Additive bases with many representations

by

PAUL ERDŐS (Budapest) and MELVYN B. NATHANSON (Bronx, N.Y.)

In additive number theory, the set $A$ of nonnegative integers is an asymptotic basis of order 2 if every sufficiently large integer can be written as the sum of two elements of $A$. Let $r_A(n)$ denote the number of representations of $n$ in the form $n = a + a'$, where $a, a' \in A$ and $a \leq a'$. An asymptotic basis $A$ of order 2 is minimal if no proper subset of $A$ is an asymptotic basis of order 2. Erdős and Nathanson [2] proved that if $A$ is an asymptotic basis of order 2 such that $r_A(n) \geq c \cdot \log n$ for some constant $c > 1/\log(4/3)$ and every sufficiently large integer $n$, then some subset of $A$ is a minimal asymptotic basis of order 2.

It is an open problem to determine whether the set $A$ must contain a minimal asymptotic basis of order 2 if $r_A(n)$ merely tends to infinity as $n$ tends to infinity. This paper contains several results connected with this question. Let $|S|$ denote the cardinality of the set $S$. For any set $A$ of nonnegative integers, let

$$S_A(n) = \{a \in A | n - a \in A\};$$

be the solution set of $n$ in $A$. Erdős and Nathanson [3] proved that there exists a probability measure on the space of all sets of positive integers such that, with probability 1, a random set $A$ has the properties that $r(n) \to \infty$ and $|S_A(m) \cap S_A(n)|$ is bounded for all $m \neq n$. We shall show that the following weaker condition suffices to prove the existence of a minimal asymptotic basis: If $r_A(n) \to \infty$ and if $|S_A(m) \cap S_A(n)| < (1/2 - \delta) |S_A(n)|$ for some $\delta > 0$ and all sufficiently large integers $m$ and $n$ with $m \neq n$, then $A$ contains a minimal asymptotic basis. On the other hand, we shall prove that for any integer $i$ there exists an asymptotic basis $A$ of order 2 such that every sufficiently large integer has at least $i$ distinct representations as a sum of two elements of $A$, but $A$ contains no minimal asymptotic basis of order 2. The proof will use a refinement of a method applied previously by the authors to construct an asymptotic basis $A$ of order 2 with the property that $A \setminus S$ is an asymptotic basis of order 2 if and only if the set $A \cap S$ is finite [1].
Erdős and Nathanson [4] have recently written a survey of results and open problems concerning minimal asymptotic bases.

**Notation.** Let \( A \) and \( B \) be sets of integers. Denote by \( A + B \) the set of all integers \( n \) of the form \( n = a + b \), with \( a \in A \) and \( b \in B \). Let \( 2A = A + A \). Let \( S_A(n) = \{a \in A \mid n - a \in A\} \), and let \( S_A(n) = \{a \in S_A(n) \mid a > n/2\} \). When \( r_A(n) = |S_A(n)| = \left|\lfloor S_A(n)/2\rfloor + 1\right|/2 \). Let \( S \) be any subset of \( A \). We write that “\( S \) destroys \( n \)” if, whenever \( n = a + a' \) with \( a, a' \in A \), then \( a \in S \) or \( a' \in S \). For any real numbers \( a \) and \( b \), let \([a, b]\) denote the set of integers \( n \) such that \( a \leq n \leq b \).

**Lemma 1.** Let \( A \) be a set of nonnegative integers. If
\[
|S_A(n) \cap S_A(u)| < (1/2)|S_A(n)|,
\]
then \( n \notin 2(A \setminus S_A(u)) \).

**Proof.** If \( n \notin 2(A \setminus S_A(u)) \), then \( S_A(u) \) destroys \( n \), and so \( S_A(u) \) contains at least one element of each pair \([a, a']\) of elements of \( A \) such that \( a + a' = n \). It follows that
\[
|S_A(n) \cap S_A(u)| < (1/2)|S_A(n)|/2,
\]
which contradicts the hypothesis of the lemma.

**Theorem 1.** Let \( A \) be an asymptotic basis of order 2 such that
(i) \( r_A(n) \to \infty \) as \( n \to \infty \), and
(ii) there exists \( \delta > 0 \) and \( N_0 \) such that for all \( m, n \geq N_0, m \neq n \),
\[
|S_A(n) \cap S_A(m)| < (1/2-\delta)|S_A(n)|.
\]
Then \( A \) contains a minimal asymptotic basis of order 2.

