An effective estimate for the density of zeros of 
Hecke–Landau zeta-functions

by

K. M. BARTZ and T. L. FRYNSKA (Poznań)

1. Let $K$ be an algebraic number field of finite degree $n \geq 2$ and 
   absolute value of discriminant $d$. Denote by $I$ a given nonzero integral 
   ideal of the ring of algebraic integers $R_k$ of $K$. Let $\chi(C)$ be a 
   Dirichlet character of the abelian group $H^*(I)$ of ideal classes $C \, (\text{mod} \, I)$ in the "narrow" sense. For 
   an integral ideal $\alpha$ of $R_k$ let $\chi(\alpha)$ be the usual extension of $\chi(C)$ (see [8], Def. 
   LVII) and $\chi^*$ the primitive character mod $I_\alpha$ induced by $\chi(\text{mod} \, I)$, $I_\alpha \subset I$. Let 
   $A = \sqrt{d} \left( \frac{5 \log d}{2(n-1)} \right)^{n-1}$ denote the constant appearing in Siegel's 
   theorem on the fundamental system of units (see [10]).

Denote by $\zeta_k(s, \chi, I)$, $s = \sigma + it$, the Hecke–Landau zeta-function 
   defined for $\sigma > 1$ by the series

$$
\zeta_k(s, \chi, I) = \sum_{\alpha} \chi(\alpha) N\alpha^{-s},
$$

where $\alpha$ runs through integral ideals of $K$. Let $N(\alpha, T, \chi)$ denote the number 
   of zeros of $\zeta_k(s, \chi, I)$, $s = \sigma + it$, in the rectangle $\alpha \leq \sigma \leq 1, 0 < t < T$. Basing 
   on some effective estimates of the order of Hecke–Landau zeta-functions near 
   the line $\sigma = 1$ (see [1], Th. 1) and using Halász–Turán ideas (see [4] and [5]) 
   we shall prove the following theorem:

Theorem. There exist absolute positive constants $c_0 > 1$ and $c_1 < 1$ such 
   that for all $\alpha$ and $T$ with

$$
1 - \min(c_1, e^{-20(A+1)}) \leq \alpha \leq 1, \quad T > c_0,
$$

the following inequality holds:

$$
\sum_{N(\alpha, T, \chi) \neq 0} \chi(\text{mod} \, I) < \exp \left[ 250 M_1(I, T)(1-\alpha)^{1/2} \log^3 \frac{1}{1-\alpha} \right],
$$

where

$$
M_1(I, T) = \max(\log^{3/2}(N(I)), A^{3/2} \log T).
$$
The star in the inner sum indicates that $\chi$ runs through primitive characters mod $a$ only.

For the Riemann zeta-function, the estimate of the form (1.1)-(1.3) is due to G. Halász and P. Turán (see [4]) and for the Dirichlet $L$-functions to the second of the present authors (see [2]). Our estimate can be compared with the following result of W. Stas [11] for the Dedekind zeta-functions $\zeta(s)$:

For all $s$ with
\begin{equation}
1 - (3n\exp(-10^n))^{-1} \leq s \leq 1
\end{equation}
and $T \geq e$ the following inequality holds:
\begin{equation}
N(\alpha, T, K) \leq \exp\exp(cn^{5600}d^{160}) T^{3.12} \log(1 - s) \log^2 \frac{1}{n^{1 - s}},
\end{equation}
where $c$ is a positive absolute constant.

Putting $l = R_k$ in (1.2) we obtain an estimate for $N(\alpha, T, K)$ which is better than (1.5) in respect of the dependence on the parameters of the field $K$, but our rectangle (1.1) is narrower than that in Stas's theorem. And vice versa, since $\prod_{j \neq 0} \zeta_k(s, \chi) = \zeta_{L}(s), n_k = nh^*(f), \Delta_L = d^{1.5} \prod_{j \neq 0} N_{L_k^j}$, where $L$ is the class-field of the group $H^*(f)$, we can obtain an estimate as close to (1.2) from (1.5), but the dependence on the parameters will be much worse.

