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An effective order of Hecke–Landau zeta functions near the line $\sigma = 1$, II (some applications)

by

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1. The present paper is a sequel to [1] and the notation of that paper is used throughout. Let K be an algebraic number field of finite degree n and absolute value of the discriminant d. Denote by $\mathfrak{f}$ a conductor of a character χ of ideal classes in the "narrow" sense.

We shall show some applications of effective order of Hecke–Landau zeta functions $\zeta_K(\sigma+it, \chi)$ near the line $\sigma = 1$, exactly for $1-1/(n+1) \leqslant \sigma \leqslant 1$, which was given in the preceding note (see [1], th. 1). We first will prove the following

THEOREM A (compare [1], th. 2 and [2]). *There exists a positive constant $c_1 > 1$, independent of K and χ such that in the region*

$$(1.1)\qquad \sigma \geqslant 1 - \frac{1}{10^4 \max\left\{\log N\mathfrak{f},\ c_1 n^{3.5} \log^{2/3}(|t|+3)\left(\log\log(|t|+3)\right)^{1/3} \max\left(1, \frac{A_1}{\log\log t}\right)\right\}}$$

the function $\zeta_K(\sigma+it, \chi)$ has no zeros except for the hypothetical real simple zero of $\zeta_K(s, \chi_1)$, χ_1 real, where $A_1 = n^{1.5}\sqrt{d}\,D$ and $D = \left(\frac{5\log d}{2(n-1)}\right)^{n-1}$ denotes the constant from Siegel's theorem on fundamental system of units (see [5]).

Remark. Putting $\mathfrak{f} = R_K$ we obtain obviously a zero-free region for the Dedekind zeta function (in this case $N\mathfrak{f} = 1$).

Next, as an application of Theorem A we get an effective version of Chebotarev density theorem. Let L be a normal extension of K with Galois group $G = G(L/K)$. Let P denote a prime ideal of K and $\left|\frac{L/K}{P}\right|$ denote the conjugacy class of Frobenius automorphisms corresponding to prime ideals

$\mathfrak{P}$ of L, $\mathfrak{P}|P$. For each conjugacy class C of G, we define

$$\Pi_C(x, L/K) = \sum_{\substack{P \text{ unramified in } L \\ N_{K/Q}P \leqslant x \\ \left|\frac{L/K}{P}\right| = C}} 1.$$

Now we can state the following explicit version of Chebotarev density theorem:

THEOREM B (compare [3], th. 1.3). *There exist absolute effectively computable constants c_2, c_3, c_4 and c_5 such that in the estimate*

$$\left|\Pi_C(x, L/K) - \frac{|C|}{|G|}\operatorname{Li}x\right| \leqslant \frac{|C|}{|G|}\operatorname{Li}(x^{\beta_0}) + R(x)$$

if $\exp(c_2 n_L^5 \sqrt{d_L}(\log d_L) D_L) \leqslant x \leqslant \exp\exp(c_3 n_L^{1.5}\sqrt{d_L}D_L)$ *then we have*

$$R(x) \ll x\exp\left(-c_4 n_L^{-3} d_L^{-0.3} D_L^{-0.6}\log^{3/5}x(\log\log x)^{2/5}\right) \tag{1.2}$$

and if $x \geqslant \exp\exp(c_3 n_L^{1.5}\sqrt{d_L}D_L)$ *then*

$$R(x) \ll x\exp\left(-c_5 n_L^{-2.1}\log^{3/5}x(\log\log x)^{-1/5}\right). \tag{1.3}$$

β_0 denotes the hypothetical real zero of Dedekind zeta function $\zeta_L(s)$ and the β_0 term is present only when β_0 does exist. n_L and d_L denote degree and absolute value of discriminant of the field L.