**Proof.** Choose \( N_0 \) such that \( n \in 2A \) for all \( n \geq N_0 \). Choose \( a_i \in A \) with \( a_i > N_0 \). Choose \( a' \in A \) with \( a' > a_i \), and let \( u_i = a_i + a' \). Then \( u_i > 2N_1 \) and \( a' \in S_A(u_i) \). We define the set \( A_i \) by
\[
A_i = (A \setminus S_A(u_i)) \cup \{a_i\}.
\]
Then \( A_i \subseteq A \setminus A_0 = A \), and \( u_i = a_i + a' \) is the unique representation of \( u_i \), as the sum of two elements of \( A_i \). Since \( a \geq u_i/2 > N_1 \) for all \( a \in A \setminus A_i \), it follows that for \( n \leq N_1 \) we have \( n \in 2A_i \) if and only if \( n \in 2A \). Let \( n > N_1 \), \( n \neq u_i \). Since
\[
|S_A(n) \cap S_A(u_i)| < (1/2-\delta)|S_A(n)| < |S_A(n)|/2,
\]
it follows from Lemma 1 that \( n \notin 2(A \setminus S_A(u_i)) \subseteq 2A_i \).

Let \( k \geq 1 \). Suppose that we have constructed a decreasing finite sequence of subsets \( A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_k \) such that \( 2A = 2A_k \). Suppose also that for \( i = 1, \ldots, k \) we have constructed integers \( a_i, a'_i \in A_i \) such that, if we define \( u_i = a_i + a'_i \), then \( u_i < \ldots < u_k \) and \( u_i = a_i + a'_i \) is the unique representation of \( u_i \) as the sum of two elements of \( A_k \). Finally, we assume that \( A_{i-1} \setminus A_i \subseteq S_{A_i}(u_i) \)

for \( i = 1, \ldots, k \).

Choose \( \tau \) such that \( 0 < \tau < 2\delta \). Since \( r_A(n) \to \infty \), there exists \( M > u_k \)

such that \( r_A(n) > (1/\tau) \sum_{i=1}^k r_A(u_i) \) for all \( n \geq M \). Choose \( a_{k+1} \in A_k \) such that \( a_{k+1} < 2M \), and define \( u_{k+1} = a_{k+1} + a'_i \).

Then \( u_{k+1} > 2M > 2u_k \) and \( a_{k+1} \in S_{A_k}(u_{k+1} ) \cap A_k \). Define the set \( A_{k+1} \subseteq A_k \) by
\[
A_{k+1} = (A \setminus S_{A_k}(u_{k+1} ) \cup \{a_{k+1}\}.
\]
Then \( u_{k+1} = a_{k+1} + a'_i \) is the unique representation of \( u_{k+1} \) as the sum of two elements of \( A_{k+1} \).

We shall show that \( 2A_{k+1} = 2A_k \). Since \( 2A = 2A_k \), it suffices to show that \( 2A_{k+1} = 2A_k \). Note that \( u_{k+1}/2 > M \), hence
\[
A \setminus A_{k+1} \subseteq S_{A_k}(u_{k+1}) \subseteq [M+1, u_{k+1}],
\]
and so, if \( n \leq M \), then \( n \notin 2A_{k+1} \). If and only if \( n \in 2A_k \). Let \( n > M, n \notin A_{k+1} \).

Then \( n \notin 2A_k \). Let \( R(n) \) denote the number of representations of \( n \) as a sum of two elements of \( A_k \) (resp. \( A_{k+1} \)). We must show that \( R(n) > 0 \). Since
\[
A \setminus A_k \subseteq \bigcup_{i=1}^k S_{A_i}(u_i),
\]
it follows that
\[
r_A(n) \leq R(n) + \sum_{i=1}^k |S_{A_i}(u_i)| = R(n) + \sum_{i=1}^k r_A(u_i) < R(n) + tr_A(n),
\]
and so \( R(n) > (1-\tau)r_A(n) \) for \( n > M \). By (1), the number of representations of \( n \) as a sum of two elements of \( A_k \) that are not representations of \( n \) as a sum of two elements of \( A_{k+1} \) is at most
\[
|S_A(n) \cap (A_k \setminus A_{k+1})| \leq |S_A(n) \cap S_{A_k}(u_{k+1})| \leq |S_A(n) \cap S_{A_k}(u_{k+1})| < (1/2-\delta)|S_A(n)|
\]
\[
\leq (1/2-\delta)|S_A(n)|< (1-2\delta)r_A(n).
\]
This implies that
\[
R(n) \geq R(n) - (1-2\delta)r_A(n)
\]
\[
> (1-\tau)r_A(n) - (1-2\delta)r_A(n) = (2\delta-\tau)r_A(n) > 0
\]
and so \( n \notin 2A_{k+1} \) for all \( n > M \). This completes the induction.
Let $A^* = \bigcap_{k=0}^{x} A_k$. Then $2A^* = 2A$ and so $A^*$ is an asymptotic basis of order 2. Moreover, $u_k = a_k + a_k'$ is the unique representation of $u_k$ as the sum of two elements of the set $A^*$.