2. The proof of (1.1)-(1.3) will rest on the following lemmas:

**Lemma 1** (see [7] and [4], p. 130). Let $G(z)$ be regular for $|z| < R, G(0) = 0$ and $|G(z)| \leq U$. Then if $0 < r < R$ and the zeros of $G(z)$ in the disc $|z| < r$ are $z_1, z_2, \ldots$ then for all non-negative integers $\mu$ we have
\begin{equation}
\left| \left( \frac{G(z)}{G(0)} \right)^{\mu \log U} \left( 1 + \frac{\log (R/r)}{\mu \log U} \right) + \frac{1}{\mu!} \left[ G(z) \right]^\mu \right| \leq \frac{2(\mu + 1) \log U}{\mu \log U} + \frac{1}{\mu!} \left[ G(z) \right]^\mu.
\end{equation}

To obtain a lower bound we will use Turán's theorem.

**Lemma 2** (Turán's second main theorem, see [13], p. 52). For any $m > 0$, positive integer $n \leq N^*$ and complex numbers $w_1, w_2, \ldots, w_n$, there is an integer $v_0$ with $m < v_0 < m + N^*$ such that
\begin{equation}
\left| \sum_{j = 1}^n w_j \right| \geq \left( \frac{N^*}{8e(m + N^*)} \right)^v \left| w_k \right|^v,
\end{equation}
where $w_k$ stands for any of the $w_j$'s.

**Lemma 3** (see [12], Lemma 6). Denoting by $N(T, \chi)$ the number of roots of the Hecke–Landau zeta-function $\zeta_k(s, \chi)$ in the region $|t| \leq T, -1 \leq \sigma \leq 1$, we have the estimate
\begin{equation}
N(T+1, \chi) - N(T, \chi) < c_2 \log(dN^{|T+1/3|}^{|T+1/3|}),
\end{equation}
where $c_2$ denotes an absolute constant.

We will also use the following estimate due to Landau (see [9]):
\begin{equation}
|H(x)| = \left( \sum_{n \leq x} 1 \right) \leq n^{x/2} + d^{2(n+1)} (\log d)^2 n^2 x^{-1/2} + c_3 x (\log d)^{2n-1},
\end{equation}
and for $\chi \neq \chi_0$.

\begin{equation}
|H(x)| = |\sum_{n \leq x} \chi(n)| < n^{x/2} + (\log d)^{2n+1} x^{-1/2} + c_4 x (\log d)^{2n-1},
\end{equation}
where $c_3, c_4$ and $c_5$ are absolute constants. An effective version of (2.4) and (2.5) can be found in [1] (Lemma 8).

**Lemma 4**. For $1 - 1/(n+1) \leq \sigma \leq 1, t \geq 1.1,$
\begin{equation}
\left| k_k(\sigma + it, \chi, f) \right| \leq \epsilon e^{dA(N)} \epsilon^{-s} e^{A_0dA(N)^2} e^{2(n+1)(\log d)^{2n+1}} (\log t)^{1/3}
\end{equation}
and for $\sigma \geq 1, t \geq 1.1$ we have
\begin{equation}
\left| k_k(s, \chi, f) \right| \leq \epsilon e^{dA(t)^{2/3}+n^{x/2} (\log d)^{2n}} \log(N),
\end{equation}
where $c_6$ and $c_7$ are absolute constants.

**Proof.** Theorem 1 in [1] yields (2.6), (2.7) can be proved similarly. For $\sigma \geq 1 - 1/(n+1), t > 1$ we have (see (4.2) in [1])
\begin{equation}
\left| k_k(\sigma + it, \chi, f) \right| \leq \left| \sum_{m \leq Y_1 \exp(D/2^{3/4})} F(m, \chi) m^{-s} \right|
\end{equation}
and
\begin{equation}
\left| k_k(s, \chi, f) \right| \leq \left| \sum_{Y_1 \exp(D/2^{3/4}) < m < Y_2} F(m, \chi) m^{-s} + b_1 (\log d)^{s-1} \right|
\end{equation}
where
\begin{equation}
Y_1 = 2^{-d} n^{x+1} d^{2n} (N), \quad Y_2 = n^{b_2} d^{2(n+1)} (\log (1+N))^{x+1} (N)
\end{equation}
and $b_1$ and $b_2$ are absolute constants.