Remark 1. For $\exp(10n_L\log^2 d_L) \leqslant x \leqslant \exp(c_6 n_L^{25} d_L^3(\log d_L)D_L^6)$ the Lagarias–Odlyzko estimate (see [3], th. 1.3) is better than ours. So, we have then

$$R(x) \ll x\exp(-c_7 n_L^{-1/2}\log^{1/2}x).$$

Remark 2. We can estimate β_0 using Stark's bound (see [6], p. 148):

$$\beta_0 < \max\left(1-(m_L\log d_L)^{-1},\ 1-(c_8 d_L^{1/n_L})^{-1}\right)$$

where

$$m_L = \begin{cases} 4 & \text{if } L \text{ is normal over } Q, \\ 16 & \text{if there is a sequence of fields} \\ & Q = k_0 \subset \ldots \subset k_r = L \text{ with each field normal} \\ & \text{over the preceding one,} \\ 4n_L! & \text{otherwise.} \end{cases}$$

Throughout this paper $c_1, c_2, \ldots$ will denote effectively computable positive absolute constants, independent of K and L. The constants implied by the notation $f \ll g$ or $f = O(g)$ are also absolute.

2. The proof of Theorems A and B will rest on the following lemmas:

LEMMA 1 (th. 1 in [1]). *For $1-1/(n+1) \leqslant \sigma \leqslant 1$, $t \geqslant 1.1$ the following inequality holds*

$$|\zeta_K(\sigma+it, \chi)| \leqslant A_2 N\mathfrak{f}^{1-\sigma} t^{A_3(1-\sigma)^{3/2}}\log^{2/3}t + A_4 N\mathfrak{f}^{1-\sigma}\log N\mathfrak{f} \tag{2.1}$$

where $A_2 = \exp(c_9\sqrt{d}Dn^5)$, $A_3 = 14\cdot 10^3 n^{25}(n+2)$, $A_4 = \sqrt{d}\log^{2n}d\cdot n^{c_{10}n}$.

LEMMA 2 (Landau). *If $F(s)$ is a function regular in the circle $|s-s_0| \leqslant r$ and satisfying the inequality $\left|\frac{F(s)}{F(s_0)}\right| \leqslant M$ in this circle, then*

$$-\operatorname{Re}\frac{F'}{F}(s_0) \leqslant \frac{4}{r}\log M - \operatorname{Re}\sum_{\varrho}\frac{1}{s_0-\varrho} \tag{2.2}$$

where ϱ runs through the zeros of $F(s)$ such that $|\varrho - s_0| \leqslant \frac{1}{2}r$ (a zero of order m being counted m times).

LEMMA 3 (see [5], Lemma 3). *If $\sigma > 1$, then for the Dedekind zeta function $\zeta_K(s)$ we have*

$$-\frac{\zeta_K'}{\zeta_K}(\sigma) < \frac{1}{\sigma} + \frac{1}{\sigma-1} + \frac{1}{2}\log\frac{d}{2^{2r_2}\pi^n} + \frac{r_1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\sigma}{2}\right) + r_2\frac{\Gamma'}{\Gamma}(\sigma) \tag{2.3}$$

where K has r_1 real and $2r_2$ complex conjugate fields.

LEMMA 4 (see [7], Lemma 6). *Denoting by $N(T, \chi)$ the number of roots of Hecke–Landau zeta function $\zeta_K(s, \chi)$ in the region $|t| \leqslant T$, $0.1 \leqslant \sigma \leqslant 1$ we have the estimate*

$$N(T+1, \chi) - N(T, \chi) \ll \log\left(dN\mathfrak{f}(|T|+3)^n\right) \tag{2.4}$$

where $\mathfrak{f}$ denotes the conductor of the character χ.

