Some new estimates in the Dirichlet divisor problem

by

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1. Introduction and statement of results. For a fixed integer \( k \geq 2 \) the (general) Dirichlet divisor problem consists of the estimation of the function

\[
D_k(x) = \sum_{n \leq x} d_k(n) - \text{Res} \ x^s \xi^k(s) s^{-1} = \sum_{n \leq x} d_k(n) - xP_{k-1} (\log x).
\]

Here \( d_k(n) \) is the divisor function which represents the number of ways \( n \) may be written as a product of \( k \) (\( \geq 2 \), fixed) factors, \( P_{k-1}(t) \) is a suitable polynomial of degree \( k-1 \) in \( t \), and \( \zeta(s) \) is the Riemann zeta-function. The function \( D_k(x) \) in (1.1) is the error term in the asymptotic formula for \( \sum_{n \leq x} d_k(n) \), that is, \( D_k(x) = o(x) \) as \( x \to \infty \). Following standard notation, we define \( \alpha_k \) and \( \beta_k \) as the infima of positive numbers \( \alpha_k \) and \( \beta_k \), respectively, for which

\[
D_k(x) \ll x^{\alpha_k}, \quad \int_1^x D_k^2(y) \, dy \ll x^{1+2\beta_k}.
\]

It is known that \( (k-1)/(2k) \leq \beta_k \leq \alpha_k \) for all \( k \geq 2 \), and it was conjectured a long time ago that \( \alpha_k = \beta_k = (k-1)/(2k) \) for all \( k \geq 2 \). For the time being the proof of this conjecture is hopeless, since \( \beta_k = (k-1)/(2k) \) (for all \( k \geq 2 \)) is equivalent to the Lindelöf hypothesis that \( \zeta(1+it) \ll t^\delta \) (see Ch. 13 of [8]).

Many authors have given upper bound estimates for \( \alpha_k \) and \( \beta_k \), and for a comprehensive account of problems involving \( D_k(x) \), we refer the reader to Ch. 12 of [8] and Ch. 13 of [5]. The latter contains the sharpest known bounds, which for \( k \geq 4 \) are as follows:

\[
\alpha_k \leq (3k-4)/(4k) (4 \leq k \leq 8), \quad \alpha_9 \leq 35/54, \quad \alpha_{10} \leq 41/60, \quad \alpha_{11} \leq 7/10,
\]

\[
\alpha_k \leq (k-2)/(k+2) (12 \leq k \leq 25), \quad \alpha_k \leq (k-1)/(k+4) (26 \leq k \leq 50),
\]

\[
\alpha_k \leq (3k-98)/(32k) (51 \leq k \leq 57), \quad \alpha_k \leq (7k-34)/(7k) (k \geq 58).
\]

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Moreover, for $k$ very large, the last bound is superseded by

$$
\alpha_k \leq 1 - \frac{1}{2} (D_k)^{-2/3},
$$

where $D > 0$ is such a constant for which

$$
(1.5) \quad \zeta(\sigma + it) \leq D(1-\sigma)^{3/2} \log^{2/3} t \quad (t \geq t_0, 1/2 \leq \sigma \leq 1)
$$

holds. From the work of H.-E. Richert [6] it is known that $D \leq 100$, and several authors (in unpublished works) have obtained smaller values of $D$. Explicit values of $\beta_k$ are also contained in [5], and they are

$$
(1.6) \quad \beta_k = \frac{(k-1)/(2k)}{k = 2, 3, 4}; \quad \beta_5 \leq 119/260 = 0.45769, \ldots
$$

$$
\beta_6 \leq 1/2, \quad \beta_7 \leq 39/70 = 0.55714, \ldots
$$

It is possible to obtain upper bounds for other $\beta_k$'s also, but a general formula seems complicated. This is due to the fact that the bounds in question depend on the functions $M(A)$ and $m(\sigma)$, which are connected with power moments of $\zeta(s)$. These functions are defined as follows: For any fixed $A \geq 4$ the number $M(A) (\geq 1)$ is the infimum of all numbers $M (\geq 1)$ such that

$$
\int_1^T \left| \zeta(\sigma + it) \right|^k \, dt \leq M^{1+k}
$$

for any $\varepsilon > 0$. Similarly, for $1/2 < \sigma < 1$ fixed we define $m(\sigma)$ (\geq 4) as the supremum of all numbers $m (\geq 4)$ such that

$$
\int_1^T \left| \zeta(\sigma + it) \right|^m \, dt \leq T^{1+k}
$$

for any $\varepsilon > 0$. Upper bounds for $\alpha_k$ and $\beta_k$ in (1.3) and (1.6) were made in [5] to depend on upper bounds for $M(A)$ and lower bounds for $m(\sigma)$, especially on the latter. Thus in order to obtain new bounds for $\alpha_k$ and $\beta_k$ we shall first refine the technique of Ch. 8 of [5] and obtain new lower bounds for $m(\sigma)$ (see § 3). Our results concerning $\alpha_k$ are contained in