For $\sigma > 1, t \geq 1.1$, we estimate $|S_1|$ trivially by partial summation using (2.5):
\begin{equation}
|S_1| \leq \sum_{m \leq Y_1 \exp(D/2^{3/4})} F(m) m^{-1} \leq n^{b_3} (\log d)^2 ((\log t)^{2/3} + \log(N))
\end{equation}
and $Y_1 > d$. The second sum $|S_2|$ is estimated in the same way as $|S_2|$ in [1]
(see (4.6)) and we obtain for \( \sigma > 1, \ t \geq 1.1 \)
\[
|\zeta_z| \leq \exp(b_3 A(N))^{1-\sigma} (\log t)^{2/3}.
\]
Finally, we get (2.7). The constants \( b_3 \) and \( b_A \) are absolute.

**Lemma 5** (see [1], Th. 2). There exists a positive constant \( c_8 > 1 \), independent of \( K \) and \( \chi \), such that in the region
\[
\sigma \geq 1 - \left( c_8 \max \{ \log (N), A(\log (||t|| + 3))^{2/3}, \log (||T|| + 3) \} \right)^{1/3},
\]
the function \( \zeta_K(\sigma + i\tau, \chi) \) has no zeros except for the hypothetical real simple zero of \( \zeta_K(s, \chi) \), \( \chi \) real.

3. Proof of the theorem. Let \( \theta \) be such that
\[
(0.07c_9 M(f, T))^{-1} \leq 1 - \theta \leq \min\{c_1, \exp(-20(A + 1))\},
\]
where \( c_1 \) is a sufficiently small absolute constant, \( c_9 \) is taken from Lemma 5 and
\[
M(f, T) = \max \{ \log (N), A(\log (||T|| + 3))^{2/3}, \log (||T|| + 3) \}^{1/3}.
\]
Further, set
\[
\lambda = (1 - \theta)^{3/2} \left( \frac{\log \frac{1}{1-\theta}}{M(f, T)} \right)^3.
\]
It is easy to notice that
\[
\frac{\lambda}{\log \frac{1}{1-\theta}} = (1 - \theta)^{3/2} \log \frac{1}{1-\theta} \geq \gamma \log M(f, T) \geq M(f, T),
\]
where \( M(f, T) \) is given by (1.3) and \( \gamma \) is an arbitrarily large absolute constant, provided \( T > c_0 \). Let \( I \) denote the segment
\[
I: \sigma = \sigma_0 = 2 - \theta, \quad T/2 \leq t \leq T.
\]

**Lemma 6.** For a suitable set \( \tilde{H}^* < I \) of measure
\[
|\tilde{H}^*| \leq n^{k^8} d^{((k+1)\log d)} \left( M(f, T) \right)^{1.5} \exp(2\lambda M(f, T))
\]
the inequality
\[
\left| \frac{\zeta}{\zeta_K}(s, \chi^*, \omega)^{(v)} \right| > \frac{\exp(-\lambda M(f, T))}{1 - \theta} v
\]
holds for all \( v \in I \backslash \tilde{H}^* \) whenever \( T > c_{10} \) (with a sufficiently large absolute constant \( c_{10} \)) for all \( \zeta_K(s, \chi^*, \omega) \) with characters \( \chi^* \pmod{a} \), where \( Na \leq Nf \), and for all \( v \) with
\[
\frac{\lambda M(f, T)}{\log \frac{1}{1-\theta}} \left( 1 + \frac{A}{\log (1/(1-\theta))} \right) \leq v \leq \frac{\lambda M(f, T)}{\log \frac{1}{1-\theta}} \left( 1 + \frac{2A}{\log (1/(1-\theta))} \right).
\]

**Proof.** For a fixed natural \( v \), consider the set \( H = H(v, f) \) of those \( s \in I \) for which
\[
\left| \frac{\zeta}{\zeta_K}(s, \chi^*, \omega)^{(v)} \right| \geq v^\frac{v!}{(\sigma_0 - 1)^v} \exp(-\lambda M(f, T))
\]
for \( \zeta_K(s, \chi^*, \omega) \) attached to some primitive character \( \chi^* \pmod{a} \), \( Na \leq Nf \) (not necessarily the same for all \( s \) in \( H \)).