In order for $A^*$ to be a minimal asymptotic basis of order 2, we impose the following additional condition on the choice of the integers $a_k$: If $a \in A^*$, then $a = a_k$ for infinitely many $k$. This means that for any $a \in A^*$ there will be infinitely many integers $u_k$ such that $u_k \notin (A^* \setminus \{a\})$. Thus, $A^*$ is minimal. This completes the proof.

**Lemma 2.** Let $I = [a, b]$ and $J = [c, d]$, where $b \leq c$. Let $k \geq 1$. If $m \in [a + c + k - 1, b + d - k + 1]$, then $m$ has at least $k$ representations in the form $m = x + y$, where $x \in I$, $y \in J$, and $x < y$. If $n \in [2a + 2k - 2, b + 2k + 2]$, then $n$ has at least $k$ representations in the form $n = x + y$, where $x \in I$, $y \in J$, and $x < y$.

**Proof.** Since $[a + c + k - 1, b + d - k + 1] = [a + k - 1, b] + [c, c + d - k + 1]$, it follows that $m = x + y$, where $x \in [a + k - 1, b]$ and $y \in [c, c + d - k + 1]$, hence $x < y$. Then $m = (x - i) + (y + i)$, where $x - i \in I = [a, b]$, $y + i \in J = [c, d]$, and $x - i \leq y + i$ for $i = 0, 1, \ldots, k - 1$.

Since $[2a + 2k - 2, b + 2k + 2] = [a + k - 1, b + k - 1] + [a + k - 1, b + k - 1]$, it follows that $n = x + y$, where $x, y \in [a + k - 1, b + k - 1]$ and $x < y$, hence $n = (x - i) + (y + i)$, where $x - i, y + i \in I$ and $x - i \leq y + i$ for $i = 0, 1, \ldots, k - 1$.

This completes the proof.

**Lemma 3.** Let $n_0 \leq n_1 \leq n_2 \leq \ldots$ be a sequence of positive integers such that $n_{k+1} \geq 3k^2 + 6k + 1$ and $n_k \geq 8n_{k-1}$ for $k \geq 1$. Let $N_k = 2n_k + 1$. For each $k \geq 1$, define the following sets of integers:

$P_k = [N_{k-1} + 1, n_k - N_{k-1}]$,

$Q_k = \{n_k - n_{k-1} - 3k + u \mid u = 1, 2, \ldots, k + 1\}$,

$R_k = [n_k + 1, n_k + N_{k-1}] \setminus \{n_k + n_{k-1} + 3k\} u = 1, 2, \ldots, k + 1$.

Let $B_k = P_k \cup Q_k \cup R_k$ and $B = \bigcup_{k=1}^{\infty} B_k$. Then

(i) $N_k \notin 2B$ for $k \geq 0$, and

(ii) If $k \geq 3$ and $n \in [N_{k-1} + 1, N_{k-1}]$, then $n$ has at least $k$ representations in the form $n = u + v$, where $u, v \in B_k \cup B_{k+1} \cup B_{k+2}$.

**Proof.** (i) Since the smallest element of $B$ is $N_0 = 1$, it is clear that $N_0 \notin 2B$. Let $k \geq 1$. Note that

$B \cap [N_{k-1} + 1, n_k] = P_k \cup Q_k$,

and

$B \cap [n_k + 1, N_k] = B \cap [n_k + 1, n_k + N_{k-1}] = R_k$.

If $N_k = 2n_k + 1 = c + d$, where $0 \leq c < d$, then $c \leq n_k$ and $d \geq n_k + 1$. If $c \in B$ and $c \notin Q_k$, then $c \leq n_k - N_{k-1}$ and so $N_k = d = n_k - c \geq n_k + N_{k-1} + 1$.