**Theorem 1.** \( \alpha_{10} \leq 27/40 = 0.675, \quad \alpha_{11} \leq 0.6957, \quad \alpha_{12} \leq 0.7130, \quad \alpha_{13} \leq 0.7306, \quad \alpha_{14} \leq 0.7461, \quad \alpha_{15} \leq 0.75851, \quad \alpha_{16} \leq 0.7691, \quad \alpha_{17} \leq 0.7785, \quad \alpha_{18} \leq 0.7868, \quad \alpha_{19} \leq 0.7942, \quad \alpha_{20} \leq 0.8009, \quad \alpha_k \leq (63k-258)/(64k) \text{ for } 79 \leq k \leq 119, \quad \alpha_k \leq 1 - 165/(28k) \text{ for } k \geq 120, \text{ and if } (1.5) \text{ holds, then}

$$
(1.7) \quad \alpha_k \leq 1 - \frac{1}{2} (D_k)^{-2/3}
$$

The bounds of Theorem 1 improve, for $k \geq 10$, all the corresponding bounds in (1.3), which give e.g. $\alpha_{10} \leq 0.68333 \ldots, \quad \alpha_{11} \leq 0.7, \quad \alpha_{12} \leq 0.71428 \ldots, \quad \alpha_{13} \leq 0.73333 \ldots, \quad \alpha_{14} \leq 0.75 \text{ etc.} \quad \text{Likewise (1.7) improves (1.4). As in [5], the bounds for } \alpha_k \text{ are not the optimal ones obtainable by our method, and small

improvements could be attained by further elaboration. It will also transpire from the proof of Theorem 1 that new bounds for $\alpha_k$ in the range $21 \leq k \leq 78$ may be obtained, but a general formula embodying the new estimates would be cumbersome, and it is for this reason that we omit it. We are also going to prove several new bounds for $\beta_k$. This is

**Theorem 2.** \( \beta_2 \leq 0.45625, \quad \beta_3 \leq 0.55469, \quad \beta_4 \leq 0.60167, \quad \beta_5 \leq 0.63809, \quad \beta_6 \leq 0.66717, \quad \text{and if (1.5) holds, then}

$$
(1.8) \quad \beta_k \leq 1 - \frac{1}{2} (D_k)^{-2/3}
$$

New bounds for $\beta_k$ when $k \geq 11$ may be also derived, but as in the case of upper bounds for $\alpha_k$, a general formula appears to be complicated. Note that $\frac{1}{2} > \frac{2}{3}$, so that the upper bound in (1.8) is smaller than the upper bound in (1.7). Our last result concerns asymptotic formulas for the mean square of $A_k(x)$ (see Ch. 13.6 of [5]). If we set

$$
(1.9) \quad R_k(x) = \int_1^x \left[ \frac{\zeta(\sigma + it)}{\zeta(\sigma)} \right]^k \, dy = \int_1^x \left( \frac{(4k-2)\pi^2}{16k} \right)^{-1} \sum_{n=1}^\infty \frac{d_k(n) n^{-k+1/2}}{n^{k-1}}, \quad \text{then it was established by K.-C. Tong [9] that under certain conditions, which involve power moments of } \zeta(s), \text{ } \frac{\zeta(\sigma + it)}{\zeta(\sigma)} \text{ of a lower order of magnitude than } x^{(2k-1)/k}, \text{ in particular, it is known that}

$$
R_k(x) \leq x \log^2 x, \quad \text{and } R_k(x) \leq x^{1/4 + \varepsilon}.
$$

It was stated in [5] that $R_k(x) \leq x^{(3k-3)/(2k-4)}$ cannot hold for any $\delta > 0$. We shall sharpen this result by proving

**Theorem 3.** If $R_k(x)$ is defined by (1.9), then for $k \geq 2$ fixed

$$
(1.10) \quad R_k(x) \leq x^{(3k-3)/(2k)} \left( \log x \right)^{(k-1)/(3-2k)} \text{ cannot hold if } B_k = (3k-3)(k \log k - k + 1)/(2k), \text{ and } D > 0 \text{ is a suitable constant.}
$$

It was conjectured in [5] that $R_k(x) \leq x^{(3k-3)/(2k)}$ for $k \geq 2$, which in view of Theorem 3 would be essentially best possible. This conjecture, if true, is very strong, since by Lemma 2 of Section 5 it immediately implies the classical conjecture $\alpha_k = (k-1)/(2k)$ for $k \geq 2$.