Let \( \tau_0 \) be the smallest \( \tau \)-value in \( H \) and \( \tau_1, \ldots, \tau_p \) being defined, let \( \tau_{p+1} \) be the smallest \( \tau \)-value in \( H \) satisfying \( \tau_{p+1} \geq \tau_{p} + 6 \) (if there is any). If \( \tau_1, \ldots, \tau_p \) are all these points then \( H = \bigcup_{i=1}^{p} [\tau_i, \tau_i + 6] \) and hence \( |H| \leq 6P \).

Analogously to [4], pp. 347–348, we get
\[
\frac{p^2 v!^2}{(\sigma_0 - 1)^v} e^{-(2\lambda M(f, T))}
\]
\[
= \left( \frac{1}{\sum_{k=0}^{d} \frac{1}{Na^{d^2}Na}} \right) \left( \sum_{k=0}^{d} \frac{\log^{2+4} Na}{Na^{d^2}} \sum_{j=1}^{p} \frac{\eta_j \chi_j^* (a) Na^{-i(j/2)}}{Na^{i(j/2)}} \right),
\]
where \( |\eta_j| = 1 \) and \( T/2 \geq \tau - \tau_j \geq 6 \). The first factor on the right-hand side is estimated using Abel's formula and inequality (2.4):
\[
\sum_{k=0}^{d} \frac{1}{Na^{d^2}Na} \leq \int_{1/2}^{\infty} H(x) \left( 1 + \frac{2}{\log x} \right) \frac{dx}{x^2 (\log x)^2}
\]
\[
\leq n^{3^a} d^{2+4(a+1)} (\log d)^{2n}.
\]
Hence we get
\[
\frac{v!^2 p^2}{(1-\theta)^{2+e}} e^{-(2\lambda M(f, T))}
\]
\[
= n^{3^a} d^{2+4(a+1)} \sum_{k=0}^{d} \frac{\log^{2+4} Na}{Na^{d^2}} \sum_{j=1}^{p} \frac{\eta_j \chi_j^* (a) \chi_j^* (a)}{Na^{i(j/2)}}
\]
\[
= n^{3^a} d^{2+4(a+1)} \sum_{k=0}^{d} \eta_j \chi_j^* \sum_{k=0}^{d} \frac{\log (Na)^{2+4(a)} \chi_j^* (a)}{Na^{d^2}}
\]
where if \( a_0 \) is the modulus of \( \chi_j^* \), \( a_2 \) is the modulus of \( \chi_j^* \), then \( \chi_j^* \) is a character modulo \( a_1 a_2 \). Next, separating the terms on the right of (3.8) with
Now, we have
\[
v \geq \frac{\lambda M_1(f, T)}{\log(3/2)} \left(1 + \frac{A}{\log(1/(1-\theta))}\right),
\]
and since \(1-\theta\) is bounded by an absolute constant, for \(T > c_1\) the estimate (3.3) allows us to reduce any numerical factor of the second expression on the right of (3.13). Hence the measure of the set \(H\) is estimated as follows:

\[
|H| \leq n^{2s_3} d^{4(n+1)} (\log d)^{4s} M_1(f, T)^{9.5} e^{2M_1(f, T)}.
\]

Let us denote by \(H^*\) the set of \(s \in I\) for which the assumptions of the lemma hold. Its complement \(H^*\) in \(I\) is certainly covered by the union of the above \(H = H(v, I)\) sets. Hence owing to (3.6) and (3.14) the lemma is proved.

4. Let us consider the horizontal strips \(I_j\) defined by
\[
T/2 + \frac{j}{M_1^2(f, T)} < t < T/2 + \frac{j+1}{M_1^2(f, T)}, \quad j = 0, 1, \ldots, \left\lceil \frac{T}{M_1^2(f, T)} \right\rceil.
\]
We call a strip \(I_j\) "good" if its intersection with \(I\) contains at least one point of the set \(H^*\), otherwise we call it "bad". By (3.14) the number of "bad" strips is
\[
\leq n^{s_3} d^{4(n+1)} (\log d)^{4s} M_1(f, T)^{13.5} e^{2M_1(f, T)}.
\]
In every "bad" strip \(I_j\) let us fix a point \(z_j = \sigma_0 + it\).

