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The distribution of powerful integers of type 4

by

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Let $k \geqslant 2$ be a fixed integer. A natural number n_k is said to be *powerful of type k* if $n_k = 1$ or if each prime factor of n_k divides it at least to the kth power. This paper is concerned only with the distribution of powerful integers n_4 of type 4. Such a number can be uniquely represented by

$$n_4 = a_0^4 a_1^5 a_2^6 a_3^7,$$

where a_1, a_2, a_3 are square-free numbers and $(a_i, a_j) = 1$ for $1 \leqslant i < j \leqslant 3$. We put

$$f_4(n) = \begin{cases} 1 & \text{for } n = n_4, \\ 0 & \text{for } n \neq n_4. \end{cases}$$

Let $N_4(x)$ denote the number of powerful integers of type 4 not exceeding x. Then

$$N_4(x) = \sum_{n_4 \leqslant x} 1 = \sum_{n \leqslant x} f_4(n).$$

For the Dirichlet series

$$F_4(s) = \sum_{n_4=1}^{\infty} \frac{1}{n_4^s} = \sum_{n=1}^{\infty} \frac{f_4(n)}{n^s}$$

we obtain

$$(1) \qquad F_4(s) = \prod_p \left(1 + \frac{p^{-4s}}{1-p^{-s}}\right) = \frac{\zeta(4s)\zeta(5s)\zeta(6s)\zeta(7s)}{\zeta(10s)} \sum_{n=1}^{\infty} \frac{c_{11}(n)}{n^s},$$

where the Dirichlet series $\sum c_{11}(n) n^{-s}$ is absolutely convergent for $\operatorname{Re}(s) > 1/11$. This shows that an asymptotic representation for $N_4(x)$ may be written in the form

$$(2) \qquad N_4(x) = \sum_{\nu=4}^{7} \gamma_{\nu,4} x^{1/\nu} + \Delta_4(x),$$

where

$$\gamma_{v,4} = \operatorname*{Res}_{s=1/v} \frac{1}{s} F_4(s).$$

We put

$$\lambda_4 = \inf\{\varrho_4 \colon \Delta_4(x) \ll x^{\varrho_4}\}.$$

The first estimate for the remainder $\Delta_4(x)$ was proved by P. Erdős and G. Szekeres in [2]. They found $\lambda_4 \leqslant 1/5$. The following improvements of this result are known;

$\lambda_4 \leqslant 1/6 = 0.1\bar{6}$,	P. Bateman–E. Grosswald [1],
$\lambda_4 \leqslant 169/1360 = 0.1242\ldots$,	E. Krätzel [6],
$\lambda_4 \leqslant 257/2072 = 0.1240\ldots$,	A. Ivić [3],
$\lambda_4 \leqslant 3187/25852 = 0.1232\ldots$,	A. Ivić [4],
$\lambda_4 \leqslant 3091/25981 = 0.1189\ldots$,	A. Ivić–P. Shiu [5],
$\lambda_4 \leqslant 5/44 = 0.1136\ldots$,	E. Krätzel [7].

It may be remarked that the last result also can be obtained by means of Theorem 2 of [10]. In this paper we shall prove

$$\lambda_4 \leqslant 21/187 = 0.1122\ldots$$

From representation (1) it is seen that we have

$$f_4(k) = \sum_{mn=k} d(4, 5, 6, 7; m)\, c_{10}(n), \tag{3}$$

where $d(4, 5, 6, 7; n)$ is the divisor function

$$d(4, 5, 6, 7; n) = \#\{(n_1, n_2, n_3, n_4) \colon n_1, \ldots, n_4 \in \mathbf{N},\ n_1^4 n_2^5 n_3^6 n_4^7 = n\},$$

and $c_{10}(n)$ is an arithmetical function with an absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty} \frac{c_{10}(n)}{n^s} = \frac{1}{\zeta(10s)} \sum_{n=1}^{\infty} \frac{c_{11}(n)}{n^s}$$

for $\mathrm{Re}(s) > 1/10$. Therefore, we need an estimation for

$$D(4, 5, 6, 7; x) = \sum_{n \leqslant x} d(4, 5, 6, 7; n).$$

This will be an immediate consequence from the following theorem.

THEOREM 1. *Let a_1, a_2, a_3, a_4 be real numbers with $1 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Put $a = (a_1, a_2, a_3, a_4)$, $A_v = a_1 + \ldots + a_v$ for $v = 1, 2, 3, 4$. Let*

$$D(a; x) = \#\{(n_1, n_2, n_3, n_4) \colon n_1, \ldots, n_4 \in \mathbf{N},\ n_1^{a_1} n_2^{a_2} n_3^{a_3} n_4^{a_4} \leqslant x\}.$$

Then

$$D(a; x) = H(a; x) + \Delta(a; x), \tag{4}$$

where the main term $H(a; x)$ is given by

$$H(a; x) = \sum_{v=1}^{4} \alpha_v x^{1/a_v}, \quad \alpha_v = \prod_{\substack{\mu=1 \\ \mu \neq v}}^{\infty} \zeta(a_\mu/a_v). \tag{5}$$

For the remainder $\Delta(a; x)$ the estimate

$$\Delta(a; x) \ll x^{42/17A_4} \log^4 x \tag{6}$$

holds under the conditions

$$6A_1 \geqslant A_4, \quad 14A_2 \geqslant 5A_4, \quad 42A_3 \geqslant 25A_4.$$

Remark. The representation (5) for the main term holds if $a_1 < a_2 < a_3 < a_4$. However, in cases of some equalities we can take the limit values.