2. Estimates of $\alpha_k$ and $\beta_k$ when $k$ is large. First we prove (1.7) and (1.8), which are of interest when $k$ is large. These estimates do not depend on power moment estimates for $\zeta(s)$ (i.e., $M(A)$ or $m(\sigma)$, but only on (1.5) (see.
Ch. 6 of [5] for a derivation and discussion of (1.5)). We shall start from the standard Perron inversion formula (see the Appendix of [5]) applied to

\[ A(s) = \zeta(s) = \sum_{n=1}^{\infty} d_n(n)^{-s} \quad \text{for} \quad \sigma = \text{Re} s > 1. \]

We have, for \( X^s \leq T < X^{1+\varepsilon}, \frac{1}{2} X \leq x \leq X, b = 1 + \varepsilon, \)

\[ \sum_{n \leq x} d_n(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta(s) x^s s^{-1} ds + O\left( X^{1+\varepsilon} T^{-1}\right). \]

Now we replace the segment of integration in the above formula by the segment \([\sigma-iT, \sigma+iT],\) where \( 1/2 < \sigma < 1 \) will be suitably chosen later. We pass over the pole \( s = 1 \) of the integrand, which gives rise to the main term in (1.1). Writing \( G = XT^{-1} \) it follows that

\[ (2.1) \quad \Delta_k(x) = (2\pi)^{-1} \int_{\sigma}^{XG} \frac{\zeta(s+i\delta) x^{s+i\delta}}{\sigma+i\delta} dt \]

\[ + O\left(GX + G \int_{\sigma}^{1+\varepsilon} |(\zeta(s+i\delta X^{-1})| x^{s-1} \delta \right). \]

Suppose now that \( G \) satisfies, besides \( X^s \leq G \leq X^{1-\varepsilon}, \) the additional condition

\[ (2.2) \quad \int_{\sigma}^{1+\varepsilon} |(\zeta(s+i\delta X^{-1})| x^{s-1} \delta \ll X^s. \]

We use then (1.5) to obtain from (2.1)

\[ (2.3) \quad \Delta_k(X) \ll X^s (G + X^s) \int_{\frac{1}{2}}^{1} t^{D(1-\varepsilon)/3 - 1} dt \ll X^s (G + X^s) X(G)^{D(1-\varepsilon)/3}. \]

We choose \( G \) so that the last two terms in (2.3) are equal. Thus \( G = X^{1-f(\sigma)}, \)

where

\[ f(\sigma) = (1-\sigma)/(1 + kD(1-\sigma)^{3/2}), \]

hence \( f'(\sigma) = 0 \) for \( \sigma = \sigma_0 = 1 - 2/3(Dk)^{-1/3}, \)

We have

\[ 1 - f(\sigma_0) = 1 - \frac{1}{3}2/3(Dk)^{-1/3}, \]

hence (1.7) follows with \( \sigma = \sigma_0 \) in (2.1), provided that (2.2) holds. To see this note that \( \zeta(s+i\delta) \ll \log^{2/3} |\delta| \) uniformly for \( \sigma \geq 1, \) and it follows from (1.5) that

\[ \max_{\sigma_0 \leq \sigma \leq 1} |(\zeta(s+i\delta X^{-1})| x^{s-1} \ll \max_{\sigma_0 \leq \sigma \leq 1} \{(X/G)^{D(1-\varepsilon)/3} X^{s-1}\} \log x \ll X^s. \]

This is because

\[ \max_{\sigma_0 \leq \sigma \leq 1} \exp \left\{ \frac{1}{3}2/3(Dk)^{1/3} (1-\alpha)^{3/2} + \alpha - 1 \right\} \log X \ll 1, \]

since

\[ \frac{1}{3}2/3(Dk)^{1/3} (1-\alpha)^{3/2} + \alpha - 1 \ll 0 \]

reduces to

\[ 1 \geq \alpha \geq 1 - 9 \cdot 2^{-4/3}(Dk)^{-2/3}, \]

and we have

\[ \alpha \geq \sigma_0 = 1 - 2/3(Dk)^{-1/3} \gg 1 - 9 \cdot 2^{-4/3}(Dk)^{-2/3}. \]

This proves (1.7).