In the following we will use a weighted prime-power-counting function $\psi_C(x, L/K)$ defined by

$$\psi_C(x, L/K) = \sum_{\substack{P \text{ unramified in } L \\ N_{K/Q}P^m \leqslant x \\ \left|\frac{L/K}{P}\right| = C}} \log(N_{K/Q}P).$$

LEMMA 5 (see [3], th. 7.1). *If $x \geqslant 2$ and $T \geqslant 2$, then*

$$\psi_C(x) - \frac{|C|}{|G|}x + S(x, T) \tag{2.5}$$

$$\ll \frac{|C|}{|G|}\left\{\frac{x\log x + T}{T}\log d_L + n_L\log x + \frac{n_L x\log x\log T}{T}\right\}$$

$$+ \log x\log d_L + n_L x\frac{\log^2 x}{T}$$

where

$$S(x, T) = \frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \left\{ \sum_{\substack{\varrho \\ |\mathrm{Im}\,\varrho| < T}} \frac{x^{\varrho}}{\varrho} - \sum_{\substack{\varrho \\ |\varrho| < 1/2}} \frac{1}{\varrho} \right\}$$

and the inner sums are taken over the nontrivial zeros ϱ *of* $\zeta_L(s, \chi)$ *and* χ *runs through the irreducible characters of the cyclic group* $H = \langle g \rangle$, *where* g *is a selected element of* C. n_L *and* d_L *denote degree and absolute value of the discriminant of* L.

3. Proof of Theorem A. Let $\beta + i\tau$ be a nontrivial root of $\zeta_K(s, \chi)$. Since $\zeta_K(\bar{s}, \bar{\chi}) = \overline{\zeta_K(s, \chi)}$ we may restrict our attention to those zeros which lie in the upper half plane. Moreover we can assume that $\tau \geqslant e^{n^2}$ since by Lemmas 8.1 and 8.2 in [3] Theorem A holds for $|\tau| \leqslant e^{n^2}$. Denote

$$M_0 = \max\left(\log N\mathfrak{f},\ c_{11} n^{3.5} \log^{2/3}\tau (\log\log\tau)^{1/3},\ c_{12} n^5 \sqrt{d}\, D \frac{\log^{2/3}\tau}{(\log\log\tau)^{2/3}}\right),$$

$$r(\tau) = \frac{(\log\log\tau)^{2/3}}{2\cdot 10^3 \log^{2/3}\tau}, \qquad \alpha_0 = 1 + \frac{1}{10^3 M_0}$$

and let $s_0 = \alpha_0 + i\tau$ and $s_0' = \alpha_0 + i2\tau$. Consider the circles $|s - s_0| \leqslant r(\tau)$ and $|s - s_0'| \leqslant r(\tau)$. Both circles lie in the region from Lemma 1.

For $\sigma > 1$

$$\frac{1}{|\zeta_K(s, \chi)|} = \left|\prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)\right| < \zeta_K(\sigma) < \left(\frac{\sigma}{\sigma - 1}\right)^n.$$

Hence by Lemma 1 we get for $|s - s_0| \leqslant r(\tau)$

$$\left|\frac{\zeta_K(s, \chi)}{\zeta_K(s_0, \chi)}\right| \leqslant 2A_1 N\mathfrak{f}^r \tau^{A_2 r^{3/2}} (\log^{2/3}\tau + \log N\mathfrak{f})(10^3 M_0)^n$$

and similarly for $|s - s_0'| \leqslant r(\tau)$

$$\left|\frac{\zeta_K(s, \chi^2)}{\zeta_K(s_0', \chi^2)}\right| \leqslant 2A_1 N\mathfrak{f}^r \tau^{A_2 r^{3/2}} (\log^{2/3}\tau + \log N\mathfrak{f})(10^3 M_0)^n.$$

Now applying Lemma 2 we have for $\beta > \alpha_0 - \frac{1}{2} r(\tau)$ the estimates

$$-\mathrm{Re}\frac{\zeta_K'}{\zeta_K}(s_0, \chi) \leqslant 6M_0 - \frac{1}{\alpha_0 - \beta} \tag{3.1}$$