Proof. The method of proof is the same as for the corresponding Theorem 2 of [10]. Let $u = (u_1, u_2, u_3, u_4)$ and

$$S(u; x) = \sum\nolimits_1 \psi\left(\left(\frac{x}{n_1^{u_1} n_2^{u_2} n_3^{u_3}}\right)^{1/u_4}\right),$$

$$\mathrm{SC}(\Sigma_1)\colon\ n_1^{u_1} n_2^{u_2} n_3^{u_3+u_4} \leqslant x, \quad 1 \leqslant n_1 (\leqslant) n_2 \leqslant n_3.$$

$n_1 (\leqslant) n_2$ means that $n_1 \leqslant n_2$ for $u_1 = a_i$, $u_2 = a_j$ and $i < j$ and $n_1 < n_2$ otherwise.

Then it is known from equations (18) and (19) of [10] that

$$\Delta(a; x) = -\sum_u S(u; x) + O(x^{2/A_4}), \tag{7}$$

where u runs over all permutations of the numbers a_1, a_2, a_3, a_4. Let $N = (N_1, N_2, N_3)$. We consider the sum

$$S(u, N; x) = \sum\nolimits_2 \psi\left(\left(\frac{x}{n_1^{u_1} n_2^{u_2} n_3^{u_3}}\right)^{1/u_4}\right),$$

$$\mathrm{SC}(\Sigma_2)\colon\ n_1^{u_1} n_2^{u_2} n_3^{u_3+u_4} \leqslant x, \quad 1 \leqslant n_1 (\leqslant) n_2 \leqslant n_3,$$

$$N_v \leqslant n_v \leqslant 2N_v \quad (v = 1, 2, 3).$$

We now use the inequality (28), Hilfssatz 6 [8] or, what is the same, inequality (6.18), Lemma 6.4 [9] to the sum over n_2, n_3. Then

$$S(u, N; x) \ll \sum_{N_1 \leqslant n_1 \leqslant 2N_1} \left(\left(\frac{x}{n_1^{u_1}}\right)^7 N_2^{10u_4-7u_2} N_3^{8u_4-7u_3}\right)^{1/17u_4} \log x$$
$$\ll (x^7 N_1^{17u_4-7u_1} N_2^{10u_4-7u_2} N_3^{8u_4-7u_3})^{1/17u_4} \log x$$
$$\ll \left(x^7 (N_1^{u_1} N_2^{u_2} N_3^{u_3+u_4})^{y_1} \left(\frac{N_1}{N_2}\right)^{y_2} \left(\frac{N_2}{N_3}\right)^{y_3}\right)^{1/17u_4} \log x.$$

Because of $u_1+u_2+u_3+u_4 = A_4$ we obtain

$$A_4 y_1 = 7(6u_4 - A_4),$$
$$A_4 y_2 = u_4(17A_4 - 42u_1),$$
$$A_4 y_3 = 3u_4(14(u_3+u_4) - 5A_4).$$

Therefore, y_1, y_2, y_3 are non-negative for all permutations of u_1, u_2, u_3, u_4 if

$$6a_1 \geqslant A_4, \quad 17A_4 \geqslant 42a_4, \quad 14(a_1+a_2) \geqslant 5A_4.$$

These inequalities follow from the conditions of the theorem. If we now use

$$N_1^{u_1} N_2^{u_2} N_3^{u_3+u_4} \ll x, \quad N_1 \ll N_2 \ll N_3,$$

then

$$S(u; N; x) \ll x^{42/17A_4} \log x, \quad S(u; x) \ll {}^{42/17A_4} \log^4 x.$$

Now (6) follows from (7).

THEOREM 2.

(8) $$\Delta_4(x) \ll x^{21/187} \log^4 x.$$

Proof. We apply Theorem 1 with $a_1 = 4$, $a_2 = 5$, $a_3 = 6$, $a_4 = 7$. Then $A_1 = 4$, $A_2 = 9$, $A_3 = 15$, $A_4 = 22$ and the conditions of the theorem are satisfied. Hence

$$D(4, 5, 6, 7; x) = \alpha_1 x^{1/4} + \alpha_2 x^{1/5} + \alpha_3 x^{1/6} + \alpha_4 x^{1/7} + O(x^{21/187} \log^4 x)$$

with

$$\alpha_1 = \zeta(\tfrac{5}{4})\zeta(\tfrac{6}{4})\zeta(\tfrac{7}{4}), \quad \alpha_2 = \zeta(\tfrac{4}{5})\zeta(\tfrac{6}{5})\zeta(\tfrac{7}{5}),$$
$$\alpha_3 = \zeta(\tfrac{4}{6})\zeta(\tfrac{5}{6})\zeta(\tfrac{7}{6}), \quad \alpha_4 = \zeta(\tfrac{4}{7})\zeta(\tfrac{5}{7})\zeta(\tfrac{6}{7}).$$

Thus, we obtain from (3)

$$N_4(x) = \sum_{mn \leqslant x} d(4, 5, 6, 7; m) c_{10}(n) = \sum_{n \leqslant x} c_{10}(n) D(4, 5, 6, 7; x/n)$$
$$= \sum_{n \leqslant x} c_{10}(n) \left\{\alpha_1 \left(\frac{x}{n}\right)^{1/4} + \alpha_2 \left(\frac{x}{n}\right)^{1/5} + \alpha_3 \left(\frac{x}{n}\right)^{1/6} + \alpha_4 \left(\frac{x}{n}\right)^{1/7}\right\}$$

$$+ O\left(\sum_{n \leqslant x} |c_{10}(n)| \left(\frac{x}{n}\right)^{21/187} \log^4 \left(\frac{x}{n}\right)\right)$$
$$= \sum_{\nu=4}^{7} \gamma_{\nu,4} x^{1/\nu} + O(x^{21/187} \log^4 x).$$

This proves (8).

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