The bound for \( \beta_k \) given by (1.8) will follow from

\[ (2.4) \quad I = \int_{\frac{1}{2}}^{X} \Delta_k^2 (x) dx \ll X^{1-2\varepsilon}, \quad \eta = 1 - \frac{1}{3}(Dk)^{-1/3}, \]

on replacing \( X \) by \( X^{2-\varepsilon} \) and summing over \( j = 0, 1, 2, \ldots \) We use (2.1), supposing again that (2.2) holds. This gives

\[ I \ll X^{1+\varepsilon} G^2 + \int_{\frac{1}{2}}^{X} \left( \int_{-\infty}^{XG} \frac{\zeta(s+i\delta) x^{s+i\delta}}{\sigma+i\delta} dt \right) dx \]

\[ = X^{1+\varepsilon} G^2 + \int_{-\infty}^{XG} \left( \int_{\frac{1}{2}}^{X} \frac{x^{2\varepsilon+i\delta} dx}{\sigma+i\delta} \right) \left( \int_{\frac{1}{2}}^{X} x^{2\varepsilon+i\delta} dx \right) dt.\]

Using \( |ab| \ll \frac{1}{2}(|a|^2 + |b|^2), \) it further follows that

\[ (2.5) \quad I \ll X^{1+\varepsilon} G^2 + X^{1+2\varepsilon} \int_{\frac{1}{2}}^{XG} \left| \zeta(s+i\delta) \right| x^{2\varepsilon + t^2 - 1} \int_{\frac{1}{2}}^{XG} \left( \frac{du}{1 + |t-u|} \right) \frac{dx}{x} \]

\[ \ll X^{1+\varepsilon} G^2 + X^{1+2\varepsilon} \log X \left( 1 + \int_{\frac{1}{2}}^{XG} \frac{dx}{x} \right) \]

\[ \ll X^{1+\varepsilon} G^2 + X^{1+2\varepsilon} \left( 1 + \int_{\frac{1}{2}}^{XG} t^{2D(1-\varepsilon)/3 - 2} dt \right) \]

\[ \ll X^{1+\varepsilon} G^2 + X^{1+2\varepsilon} + X^{2\varepsilon} + 2Dk(1-\varepsilon)^{3/2} + 2Dk(1-\varepsilon)^{3/2}, \]

provided that

\[ (2.6) \quad 2Dk(1-\varepsilon)^{3/2} > 1. \]

This time we choose \( G \) to make the first and the third term in the above estimate equal. We obtain

\[ G = X^{-\sigma(\varepsilon)}, \quad \sigma(\varepsilon) = 2(1-\varepsilon)/(1 + 2Dk(1-\varepsilon)^{3/2}), \]
so that \( g(\sigma) = 0 \) for \( \sigma = \sigma_1 = 1 - (Dk)^{-2/3} \), and (2.6) holds. Hence we choose \( G = X^{1-\nu(\sigma_1)} \), where \( 1-g(\sigma_1) = \eta \), as given by (2.4). Since \( \sigma_1 < 1-g(\sigma_1) \), (1.8) follows from (2.5), provided that (2.2) holds. This in turn follows from

\[
\max_{\sigma_1 < \varepsilon < 1} \left( (Xg^{-1})^{(1-\sigma)/2} (X)^{2\sigma - 1} \right) = \max_{\sigma_1 < \varepsilon < 1} \exp \left[ \left( \frac{3}{2} (Dk)^{1/2} (1-\sigma)^{3/2} + \sigma \right) \log X \right] \leq 1.
\]

The inequality

\[
\frac{3}{2} (Dk)^{1/2} (1-\sigma)^{3/2} + \sigma - 1 \leq 0
\]

reduces to \( \varepsilon \geq 1 - \frac{3}{2} (Dk)^{-2/3} \), and we have

\[
1 \geq \varepsilon \geq \sigma_1 = 1 - (Dk)^{-2/3} > 1 - \frac{3}{2} (Dk)^{-2/3},
\]

so that (1.8) is proved.

3. New bounds for \( M(\sigma) \). In this section we shall derive some new bounds for the function \( M(\sigma) \) (defined in Section 1), which will lead then to bounds for \( a_\sigma \) and \( b_\beta \) in Theorem 1 and Theorem 2. We shall refine the method which is exploited in Ch. 8 of [5]. Therein one of the key ingredients in estimating \( M(\sigma) \) was the following.