and

$$-\mathrm{Re}\frac{\zeta_K'}{\zeta_K}(s_0', \chi^2) \leqslant 6M_0. \tag{3.2}$$

For the principal character $\chi = \chi_0$ we have on real axis the estimate

$$-\frac{\zeta_K'}{\zeta_K}(\alpha_0, \chi_0) = \sum_{\substack{\mathfrak{p}, m \\ \mathfrak{p} \nmid \mathfrak{f}}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}^{m\alpha_0}} \leqslant \sum_{\mathfrak{p}, m} \frac{\log N\mathfrak{p}}{N\mathfrak{p}^{m\alpha_0}} = -\frac{\zeta_K'}{\zeta_K}(\alpha_0)$$

and by Lemma 3 since $1 < \alpha_0 < 1.001$, $\frac{\Gamma'}{\Gamma}\left(\frac{\alpha_0}{2}\right) < 0$ and $\frac{\Gamma'}{\Gamma}(\alpha_0) < 0$ we get

$$-\frac{\zeta_K'}{\zeta_K}(\alpha_0, \chi_0) < \frac{1.001}{\alpha_0 - 1}. \tag{3.3}$$

Now from the well-known cosinus inequality for $\sigma > 1$,

$$-3\frac{\zeta_K'}{\zeta_K}(\sigma, \chi_0) - 4\mathrm{Re}\frac{\zeta_K'}{\zeta_K}(\sigma + it, \chi) - \mathrm{Re}\frac{\zeta_K'}{\zeta_K}(\sigma + i2t, \chi^2) \geqslant 0,$$

putting $t = \tau$ and $\sigma = \alpha_0$ we get by (3.1), (3.2) and (3.3) the estimate

$$3033\, M_0 - \frac{4}{\alpha_0 - \beta} \geqslant 0.$$

Hence

$$\beta < 1 - \frac{1}{10^4 M_0}.$$

If $\beta < \alpha_0 - \frac{1}{2} r(\tau)$ we obtain a similar result. It means that the proof of Theorem A is complete.

4. Proof of Theorem B. The asymptotic formula $\pi_C(x) \sim \frac{|C|}{|G|}\mathrm{Li}(x)$ with an explicit remainder term is derived by partial summation from that for $\psi_C(x)$. We first prove

LEMMA 6 (compare [3], th. 9.2). *There exist absolute constants* c_{13}, c_{14}, c_{15}, c_{16} *and* c_{17} *such that if* $x \geqslant \exp\exp(c_{13} n_L^{1.5} \sqrt{d_L}\, D_L)$ *then*

$$\psi_C(x) = \frac{|C|}{|G|} x - \frac{|C|}{|G|} \chi_0(g) \frac{x^{\beta_0}}{\beta_0} + R(x)$$

where

$$R(x) \ll x \exp\left(-c_{14} n_L^{-2.1} \log^{3/5} x (\log\log x)^{-1/5}\right) \tag{4.1}$$

and if $\exp(c_{15} n_L^5 \sqrt{d_L} (\log d_L) D_L) \leqslant x \leqslant \exp\exp(c_{16} n_L^{1.5} \sqrt{d_L}\, D_L)$ *then*

$$R(x) \ll x \exp\left(-c_{17} n_L^{-3} d_L^{-0.3} D_L^{-0.6} \log^{3/5} x (\log\log x)^{2/5}\right). \tag{4.2}$$

β_0 *denotes the hypothetical real zero of* $\zeta_L(s)$ *and the* β_0 *term is present* *only when β_0 does exist. χ_0 is a real character of the cyclic group $H = \langle g \rangle$ for which $\zeta_{L/E}(s, \chi_0)$ has β_0 as a zero, E is a fixed field of H.*

Proof. We have

$$\left|\psi_C(x) - \frac{|C|}{|G|}x + \frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right|$$

$$\leqslant \left|\psi_C(x) - \frac{|C|}{|G|}x + S(x, T)\right| + \left|\frac{|C|}{|G|}\chi_0(g)\frac{x^{\beta_0}}{\beta_0} - S(x, T)\right| = W_1 + W_2.$$