**Lemma 7.** For all \(z_k(s, \lambda, \omega)\) functions, \(N_\omega \leq N_f\), except at most
\[
n^{d_3} d^{4(n+1)} (\log d)^{4s} M_1(f, T)^{9.5} e^{2M_1(f, T)},
\]
the inequality (3.5) holds at \(z_j\) for all \(v\) satisfying (3.6), provided \(T\) is sufficiently large.

**Proof.** Let
\[
x_v^i (\mod \omega_i), \quad N_\omega \leq N_f, \quad i = 1, \ldots, N,
\]
be all primitive characters for which
\[
\left| \zeta_k^v(z_j^*, \zeta_*^*; \omega^v) \right| \geq \frac{v! \exp(-\lambda M_1(f, T))}{(1-\theta)^v},
\]
for a fixed \(v\) from (3.6). Analogously to (3.9) we get
\[
\frac{N_\omega^2}{(1-\theta)^v} e^{-2M_1(f, T)} \leq n^{s_3} d^{4(n+1)} (\log d)^{4s} \left( \max_{N_\omega \leq N_f} \left| \zeta_k^v(z_j^*; \omega^v) \right| \right) + \frac{N_\omega^2}{(1-\theta)^v} e^{2M_1(f, T)}.
\]
Now, the first expression on the right of (4.3) is estimated using (3.10), and the
second using Cauchy's inequality for the circle \(|s - (2\sigma_0 - 1)| \leq 2\sigma_0 - 1 - \theta = 3(1 - \theta)\). In this circle, if \(\chi \neq \chi_0\) and \(N \leq N_1^2\) we have

\[
|\zeta(s, \chi, \alpha)| \leq \frac{n^{11/12} d^{2(\alpha + 1)}}{1 - \theta} (\log d)^{2s} (N! (\log N)!^{\alpha + 1/2})^{1 - \theta}.
\]

To prove this we take \(\sigma > 1\) and use Abel's formula. We obtain

\[
\zeta(s, \chi, \alpha) = s \int_1^\infty H(\xi, \chi) \xi^{-s - 1} d\xi + s \int_2^\infty H(\xi, \chi) \xi^{-s - 1} d\xi.
\]

By (2.4), for \(\sigma > \theta \geq 1 - 1/(n + 1)\) the first integral is estimated as follows:

\[
|s \int_1^\infty H(\xi, \chi) \xi^{-s - 1} d\xi| \leq |s| n^{1/2} (d(n!)(\log d)^{s}) x^{-1 - \theta}.
\]

Similarly, (2.5) gives for the second integral (for \(\sigma > \theta \geq 1 - 1/(n + 1)\)) the bound

\[
|s \int_2^\infty H(\xi, \chi) \xi^{-s - 1} d\xi| \leq |s| n^{1/2} (d(n!)(\log d)^{s}) x^{-1 - \theta}.
\]

Putting \(x = \sqrt{N}\), where \(N \neq N_1^2\), we get (4.5).

Hence, if \(\chi \neq \chi_0\) and \(N \leq N_1^2\), Cauchy's inequality yields by (4.5)

\[
|\zeta(2\sigma_0 - 1, \chi, \alpha)|^{2(\alpha + 1)} \leq \frac{\theta + 5}{3 + 2\theta + 5 + (1 - \theta)^{2(\alpha + 1)}} (N! (\log N)!^{\alpha + 1/2})^{1 - \theta}
\]

and this shows that using (4.4) we get the inequality

\[
N_{\sigma_0} = 2^{t \alpha_1} M_1(t, T) \leq n^{11/12} d^{2(\alpha + 1)} (\log d)^{4s} \left(1 - \frac{5}{3 + 2\theta + 5} (N! (\log N)!^{\alpha + 1/2})^{1 - \theta}\right)
\]

analogous to (3.12). Thus for sufficiently large \(T\) we have for \(N\) the same estimate as before for \(P\) and the lemma is proved.