**Lemma 1.** Let \( t_1 < \ldots < t_R \) be real numbers such that \( T < t_r \leq 2T \) for \( r = 1, \ldots, R \) and \( \left| t_r - t_{r+1} \right| \geq \log^4 T \) for \( 1 \leq r \neq s \leq R \). If

\[
T^x < V \leq \sum_{M \leq n < 2M} a(n) n^{-\sigma + it} \eta
\]

where \( a(n) \leq M^x \) for \( M < n \leq 2M, 1 \leq M \leq T^{\sigma} (C > 0 \text{ a fixed number}) \), then

\[
R \leq T^x (M^{2-2\sigma} V^{-2} + TV^{-f(\sigma)}),
\]

where

\[
f(\sigma) = \begin{cases} 
2(3-4\sigma) & \text{for } 1/2 < \sigma \leq 2/3, \\
10(7-8\sigma) & \text{for } 2/3 < \sigma \leq 11/14, \\
34/(15-16\sigma) & \text{for } 11/14 < \sigma \leq 13/15, \\
98/(31-32\sigma) & \text{for } 13/15 < \sigma \leq 57/62, \\
51/(1-\sigma) & \text{for } 57/62 < \sigma \leq 1 - \varepsilon.
\end{cases}
\]

We shall indicate how for \( \sigma \) relatively close to \( 1 \) the last expression for \( f(\sigma) \) may be replaced by a better one. Namely, one can take

\[
f(\sigma) = \frac{2(l-2) + 2}{2l-1-2\sigma} \text{ for } 1 - \frac{l-1}{2l-1} \leq \sigma \leq 1 - \frac{l}{2l+1-2l}.
\]

for any \( l = 3, 4, \ldots \), and also for \( k \geq 3 \)

\[
f(\sigma) = \frac{k}{1-\sigma} \text{ for } 1 - \frac{k}{2k+1-2} \leq \sigma \leq 1 - \varepsilon.
\]

for any fixed \( \varepsilon > 0 \). Therefore the last value of \( f(\sigma) \) in (3.1) may be replaced by an arbitrary value of numbers furnished by (3.2) for \( l \geq 6 \), plus a value of \( f(\sigma) \) furnished by (3.3) with a suitable \( k \). The proof is analogous to the proof of (3.1) given in [5], and therefore the details will be omitted. If usual one defines

\[
\mu(\sigma) = \inf \{ c > 0 : \zeta(\sigma + it) \ll |t|^c \}
\]

for a given real \( \sigma \), and \( c(\sigma) \) is an upper bound for \( \mu(\sigma) \), then it was shown in [5] that \( f(\sigma) \) of Lemma 1 may be determined by the equations

\[
2c(\theta) + 1 + \theta - 2(1 + c(\theta)) \sigma = 0,
\]

\[
f(\sigma) = \frac{2(1 + c(\theta))}{c(\theta)}.
\]

Using the classical estimates (see [8]) \( \mu(\sigma) \ll 1/(2L-2) \) for \( \sigma = 1 - l/(2L-2), L = 2^{\nu_1}, l \geq 3, \) and convexity of \( \mu(\sigma) \) it follows that one may take

\[
c(\theta) = \frac{2l-1 - 2^{-l}}{2^{l-2} - 2} \text{ for } 1 - \frac{1}{2^{l-2} - 2} \leq \theta \leq 1 - \frac{l}{2^{l-2} - 1},
\]

and similarly one can take

\[
c(\theta) = \frac{1 - \theta}{k} \text{ for } 1 - \frac{k}{2^{l-2} - 1} \leq \theta \leq 1.
\]

Substituting (3.6) and (3.7) in (3.4) and (3.5), we obtain (3.2) and (3.3), respectively.

We are now going to bound the function \( m(\sigma) \) for the values \( \sigma = \frac{2l}{2L}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \) and \( \frac{5l}{2} \). It was shown in [5], Ch. 8 that to obtain bounds for \( m(\sigma) \) it suffices to obtain bounds of the form \( R \leq T^{1+\varepsilon} V^{-f(\sigma)} \), where \( R \) is the number of points \( t_r \) \( (r = 1, \ldots, R) \) such that \( |t_r| \leq T, |t_r - t_{r+1}| \geq \log^4 T \) for \( 1 \leq r \neq s \leq R \) and \( \left| \zeta(\sigma + it_r) \right| \geq V > 0 \) for any given \( V \). Moreover, by (8.97) of [5] we have with \( T^x \) omitted for brevity

\[
R \leq T \left( V^{-2} + \frac{4l}{(2l-1) + 2T} \right) \left( V^{-2} + \frac{4l}{(2(2l-1) + 2T) \log^4 T} \right) \leq R_1 + R_2 + R_3,
\]

say. Here \( f(\sigma) \) has the same meaning as in Lemma 1, and \( (\kappa, \lambda) \) is an exponent pair (see e.g. Ch. 2 of [5] for the definition and properties of exponent pairs). To avoid unwieldy expressions, we shall work primarily with
the exponent pair \((\alpha, \lambda) = \left(\frac{11}{13}, \frac{33}{13}\right)\) is \(BABA^2B\) for our purposes.