Lemma 5 gives the required estimate for W_1 and W_2 we can estimate as follows

$$W_2 = \frac{|C|}{|G|}\left|\sum_{\chi}\bar{\chi}(g)\left\{\sum_{\substack{\varrho \\ |\mathrm{Im}\,\varrho| < T}}\frac{x^{\varrho}}{\varrho} - \sum_{\substack{\varrho \\ |\varrho| < 1/2}}\frac{1}{\varrho}\right\} - \chi_0(g)\frac{x^{\beta_0}}{\beta_0}\right|$$

$$\ll \frac{|C|}{|G|}\sum_{\chi}\left(\sum_{\substack{|\varrho| \geqslant 1/2 \\ \varrho \neq \beta_0 \\ |\mathrm{Im}\,\varrho| < T}}\frac{|x^{\varrho}|}{|\varrho|} + \sum_{\substack{|\varrho| < 1/2 \\ \varrho \neq 1-\beta_0}}\left(\frac{|x^{\varrho}|}{|\varrho|} + \frac{1}{|\varrho|}\right) + \left|\frac{x^{1-\beta_0}}{1-\beta_0} - \frac{1}{1-\beta_0}\right|\right)$$

where

$$\frac{x^{1-\beta_0} - 1}{1-\beta_0} \leqslant \frac{x^{1/\log x} - 1}{1/\log x} \leqslant x^{1/2}$$

and by Lemma 4 and the fact that for $\varrho \neq 1-\beta_0$, $|\varrho| \geqslant 1/(4\log d_L)$ (see [3], Lemma 8.2) using the conductor-discriminant formula $\sum_{\chi}\log(d_E N_{E/Q}\mathfrak{f}(\chi)) = \log d_L$, we obtain

$$\sum_{\chi}\sum_{\substack{\varrho \neq 1-\beta_0 \\ |\varrho| < 1/2}}\left\{\left|\frac{x^{\varrho}}{\varrho}\right| + \frac{1}{|\varrho|}\right\} \leqslant 2x^{1/2}\sum_{\substack{\varrho \neq 1-\beta_0 \\ |\varrho| < 1/2}}\frac{1}{|\varrho|} \ll x^{1/2}(\log d_L)^2$$

and

$$\sum_{\chi}\sum_{\substack{|\varrho| > 1/2 \\ \varrho \neq \beta_0 \\ |\mathrm{Im}\,\varrho| < T}}\left|\frac{x^{\varrho}}{\varrho}\right| \ll x^{\beta}\log T(\log d_L T^{n_L})$$

where by Theorem A for $T \geqslant 3$

$$x^{\beta} \leqslant x\exp\left(-\frac{\log x}{c_1 n_L^{3.5}\log^{2/3}T(\log\log T)^{1/3}\max\{1, A_1/\log\log T\}}\right).$$

Hence for $\log\log T \leqslant n_L^{1.5}\sqrt{d_L}\,D_L$,

$$W_2 \ll x\log T(\log d_L T^{n_L})\exp\left(-\frac{\log x(\log\log T)^{2/3}}{c_{18}n_L^5\sqrt{d_L}\,D_L\log^{2/3}T}\right) + x^{1/2}(\log d_L)^2.$$

We now choose

$$\log T = \frac{\log^{3/5}x(\log\log x)^{2/5}}{c_{19}n_L^3 D_L^{0.6}d_L^{0.3}}$$

and then

$$W_1 \ll x\exp\left(-\frac{\log^{3/5}x(\log\log x)^{2/5}}{c_{20}n_L^3 d_L^{0.3}D_L^{0.6}}\right) \quad \text{if } \frac{\log x}{\log\log x} \geqslant c_{21}n_L^5\sqrt{d_L}\,D_L,$$

$$W_2 \ll x\exp\left(-\frac{\log^{3/5}x(\log\log x)^{2/5}}{c_{22}n_L^3 d_L^{0.3}D_L^{0.6}}\right) \quad \text{if } \log\log x \leqslant \frac{5}{3}n_L^{1.5}\sqrt{d_L}\,D_L.$$

