k \subseteq 2B_k$$

for $u = 1, 2, \ldots, k$. Let $n \in [N_{k-1}+1, N_{k-1}+2k-2]$. Then $n = N_{k-1}+w$ for some $w \in [1, 2k-2]$ and

$$n = (n_{k-1}-n_{k-2}-3(k-1)u+1)+(n_{k-1}+n_{k-2}+3(k-1)u+w) \in Q_{k-1}+R_{k-1} \subseteq 2B_{k-1}$$

for $u = 1, 2, \ldots, k$. Thus, if $n \in [N_{k-1}+1, N_k-1]$, then n has at least k

representations as a sum of two elements of $B_k \cup B_{k-1} \cup B_{k-2}$. This completes the proof of Lemma 3.

LEMMA 4. *Let B be the set of integers defined in Lemma 3. Let $r_B(n)$ denote the number of representations of n in the form $n = b+b'$, where $b, b' \in B$ and $b \leqslant b'$. Then $r_B(N_k) = 0$ for all k, and $r_B(n) \to \infty$ as $n \to \infty$, $n \neq N_k$.*

Proof. This follows immediately from Lemma 3, since $r_B(n) \geqslant t$ for $n > N_{t-1}$, $n \neq N_k$.

THEOREM 2. *For any integer t, there exists a set A of nonnegative integers such that $r_A(n) \geqslant t$ for all sufficiently large n, and, for any subset S of A, the set $A \setminus S$ is an asymptotic basis of order 2 if and only if S is finite. In particular, A does not contain a minimal asymptotic basis of order 2.*

Proof. Let $\{n_k\}$ be a sequence of integers that satisfies the conditions of Lemma 3. Let B be the corresponding set of integers constructed in Lemma 3 from this sequence $\{n_k\}$. Then $n_k \geqslant 8n_{k-1}$ implies that

$$B \cap [N_k - N_{k-1}, N_k] \subseteq B \cap [n_k + N_{k-1} + 1, N_k] = \emptyset$$

for all $k \geqslant 1$. Choose j so large that $|B \cap [1, N_{j-1}]| \geqslant t$. Let F_j be a subset of $B \cap [1, N_{j-1}]$ such that $|F_j| = t$. Let $G_j = \{N_j - f \mid f \in F_j\}$, and define $A_j = B \cup G_j$. Then $G_j = A_j \cap [N_j - N_{j-1}, N_j]$. It follows that $N_j \in 2A_j$ and $r_{Aj}(N_j) = t$.

Suppose that for $i = j, j+1, \ldots, k$ we have determined finite sets F_i and G_i and infinite sets $B = A_{j-1} \subseteq A_j \subseteq A_{j+1} \subseteq \ldots \subseteq A_k$ such that

$$F_i \subseteq A_{i-1} \cap [1, N_{i-1}], \quad G_i = \{N_i - f \mid f \in F_i\}, \quad A_i = A_{i-1} \cup G_i$$

and $|F_i| = |G_i| = t$. Then $r_{A_i}(N_i) = t$. Choose $F_{k+1} \subseteq A_k \cap [1, N_k]$ such that $|F_{k+1}| = t$. An additional condition on the choice of the subset F_{k+1} will be imposed shortly. Let $G_{k+1} = \{N_{k+1} - f \mid f \in F_{k+1}\}$. Let $A_{k+1} = A_k \cup G_{k+1}$. Then $|G_{k+1}| = t$ and $G_{k+1} \subseteq [N_{k+1} - N_k, N_{k+1}]$. Since

$$A_k \setminus B = G_j \cup G_{j+1} \cup \ldots \cup G_k \subseteq [1, N_k]$$

and

$$B \cap [N_{k+1} - N_k, N_{k+1}] = A_k \cap [N_{k+1} - N_k, N_{k+1}] = \emptyset,$$

it follows that $r_{A_{k+1}}(N_{k+1}) = t$. By induction, we obtain sets F_k, G_k, and A_k for all $k \geqslant j$. Define the set A by

$$A = \bigcup_{k=j}^{\infty} A_k = B \cup \left(\bigcup_{k=j}^{\infty} G_k\right).$$

Then A is an asymptotic basis of order 2 such that $r_A(N_k) = t$ for all $k \geqslant j$, and $r_A(n) \to \infty$ as $n \to \infty$, $n \neq N_k$.

We now impose the following additional condition on the choice of the sets F_k: We must choose every t-element subset F of A exactly once. Thus, if $F \subseteq A$ and $|F| = t$, then $F = F_k$ for some unique integer $k \geqslant j$.

Let S be a subset of A. Suppose that S is finite. Since $r_A(n) \to \infty$ as $n \to \infty$, $n \neq N_k$, it follows that $n \in A \setminus S$ for all n sufficiently large, $n \neq N_k$. Since S contains only finitely many subsets F with $|F| = t$, and since each such F destroys exactly one N_k with $k \geqslant j$, it follows that $A \setminus S$ is an asymptotic basis of order 2. If S is infinite, however, then S contains infinitely many subsets F with $|F| = t$, and so S destroys infinitely many integers N_k, hence $A \setminus S$ is not an asymptotic basis of order 2.