For \(\sigma = \frac{13}{11}\), we obtain \(f(\frac{13}{11}) = \frac{4}{3}, \ c(\frac{13}{11}) = \frac{4}{5}\), hence \(R_1 = TV^{-\frac{25}{2}}, \ R_2 = TV^{-\frac{26}{47}}\) and \(TV^{-15} \leq \frac{21}{47\alpha - 240} \leq 0.101329\ldots\). Whence \(R_2 \leq TV^{-10.1329\ldots}\).

Taking \((\alpha, \lambda) = \left(\frac{14}{13}, \frac{35}{13}\right)\) and \((\frac{1}{3}, \frac{1}{3})\) respectively, we obtain

\(m(\frac{13}{11}) \geq 133 \leq 12.090909\ldots, \ m(\frac{3}{4}) \geq 2328 \leq 14.28220859\ldots\)

and

\(m(\frac{15}{11}) \geq 113 \leq 12.090909\ldots, \ m(\frac{3}{4}) \geq 2328 \leq 14.28220859\ldots\)

Similar calculations with \((\alpha, \lambda) = \left(\frac{13}{11}, \frac{33}{13}\right)\) yield \(m(\frac{13}{11}) \geq 26.881578\ldots, \ m(\frac{3}{4}) \geq 2.328 \leq 14.28220859\ldots\). All these values improve the corresponding ones in Ch. 8 of [5], and for intermediate values of \(\sigma\) one may use the properties of \(m(\sigma)\). Namely by Th. 8.1 of [5] one has for \(1/2 \leq \sigma < \sigma_2 < 1\),

\[m(\sigma) \geq m(\sigma_1)m(\sigma_2) - m(\sigma_1)m(\sigma_2)(\sigma - \sigma_1).
\]

Further slight improvements on the above estimates could be obtained by using the recent algorithm of S. W. Graham [2] for minimizing certain expressions involving exponent pairs. For values of \(\sigma\) between 14/15 and 1, we use the bound

\[c(\sigma) = \frac{1}{6}(1 - \sigma) \quad \left(\frac{28}{31} \leq \sigma \leq 1\right)
\]

and

\[R \leq TV^{-\frac{19}{18} + T^{3/2} - 2n/(4\sigma - 1)\ V^{3/2} - 2n/(4\sigma - 1)\ + T(12 - 2n)/(34\sigma - 15)\ V^{3/2} - 38/(34\sigma - 15)\).
\]

This is (8.99) and (8.100) of [5], and it gives \(R \leq TV^{-\lambda}\) for

\[x = \min \left(\frac{30\sigma - 12}{4(\sigma - 1)(1 - \sigma)}, \frac{238\sigma - 12}{4(\sigma - 1)(1 - \sigma)}\right).
\]

Hence using (3.2) with \(l = 6\) and (3.3) with \(k = 6\) we obtain

\[m(\sigma) \geq \frac{258}{63 - 64\sigma} \quad \text{for} \ 14/15 \leq \sigma \leq c_0,
\]

where \(c_0 = \frac{1}{22}(171 + \sqrt{1602}) = 0.95056\ldots\)

4. Proof of other bounds for \(\alpha_k\) and \(\beta_k\). To obtain the remaining bounds for \(\alpha_k\) in Theorem 1 we use

\[A_k(\sigma) \leq \sigma^{\sigma - \epsilon},
\]

which is the estimate proved in Ch. 13.3 of [5]. Here \(1/2 < \sigma < 1\) is a constant for which \(m(\sigma) = k\), where for \(m(\sigma) = k\) one may take lower bounds for this function, such as those furnished by Section 3 and convexity. All the latter are easily seen to satisfy \(m(\sigma) \leq 1/c(\sigma)\), where \(c(\sigma)\) is given by (3.6) and (3.7), and this condition is necessary for (4.1) to hold. Using only \(m(\sigma_0) > 10\), \(m(\frac{13}{11}) \geq 133/11\) and the bound in (3.8) we obtain

\[m(\sigma) \geq \frac{1463}{581 - 644\sigma} \quad \text{for} \ \frac{27}{40} \leq \sigma \leq \frac{5}{7}.
\]