We call a zeta function "good" in a "bad" strip \(\alpha\) if it satisfies inequality (3.5) at \(z = \alpha\) for all \(v\) from (3.6), and "good" in the opposite case.

Owing to Lemma 2, from (4.2) and (4.3) the number of zeros of "bad" \(\zeta\)-functions in all "bad" strips of the rectangle

\[
1/0.07c_0 M(f, T) \leq 1 - \theta \leq \min(c_1, e^{-20\alpha + 1}), \quad T/2 \leq t \leq T, \quad T > c_13
\]

cannot exceed

\[
N^{e^{1/4} d^{2(n + 1)} \log^{8} d M_1^{24}(f, T) e^{2\alpha_1 M_1(t, T)}
\]

5. Let \(z^* = \sigma_0 + \alpha\) be a point of \(H_1\) in any fixed "good" strip or the point \(z^*_j\) in any fixed "bad" strip. Hence for all \(v\) from the interval (3.6) we have

\[
|\zeta''(\zeta^*, \chi, \alpha)|^{1 + v/4} \leq \frac{4(\alpha + 1) \log U}{(1 - \theta)^{v/4}} + \frac{1}{e^{v/4}} \left|\zeta''(\zeta^*, \chi, \alpha)\right|
\]

where \(\zeta\) is any \(\zeta\)-function in the first case, and a "good" one in the second case. We shall apply Lemma 1 with \(r = e(1 - \theta), R = e(1 - \theta), G(z) = \zeta(z + z^*, \chi, \alpha)\). We have

\[
|\zeta''(\zeta^*, \chi, \alpha)|^{1 + v/4} \leq \frac{4(\alpha + 1) \log U}{(1 - \theta)^{v/4}} + \frac{1}{e^{v/4}} \left|\zeta''(\zeta^*, \chi, \alpha)\right|
\]

where \(v\) runs through the zeros of \(\zeta\) in the disc \(|z^* - z| \leq e(1 - \theta)\) and

\[
U = \max_{|z^* - z| \leq e(1 - \theta)} \left|\zeta(z + z^*, \chi, \alpha)\right|
\]

Owing to (2.4) we get

\[
|\zeta''(\zeta^*, \chi, \alpha)|^{1 + v/4} \leq \frac{4(\alpha + 1) \log U}{(1 - \theta)^{v/4}} + \frac{1}{e^{v/4}} \left|\zeta''(\zeta^*, \chi, \alpha)\right|
\]

and using Lemma 4,

\[
|\zeta''(\zeta^*, \chi, \alpha)|^{1 + v/4} \leq e^{1/4} M_2^2(f, T) (N!^{12} \log^{12} d M_1^{32} M_1(t, T))
\]

Hence

\[
U \leq e^{1/4} M_2^2(f, T) (N!^{12} \log^{12} d M_1^{32} M_1(t, T))
\]

and \(\log U \leq M_1(t, T)\).

So the first expression on the right of (5.2) is arbitrarily small and the second can be estimated using (5.1). Therefore for \(T > c_13\) and for all \(v\) permitted by (3.6) we have

\[
\left|\zeta''(\zeta^*, \chi, \alpha)\right|^{1 + v/4} \leq \frac{1}{e^{1/4} M_1(t, T)}
\]

In order to estimate the sum in (5.4) from below we shall apply Turán's theorem (Lemma 2). To estimate the number of terms in (5.4) we apply Jensen's inequality, which gives for the number of zeros of the regular
function \( f(s) \) in the disc \( |s-s_0| \leq R \) \((0 < \theta < 1)\) the bound

\[
\frac{1}{\log(1/\theta)} \max_{|s-s_0| \leq R} \log \left| \frac{f(s)}{f(s_0)} \right|
\]