The estimate (4.2) of Lemma 6 is proved. Now we consider the second case. Let $\log\log T > n_L^{1.5}\sqrt{d_L}\,D_L$. Then

$$W_2 \ll x\log T\log(d_L T^{n_L})\exp\left(-\frac{\log x}{c_{23}n_L^{3.5}\log^{2/3}T(\log\log T)^{1/3}}\right)$$

and we choose

$$\log T = \frac{\log^{3/5}x(\log\log x)^{-1/5}}{c_{24}n_L^{2.1}} \quad \text{and} \quad \log x \geqslant \exp(c_{25}n_L^{1.5}\sqrt{d_L}\,D_L).$$

Then

$$W_1 \ll x\exp\left(2\log\log x - \frac{\log^{3/5}x(\log\log x)^{-1/5}}{c_{26}n_L^{2.1}}\right)$$

$$\ll x\exp\left(-\frac{\log^{3/5}x(\log\log x)^{-1/5}}{c_{27}n_L^{2.1}}\right)$$

and

$$W_2 \ll x\exp\left(2\log\log T - \frac{\log x}{c_{28}n_L^{3.5}\log^{2/3}T(\log\log T)^{1/3}}\right)$$

$$\ll x\exp\left(-\frac{\log^{3/5}x(\log\log x)^{-1/5}}{c_{29}n_L^{3.5}}\right)$$

and we get the result (4.1) of Lemma 6.

The method of the proof of Theorem B is standard. We first define the function

$$\theta_C(x) = \sum_{\substack{P \text{ unramified in } L \\ N_{K/Q}P \leqslant x \\ \left[\frac{L/K}{P}\right] = C}}\log N_{K/Q}P$$

and since there are at most n_K ideals P^m (P prime) of a given norm in K, we have

$$\psi_C(x) = \theta_C(x) + \sum_{\substack{P \text{ unramified in } L \\ N_{K/Q} P^m \leqslant x,\, m \geqslant 2 \\ \left[\frac{L/K}{P}\right] = C}} \log N_{K/Q} P = \theta_C(x) + O(n x \log x)$$

and this shows that the estimates of Lemma 6 hold when $\psi_C(x)$ is replaced by $\theta_C(x)$.

Theorem B now follows from Lemma 6 by a modified form of partial summation (see [4], Lemma 7.3).

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On the linear independence of roots of unity over finite extensions of Q

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The problem we shall treat in the present paper seems to have been first considered by H. B. Mann.

In [4], among other things, the following theorem is proved:

Let

$$\alpha_0 + \alpha_1 \zeta^{n_1} + \ldots + \alpha_{k-1} \zeta^{n_{k-1}} = 0 \tag{1}$$

be an equation, where ζ *is a primitive* N*-th root of unity, the* α_i *are rational numbers, such that no proper subsum of its left-hand side vanishes* (Mann calls such an equation "irreducible").

Then $N/(N, n_1, \ldots, n_{k-1})$ *divides the product of prime numbers up to* k.

This result was improved in one direction by Conway and Jones who showed in [2] that, if $p_1, \ldots, p_s$ are the primes dividing $N/(N, n_1, \ldots, n_{k-1})$ then

$$\sum (p_i - 2) \leqslant k - 2.$$

In another direction Schinzel considered recently the analogous problem to obtain an estimate for the above quotient assuming the coefficients α_i to be elements of some algebraic extension L of the rationals. (A particular case of this had been treated by Loxton [3]: he assumes $\alpha_0 \in L$, while $\alpha_j \in Q$ for $1 \leqslant j \leqslant k-1$.)

Schinzel proves in [5] that there is some bound for the quotient which depends only on k and on the degree $d = [L:Q]$.

However his proof uses van der Waerden's Theorem on arithmetic progressions and so leads to extremely large values for such a bound.

The question arises whether Mann's method (for instance), which is different from Schinzel's one, can be adapted to obtain a more satisfactory estimate.

In this paper we show that the answer is to some extent affirmative.

We remark that the problem is simplified if one looks for bounds