Setting the right-hand side equal to 11 and 12 and solving for \(\sigma\) we obtain \(\sigma_1 < 0.695652\ldots\) and \(\sigma_2 < 0.712862\ldots\). In general, from (3.8) and (4.1) we obtain

\[\sigma_k \leq k \frac{m(\sigma_2)(\sigma_2 - \sigma_1)}{m(\sigma_1) - m(\sigma_2)(\sigma_2 - \sigma_1)}
\]

for \(13 < k < 26\), where \(\sigma_1 = 5/7, \ \sigma_2 = 3/4\) or \(\sigma_1 = 3/4, \ \sigma_2 = 5/6\). Hence from (4.2) we easily obtain the remaining upper bounds stated in Theorem 1 for \(13 < k < 20\). It is obvious that, using the remaining values of \(m(\sigma)\) calculated in Section 3 and (4.2), one can improve all the bounds given in (1.3). In particular, from the first bound in (3.9) one has

\[m(\sigma) \geq \frac{258}{63 - 64\sigma} \quad \text{for} \ 14/15 \leq \sigma \leq c_0,
\]
implying by (4.1)

\[(4.3) \quad \alpha_k \leq \frac{63k - 258}{64k} \quad (79 \leq k \leq 119).\]

Likewise for \(\sigma \geq 19/20 = 0.95\) we have \((30\sigma - 12)/(4\sigma - 1) \geq 165/28\), hence

\[(4.4) \quad \alpha_k \leq \frac{28k - 165}{28k} \quad (k \geq 120).\]

The bounds in (4.3) and (4.4) complete the proof of Theorem 1.

To obtain upper bounds for \(\beta_k\) one may note that \(\beta_k \leq \sigma_1(k)\), if \(\sigma_1\) satisfies

\[(4.5) \quad \int_1^T |\zeta(s_1 + it)|^{2k} dt \leq T^{2 - \delta},\]

for some \(\delta = \delta(k) > 0\). This follows e.g. from Lemma 13.1 of [5], and was used in the proof of Th. 13.4 of [5]. To prove \(\beta_k \leq 73/160\) we observe first that, from \(\mu(27/40) > 10\) and the functional equation for \(\zeta(s)\), we have

\[(4.6) \quad \int_1^T |\zeta(s_1 + it)|^{10} dt \leq T^{11/4 + \epsilon},\]

while

\[(4.7) \quad \int_1^T |\zeta(s_1 + it)|^{20} dt \leq T^{7/4 + \epsilon}\]

by Th. 8.3 of [5]. Combining the preceding estimates by convexity we obtain

\[(4.8) \quad \int_1^T |\zeta(s_1 + it)|^{10} dt \leq T^{129 - 160\epsilon/28 + \epsilon}\]

(13/40 \leq \epsilon \leq 1/2).

Since \((129 - 160\epsilon)/28 < 2\) for \(\sigma > 73/160\), one obtains \(\beta_2 \leq 73/160 = 0.45625\) from (4.5). For the time being it does not seem possible to improve the bound \(\beta_2 \leq 1/2\) of [5], but for \(k > 6\) one can improve all the existing upper bounds for \(\beta_k\) by using the improved estimates for \(\mu(s)\), which were derived in Section 3. For \(k\) fixed let \(c = c(k)\) be such a constant for which \(M(2k) \leq 1 + c\), and let \(\sigma_0 = \sigma_0(k) > 1/2\) satisfy \(\mu(\sigma_0) \geq 2k\). Then we can show that

\[(4.9) \quad \beta_k \leq \frac{(c - 1)\sigma_0 + 1/2}{c}.

Indeed, if

\[F(\sigma) = \frac{2c(\sigma_0 - \sigma) + 2\sigma_0 - 1}{2\sigma_0 - 1},\]

then \(F(1) = 1 + c\) and \(F(\sigma_0) = 1\). Hence by convexity

\[\int_1^T |\zeta(\sigma + it)|^{2k} dt \leq T^{11/4 + \epsilon}\]

and \(F(\sigma) < 2\) for \(\sigma > (\sigma_0 - \sigma_0 + 1)/c\), so that (4.6) follows from (4.5).

Following the proof of Th. 8.3 of [5] and using the new bound \(\mu(\sigma) \leq 9/56\) of E. Bombieri and H. Iwaniec [1], we obtain

\[(4.10) \quad M(2k) \leq 1 + c(k) (k \geq 7),\]

whence \(c = c(k) = \frac{9}{56} (k - 3)\). From the proof of the upper bounds for \(\alpha_k\) we readily find that

\[\sigma_0(7) = 0.7461, \quad \sigma_0(8) = 0.7691, \quad \sigma_0(9) = 0.7868, \quad \sigma_0(10) = 0.8009.\]

It follows then immediately from (4.6) and (4.7) that

\[\beta_1 \leq 0.5584, \quad \beta_2 \leq 0.6016, \quad \beta_9 \leq 0.63808, \quad \beta_{10} \leq 0.66716, \ldots\]

and upper bounds for \(\beta_k\) when \(k \geq 11\) may be calculated analogously.