This means that the sum (5.4) has at most \( \log U \) terms. Owing to (5.3),

\[
\log U \leq \frac{\lambda M_1}{\log^2 \frac{1}{1-\theta}} \left( \frac{c_{15} A \log^3 (1-\theta)^{3/2} \log^2 \frac{1}{1-\theta} M_1}{(1-\theta)^{3/2} \log^2 \frac{1}{1-\theta} M_1} + \frac{2 \log M_1}{(1-\theta)^{3/2} \log^2 \frac{1}{1-\theta} M_1} \right)
+ \frac{(e^2-1) \log N_1}{(1-\theta)^{1/2} \log^2 \frac{1}{1-\theta} M_1} + \frac{14 \cdot 10^3 n^{2.5} (n+2) (e^2-1)^{3/2} \log T \log \frac{1}{\theta}}{M_1 \log^2 \frac{1}{1-\theta}}
\]

It is easy to notice that the first three terms on the right-hand side are, owing to (3.3), arbitrarily small, provided \( T \) is sufficiently large. Similarly, the last term can be made arbitrarily small for sufficiently large \( T \), provided \( 1-\theta \) is sufficiently small. Hence we can choose in Lemma 2

\[
N^* = \frac{\lambda M_1 A}{\log^2 \log \frac{1}{1-\theta}}
\]

and according to (3.6)

\[
m = \frac{\lambda M_1}{\log^2 \log \frac{1}{1-\theta}} \left( 1 + \frac{A}{\log \frac{1}{1-\theta}} \right)
\]

Moreover, in any strip \( l_j \) we have, owing to (3.6),

\[
\left| \frac{1-\theta + 1-\sigma^e}{z^* - q} \right| \geq \frac{1}{2}
\]

provided \( T \) is sufficiently large. Hence by Lemma 2 there exists a \( \nu^* \) in the interval (3.6) such that

\[
\sum_{\nu \leq \nu^*} \left( \frac{1-\theta}{z^* - q} \right)^{\nu+1} \geq \frac{1}{2} \exp \left( -N^* \log \left( 8e \left( 1 + \frac{m}{N^*} \right) \right) - (\nu_0 + 1) \log \left( 1 + \frac{1-\sigma^e}{1-\theta} \right) \right)
\]

where \( \sigma^e = \sigma + it \sigma \) denotes the zero with the greatest real part in the strip \( l_j \).
6. Now, as in the proof of (4.6) we shall show that in the rectangle

\[ 1 - \min(c_1, \exp(-20(A+1))) \leq \alpha \leq 1, \quad 0 \leq t \leq c_1 \]

the inequality

\[
\sum_{N_0 < N \leq N_1} \sum_{I \bmod q} N(z, c_1, \chi) = n^{\frac{1}{2}} \sum_{n \leq N \leq N_1} \sum_{I \bmod q} \log^{14}(dN) \exp(2\lambda \log^{3/2}(dN)) \leq \exp((2 + \epsilon) \lambda \log^{3/2}(dN))
\]

holds for an arbitrarily small \( \epsilon \).

Since there exists a numerical constant \( c_{19} > 0 \) such that in the region

\[
s \geq 1 - \frac{c_{19}}{n \log(dN)}, \quad |t| \leq c_1 \]

the function \( \Phi(s, t) = \prod_{N_0 < N \leq N_1} \prod_{I \bmod q} \zeta_K(s, \chi, \alpha) \) has at most one zero (see [3]), let \( \theta \) be such that

\[
\frac{10c_{19}}{n \log(dN)} \leq 1 - \theta \leq \min(c_1, e^{-20(A+1)}).
\]

We obtain for the same \( \lambda \) as before

\[
\frac{\lambda \log(dN)^{3/2}}{\log(1/(1-\theta))} \geq \gamma n d^{1/4} \log(n \log(dN))
\]

where \( \gamma \) is a sufficiently large constant, provided either \( n, d \), or \( N \) is sufficiently large.

We divide the segment \( \sigma_0 = 2 - \theta, 0 \leq t \leq c_1 \) using the points

\[
s_j = \sigma_0 + i\pi = 2 - \theta + i \frac{j c_{19}}{(n \log(dN))^3}
\]

where \( j = 0, 1, \ldots, (c_{19} 10n \log(dN))^3/c_{19} \).