Since $B \cap [n_k + N_{k-1} + 1, N_k] = \emptyset$, it follows that $d \notin B$. If $c \in Q_k$, then $c = n_k - n_{k-1} - 3k + u$ for some $u \in [1, k + 1]$, hence $d = n_k - c = n_k + n_{k-1} + 3k + u \in [n_k + 1, N_k]$. Since $d \notin R_k$, it follows that $d \notin B$ and so $N_k \notin 2B$.

(ii) Let $k \geq 3$. We apply Lemma 2 to the set $P_k$.

Let $n \in [2N_{k-1} + 2k, N_k - 2N_{k-1} - 2k + 1]$, then $n$ has at least $k$ distinct representations as the sum of two elements of $P_k$. Define the sets $S_k$ and $T_k$ by

$S_k = \{n + n_{k-1} + k + 1\}$,

$T_k = \{n + n_{k-1} + 3k + 1, n_k + n_{k-1}\}$.

Then $S_k \cup T_k \subseteq R_k$. Since

$N_k - n_k + n_{k-1} + 3k + 1 \leq N_k - 2N_{k-1} - 2k + 1$,

it follows from Lemma 2, applied to the sets $P_k$ and $T_k$, that if

$n \in [2N_{k-1} + 2k, N_k - 2N_{k-1} - 2k + 1]$,

then $n$ has at least $k$ distinct representations in the form $n = x + y$, where $x \in P_k$ and $y \in T_k \subseteq R_k$. Similarly, Lemma 2, applied to the set $S_k$, implies that if

$n \in [N_k - 2N_{k-1} - 2k, N_k - 2k + 1]$,

then $n$ has at least $k$ distinct representations as the sum of two elements of $S_k$. Finally, Lemma 2, applied to the sets $P_k$ and $P_{k-2}$, shows that if

$n \in [N_{k-1} + 1, N_{k-2} + 2, 2N_{k-1} + 2k + 1]$,

then $n$ has at least $k$ distinct representations in the form $n = x + y$, where $x \in P_k \cup P_{k+2}$. From (2)-(5), we conclude that if $n \in [N_{k-1} + 2k - 1, N_{k-1} - k]$, then $n$ has at least $k$ distinct representations as a sum of two elements of $B_k \cup B_{k+1} \cup B_{k+2}$.

Let $n \in [N_{k-1} - k + 1, N_k - 1]$. Then $n = N_{k-1} - w$ for some $w \in [1, k - 1]$ and

$n = (n_k - n_{k-1} - 3k + u + 1) + (n_k - n_{k-1} + 3k + u) \in Q_k \cup R_k \subseteq 2B_k$,

for $u = 1, 2, \ldots, k$. Let $n \in [N_{k-1} + 1, N_k - 1 + 2k - 2]$. Then $n = N_{k-1} + w$ for some $w \in [1, 2k - 2]$ and

$n = (n_k - n_{k-1} - 3(k - 1) + u + 1) + (n_k - n_{k-1} + 3(k - 1) - u + 1) \in Q_k \cup R_k \subseteq 2B_k$,

for $u = 1, 2, \ldots, k$. Thus, if $n \in [N_{k-1} + 1, N_k - 1]$, then $n$ has at least $k$
representations as a sum of two elements of \( B_k \cup B_{k-1} \cup B_{k-2} \). This completes the proof of Lemma 3.

**Lemma 4.** Let \( B \) be the set of integers defined in Lemma 3. Let \( r_B(n) \) denote the number of representations of \( n \) in the form \( n = b + b' \), where \( b, b' \in B \) and \( b \leq b' \). Then \( r_B(n_k) = 0 \) for all \( k \) and \( r_B(n) \to \infty \) as \( n \to \infty \), \( n \neq N_k \).

**Proof.** This follows immediately from Lemma 3, since \( r_B(n) \geq t \) for \( n > N_{k-1} \), \( n \neq N_k \).

**Theorem 2.** For any integer \( t \), there exists a set \( A \) of nonnegative integers such that \( r_A(n) \geq t \) for all sufficiently large \( n \), and for any subset \( S \) of \( A \), the set \( A \setminus S \) is an asymptotic basis of order 2 if and only if \( S \) is finite. In particular, \( A \) does not contain a minimal asymptotic basis of order 2.