5. Proof of Theorem 3. For the proof of Theorem 3 we need the following

**Lemma 2.** For \(x^e \leq H \leq x\) and \(k \geq 2\) fixed we have uniformly

\[(5.1) \quad A_k(x) = H^{-1} \int_x^{x+H} A_k(y) dy + O(H \log^{k-1} x).\]

**Proof.** We have

\[H^{-1} \int_x^{x+H} A_k(y) dy - A_k(x) = H^{-1} \int_x^{x+H} (A_k(y) - A_k(x)) dy =
\]

\[\ll H \log^{k-1} x + H^{-1} \int_x^{x+H} \sum_{x < n \leq x+H} d_k(n) dy =
\]

\[\ll H \log^{k-1} x + H^{-1} \int_x^{x+H} \sum_{x < n \leq x+H} d_k(n) dy =
\]

\[\ll H \log^{k-1} x.\]
Here we used (1.1) and the estimate
\[ \sum_{x < n < x + H} d_k(n) \ll H \log^{k-1} x \quad (x^\epsilon \leq H \leq x), \]
which follows from the work of P. Shiu [7].

We proceed now to the proof of Theorem 3. Suppose that we have
\[ (2.2) \quad \int_1^x A_k^2(y) \, dy = A_k x^{2k-1}/y + O\left(x^{(3k-3)/(2k)} G_k(x)^2\right), \]
where
\[ A_k = (4k-2)^{-1} \frac{1}{k-2} \sum_{n=1}^{\infty} d_k(n) n^{-1/k+1/2k}, \]
and \( G_k(x) \) is a decreasing function for \( x \geq x_0(k) \) such that \( \log^{1-k} x \ll G_k(x) \ll 1 \). We use (5.1) and the Cauchy–Schwarz inequality. Then (5.2) gives, for \( x^\epsilon \leq H \leq x \),
\[ (5.3) \quad A_k^2(x) \ll H^{-1} \int_1^x A_k^2(y) \, dy + H^2 \log^{2k-2} x \]
\[ = H^{-1} A_k \left[ (x+H)^{2k-1}/y - x^{2k-1}/y \right] \]
\[ + O(x^{(3k-3)/(2k)} G_k(x) x^{-1} + H^2 \log^{2k-2} x) \]
\[ \ll x^{(k-1)/k} + x^{(2k-3)/(2k)} G_k(x) x^{-1} + H^2 \log^{2k-2} x. \]

Choosing
\[ H = x^{(k-1)/(2k)} (G_k(x) \log^{2-2k} x)^{1/3} \]
we obtain from (5.3)
\[ (5.4) \quad A_k(x) \ll x^{(k-1)/(2k)} \left(1 + (G_k(x) \log^{k-1} x)^{1/3}\right) \]
\[ \ll x^{(k-1)/(2k)} (G_k(x) \log^{k-1} x)^{1/3}. \]

On the other hand, it is known (see J. L. Hafner [3], [4]) that, for \( k \geq 2 \),
\[ (5.5) \quad A_k(x) = \Omega_+ \left(\left(\log x^{(k-1)/(2k)} (\log \log x)^{\gamma_k} \exp\left(-C (\log \log \log x)^{1/2}\right)\right), \right) \]
where \( \gamma_k = (k-1)(k \log k - k + 1)/(2k) + k - 1 \), \( C > 0 \). Comparing (5.4) and (5.5) we obtain
\[ (5.6) \quad (\log x)^{(k-1)/(2k)} (\log \log x)^{\gamma_k} \exp\left(-C (\log \log \log x)^{1/2}\right) \]
\[ \ll (G_k(x) \log^{k-1} x)^{1/3}. \]

Thus if we choose
\[ G_k(x) = (\log x)^{(k-1)/(3-2k)/(2k)} (\log \log x)^{3\gamma_k} \exp\left(-D (\log \log \log x)^{1/2}\right) \]
then \( G_k(x) \) is decreasing for \( x \geq x_0(k, D) \) and satisfies \( \log^{1-k} x \ll G_k(x) \ll 1 \), but (5.6) is false with a suitable \( D > 0 \). Hence we obtain the assertion of the theorem.

References


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