Similarly to Lemma 7, we obtain

**Lemma 8.** For all \( \zeta_K(s, \chi, \alpha) \) functions, \( N_0 < N < N_1 \), except at most

\[
n^{1/2} \sum_{n} \log(dN)^{1/2} \log(dN)^{1/2} \exp(2\lambda \log(dN)^{3/2})
\]

the inequality

\[
\frac{\zeta_K(s, \chi, \alpha)^{1/n}}{\zeta_K(s, \chi, \alpha)^{1/n}} \leq \frac{\exp(-\lambda \log(dN)^{3/2})}{(1-\theta)^n}
\]

holds for all \( n \) with

\[
\frac{\lambda \log(dN)^{3/2}}{\log(1/(1-\theta))} \leq \frac{\lambda \log(dN)^{3/2}}{\log(1/(1-\theta))}
\]

provided either \( n, d \) or \( N \) is sufficiently large.

Using Landau's and Turán's theorems (Lemmas 1 and 2), we find that in the strip

\[
|t - \tau_j| < 500 c_{19}^3 (n \log(dN))^{-3}
\]

zeros \( \sigma^* = \sigma + i\pi \) of such zeta functions satisfy

\[
1 - \sigma^* \geq 0.1(1-\theta).
\]

Putting \( \alpha = 1 - 0.1(1-\theta) \) we see that if \( 1 - \theta \leq c_20 \) and one of the parameters \( n, d \) or \( N \) is sufficiently large, all zeros of \( \Phi(s, t) \) for \( \sigma \geq \alpha \) are zeros of "bad" \( \zeta_K(s, \chi, \alpha) \)-functions, which we estimate using (6.3) and Lemma 3 and finally get (6.1).

If \( n, d \) and \( N \) are all bounded we take \( 1 - \theta < c_{21} \) where \( c_{21} \) is a sufficiently small absolute constant such that the rectangle \( 0 \leq 1 - \theta < c_{21}, t \leq c_{17} \) is narrower than the rectangle (6.2). This means that there is at most one zero there.

**References**

On representation of $r$-th powers by subset sums

by

E. Lipkin* (Tel-Aviv)

Let $A$ be a set of $x$ natural numbers

(1) \[ A = \{a_1, \ldots, a_x\}, \quad 1 < a_1 < a_2 < \ldots < a_x < l, \quad |A| = x. \]

Let $\mathcal{M}$ be a given set of integers. Denote by $f(l, \mathcal{M})$ the maximum cardinality of a set $A$ which contains no subset $B \subseteq A$ such that $\sum_{a_k \in B} a_k \in \mathcal{M}$.

Recently Erdős and Freud, and N. Alon proposed the following four similar problems:

1. Let $a_x \leq 3(x-1)$. Does there exist a subset $B \subseteq A$ such that $\sum_{a_k \in B} a_k$ is a power of two? ([Er].)

2. Let $a_x \leq 4(x-1)$. Does there exist a subset $B \subseteq A$ such that $\sum_{a_k \in B} a_k$ is a square-free number? ([Er].)

3. What is a maximal cardinality of set $A$ which contains no subset $B \subseteq A$ such that $\sum_{a_k \in B} a_k$ is a square? In other words what is $f(l, \mathcal{M})$ if $\mathcal{M} = M_2$ is the set of all squares? ([Er].)

4. Let $f(l, m)$ denote for $m \geq 1$ the maximum cardinality of a set $A \subseteq \{1, \ldots, l\}$ which contains no subset $B \subseteq A$ such that $\sum_{a_k \in B} a_k = m$.

Conjecture of N. Alon is that if $l^{1.1} \leq m \leq l^{1.6}$, then

\[ f(l, m) = (1+o(1))\frac{l}{m} \quad \text{as } l \to \infty; \]

$m$ denotes the smallest integer that does not divide $m$. ([AI])

G. Freiman stated a natural generalization of problem 3 of P. Erdős:

3'. What is $f(l, \mathcal{M})$ in the case when $\mathcal{M} = M_r$ is the set of all $r$th powers?

Problems 1 and 2 are considered in [AI] and [EF]. In [AI] it is shown

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