**Proof.** Let \( \{n_k\} \) be a sequence of integers that satisfies the conditions of Lemma 3. Let \( B \) be the corresponding set of integers constructed in Lemma 3 from this sequence \( \{n_k\} \). Then \( n_k \geq 8n_{k-1} \) implies that

\[
B \cap [N_k - N_{k-1} - 1, N_k] \subseteq B \cap [n_k + N_{k-1} + 1, N_k] = \emptyset
\]

for all \( k \geq 1 \). Choose \( j \) so large that \( |B \cap [1, N_{j-1}]| \geq t \). Let \( F_j \) be a subset of \( B \cap [1, N_{j-1}] \) such that \( |F_j| = t \). Let \( G_j = \{N_j - j \mid j \in F_j\} \), and define \( A_j = B \cup G_j \). Then \( G_j = A_j \cap [N_j - N_{j-1}, N_j] \). It follows that \( N_j \in 2A_j \) and \( r_{A_j}(N_j) = t \).

Suppose that for \( i = j, j+1, \ldots, k \) we have determined finite sets \( F_i \) and \( G_i \) and infinite sets \( B = A_{j-1} \supseteq A_j \supseteq \cdots \supseteq A_k \) such that

\[
F_i \subseteq A_{i-1} \cap [1, N_{i-1}], \quad G_i = \{N_i - j \mid j \in F_i\}, \quad A_i = A_{i-1} \cup G_i
\]

and \( |F_j| = |G_j| = t \). Then \( r_{A_j}(N_j) = t \). Choose \( F_{k+1} \subseteq A_k \cap [1, N_k] \) such that \( |F_{k+1}| = t \). An additional condition on the choice of the subset \( F_{k+1} \) will be imposed shortly. Let \( G_{k+1} = \{N_{k+1} - j \mid j \in F_{k+1}\} \). Let \( A_{k+1} = A_k \cup G_{k+1} \). Then \( |G_{k+1}| = t \) and \( A_{k+1} = A_k \cup G_{k+1} \).

Then \( A_k \setminus B = G_j \cup G_{j+1} \cup \cdots \cup G_k \subseteq [1, N_k] \)

and

\[
B \cap [N_k + N_{k+1} - N_k, N_k + N_{k+1}] = A_k \cap [N_k + N_{k+1} - N_k, N_k + N_{k+1}] = \emptyset,
\]

it follows that \( r_{A_{k+1}}(N_{k+1}) = t \). By induction, we obtain sets \( F_k, G_k \), and \( A_k \) for all \( k \geq j \). Define the set \( A \) by

\[
A = \bigcup_{k=j}^{\infty} A_k = B \cup \bigcup_{k=j}^{\infty} G_k.
\]

Then \( A \) is an asymptotic basis of order 2 such that \( r_A(N_k) = t \) for all \( k \geq j \), and \( r_A(n) \to \infty \) as \( n \to \infty \), \( n \neq N_k \).

We now impose the following additional condition on the choice of the sets \( F_j \): We must choose every \( t \)-element subset \( F \) of \( A \) exactly once. Thus, if \( F \subseteq A \) and \( |F| = t \), then \( F = F_k \) for some unique integer \( k \geq j \).

Let \( S \) be a subset of \( A \). Suppose that \( S \) is finite. Since \( r_B(n) \to \infty \) as \( n \to \infty \), \( n \neq N_k \), it follows that \( n \in A \setminus S \) for all \( n \) sufficiently large, \( n \neq N_k \). Since \( S \) contains only finitely many subsets \( F \) with \( |F| = t \), and since each such \( F \) destroys exactly one \( N_k \) with \( k \geq j \), it follows that \( A \setminus S \) is an asymptotic basis of order 2. If \( S \) is infinite, however, then \( S \) contains infinitely many subsets \( F \) with \( |F| = t \), and so \( S \) destroys infinitely many integers \( N_k \), hence \( A \setminus S \) is not an asymptotic basis of order 2.

Since the infinite set \( A \) contains no maximal finite subset \( S \), it follows that \( A \) does not contain a minimal asymptotic basis of order 2. This completes the proof of Theorem 2.

**Definition.** Let \( t \geq 1 \). An asymptotic basis \( A \) of order 2 is \( t \)-minimal if \( A \setminus S \) is an asymptotic basis of order 2 if and only if \( |A \setminus S| < t \).