Since the infinite set A contains no maximal finite subset S, it follows that A does not contain a minimal asymptotic basis of order 2. This completes the proof of Theorem 2.

DEFINITION. Let $t \geqslant 1$. An asymptotic basis A of order 2 is *t-minimal* if $A \setminus S$ is an asymptotic basis of order 2 if and only if $|A \cap S| < t$.

THEOREM 3. *For any integer t, there exists a set A of nonnegative integers such that $r_A(n) \geqslant t$ for all sufficiently large n, and A is t-minimal.*

Proof. The construction of A is exactly the same as in Theorem 1, but with a different condition on the choice of the finite sets F_k: We must now choose every t-element subset S of A infinitely often. This means that if $S \subseteq A$ and $|S| = t$, then $S = F_k$ for infinitely many k, and so S destroys infinitely many integers N_k. Since $r_A(n) \geqslant t$ for all sufficiently large n, it follows that if $|S| < t$, then S destroys at most finitely many n and so $A \setminus S$ is an asymptotic basis or order 2. This completes the proof.

The following simple observation is interesting as a contrast to Theorem 2.

THEOREM 4. *Let A be an asymptotic basis of order 2 such that $r_A(n) \to \infty$. Then there exists an infinite subset I of A such that $A \setminus I$ is an asymptotic basis of order 2, and $r_{A \setminus I}(n) \to \infty$.*

Proof. If F is any finite subset of A, then $r_{A \setminus F}(n) \geqslant r_A(n) - |F|$, and so $r_{A \setminus F}(n) \to \infty$.

We shall construct an infinite subset $I = \{a_1, a_2, \ldots\}$ of A and an increasing sequence of positive integers $N_1, N_2, \ldots$ such that $N_1 < a_1 < N_2 < a_2 < N_3 < \ldots$, and such that, if we define $A_k = A \setminus \{a_1, a_2, \ldots, a_k\}$, then $r_{Ak}(n) \geqslant k$ for all $n \geqslant N_k$.

Choose N_1 such that $r_A(n) \geqslant 2$ for all $n \geqslant N_1$. Let $a_1 \in A$ with $a_1 > N_1$. Define $A_1 = A \setminus \{a_1\}$. Then $r_{A1}(n) \geqslant r_A(n) - 1 \geqslant 1$ for all $n \geqslant N_1$. Suppose that for some $k \geqslant 1$ we have determined integers $a_1, \ldots, a_k \in A$ and integers $N_1, \ldots, N_k$ such that $0 < N_1 < a_1 < \ldots < N_k < a_k$ and, for $j = 1, \ldots, k$, if $A_j = A \setminus \{a_1, a_2, \ldots, a_j\}$, then $r_{A_j}(n) \geqslant j$ for all $n \geqslant N_j$. Since $r_{A_k}(n) \geqslant r_A(n) - k$, it follows that $r_{A_k}(n) \to \infty$, and so there exists $N_{k+1} > a_k$ such that $r_{A_k}(n) \geqslant k+2$ for all $n \geqslant N_{k+1}$. Choose $a_{k+1} > N_{k+1}$ and let $A_{k+1} = A_k \setminus \{a_{k+1}\}$. Then $r_{A_{k+1}}(n) \geqslant k+1$ for all $n \geqslant N_{k+1}$. This completes the induction.

Let $I = \{a_1, a_2, a_3, \ldots\}$ and define $A^* = A \setminus I$. Since $A^* \cap [0, N_{k+1}] = A_k \cap [0, N_{k+1}]$, it follows that if $N_k \leqslant n < N_{k+1}$, then $r_{A^*}(n) = r_{A_k}(n) \geqslant k$, and so $r_{A^*}(n) \to \infty$. This completes the proof.

Erdös and Nathanson [5] proved that if A is an asymptotic basis of order 2 such that $r_A(n) \geqslant c \cdot \log n$ for some $c > 1/\log(4/3)$ and $n \geqslant n_0$, then A can be partitioned into two disjoint sets, each of which is an asymptotic basis of order 2. The following result is a simple corollary of Theorem 2.

THEOREM 5. *For any integer t, there exists an asymptotic basis A of order 2 such that $r(n) \geqslant t$ for all $n \geqslant n_0$, but A is not the union of two disjoint sets, each of which is an asymptotic basis of order 2.*

Proof. Let A be a minimal asymptotic basis of order 2 such that $r(n) \geqslant t$ for all $n \geqslant n_0$. Since no subset of A is an asymptotic basis, it is clear that A cannot be partitioned into a disjoint union of two asymptotic bases of order 2.

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