**Theorem 3.** For any integer \( t \), there exists a set \( A \) of nonnegative integers such that \( r_A(n) \geq t \) for all sufficiently large \( n \), and \( A \) is \( t \)-minimal.

**Proof.** The construction of \( A \) is exactly the same as in Theorem 1, but with a different condition on the choice of the finite sets \( F_j \). We must now choose every \( t \)-element subset \( S \) of \( A \) infinitely often. This means that if \( S \subseteq A \) and \( |S| = t \), then \( S = F_k \) for infinitely many \( k \), and so \( S \) destroys infinitely many integers \( N_k \). Since \( r_A(n) \geq t \) for all sufficiently large \( n \), it follows that if \( |S| < t \), then \( S \) destroys at most finitely many \( n \) and so \( A \setminus S \) is an asymptotic basis or order 2. This completes the proof.

The following simple observation is interesting as a contrast to Theorem 2.

**Theorem 4.** Let \( A \) be an asymptotic basis of order 2 such that \( r_A(n) \to \infty \). Then there exists an infinite subset \( I \) of \( A \) such that \( A \setminus I \) is an asymptotic basis of order 2, and \( r_{A \setminus I}(n) \to \infty \).

**Proof.** If \( F \) is any finite subset of \( A \), then \( r_{A \setminus F}(n) \geq r_A(n) - |F| \), and so \( r_{A \setminus F}(n) \to \infty \).

We shall construct an infinite subset \( I = \{a_1, a_2, \ldots\} \) of \( A \) and an increasing sequence of positive integers \( N_1, N_2, \ldots \) such that \( N_1 < a_1 < N_2 < a_2 < N_3 < \ldots \), and such that, if we define \( A_k = A \setminus \{a_1, a_2, \ldots, a_k\} \), then \( r_{A_k}(n) \geq k \) for all \( n \geq N_k \).

Choose \( N_1 \) such that \( r_A(n) \geq 2 \) for all \( n \geq N_1 \). Let \( a_1 \in A \) with \( a_1 > N_1 \). Define \( A_1 = A \setminus \{a_1\} \). Then \( r_{A_1}(n) \geq r_A(n) - 1 \geq 1 \) for all \( n \geq N_1 \). Suppose that for some \( k \geq 1 \) we have determined integers \( a_1, \ldots, a_k \in A \) and integers \( N_1, \ldots, N_k \) such that \( 0 < N_1 < a_1 < \ldots < N_k < a_k \) and, for \( j = 1, \ldots, k \), if \( A_j = A \setminus \{a_1, a_2, \ldots, a_j\} \), then \( r_{A_j}(n) \geq j \) for all \( n \geq N_j \). Since \( r_{A_k}(n) \geq r_A(n) - k \),
it follows that $r_{A_k}(n) \to \infty$, and so there exists $N_{k+1} > a_k$ such that $r_{A_k}(n) 
less k + 1$ for all $n \nless N_{k+1}$. Choose $a_{k+1} > N_{k+1}$ and let $A_{k+1} = A_k \setminus \{a_{k+1}\}$. Then $r_{A_{k+1}}(n) \to k + 1$ for all $n \nless N_{k+1}$. This completes the induction.

Let $I = \{a_1, a_2, a_3, \ldots\}$ and define $A^* = A \setminus I$. Since $A^* \cap [0, N_{k+1}] = A_k \cap [0, N_{k+1}]$, it follows that if $N_k \leq n < N_{k+1}$, then $r_{A_k}(n) = r_{A_k}(n) \nless k$, and so $r_{A_k}(n) \to \infty$. This completes the proof.

Erdős and Nathanson [5] proved that if $A$ is an asymptotic basis of order 2 such that $r_{A_k}(n) \geq c \cdot \log n$ for some $c > 1/\log(4/3)$ and $n \nless n_0$, then $A$ can be partitioned into two disjoint sets, each of which is an asymptotic basis of order 2. The following result is a simple corollary of Theorem 2.

**Theorem 5.** For any integer $i$, there exists an asymptotic basis $A$ of order 2 such that $r(n) \nless i$ for all $n \nless n_0$, but $A$ is not the union of two disjoint sets, each of which is an asymptotic basis of order 2.

**Proof.** Let $A$ be a minimal asymptotic basis of order 2 such that $r(n) \nless i$ for all $n \nless n_0$. Since no subset of $A$ is an asymptotic basis, it is clear that $A$ cannot be partitioned into a disjoint union of two asymptotic bases of order 2.

References


